QUASISTATIC DROP MODEL OF THE NUCLEUS AS AN APPROXIMATION TO THE STATISTICAL MODEL

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It is shown that the problem of determining the equilibrium distribution of nucleon density in the quasiclassical statistical model of nuclear matter under the assumption of low compressibility reduces to determining the equilibrium shape of an effective nuclear surface. In this approximation the statistical and the drop models are identical. Corrections to the drop model due to the deformation of the diffuse layer and to finite compressibility are derived.

In the statistical model the energy of a nucleus may be written as a functional of the nucleon density:

$$E = \int dv\, [\mathscr{E}(\rho) + F(\rho)(\nabla\rho)^2 + \tfrac{1}{2}\Phi\rho_Z], \qquad (1)$$

where ρ and ρ_Z are respectively the average densities of the total number of particles and of the charge. For the sake of simplicity we shall take ρ_Z to be proportional to ρ:

$$\rho_Z = Ze\rho/A \qquad (2)$$

(Z is the number of protons, A is the total number of nucleons in the nucleus.)

In Eq. (1) Φ is the Coulomb potential, and $\mathscr{E}(\rho)$ and $F(\rho)$ are functions of the density. We assume (cf., for example, [1-5]) that

$$F(\rho) = \zeta\hbar^2/8M\rho, \qquad (3)$$

where M is the nucleon mass, ζ is a numerical factor of the order of unity. The specific form of the function $F(\rho)$ is not essential for further development. Inside the nucleus the term containing $\mathscr{E}(\rho)$ is the principal one. The function $\mathscr{E}(\rho)/\rho$ (energy per particle) has a minimum at a certain value of the density $\rho = \rho^*$. For values of ρ which differ little from ρ^* it can be written in the form

$$\mathscr{E}(\rho)/\rho \approx E^* + \varkappa(\rho - \rho^*)^2, \qquad (4)$$

where ρ^* corresponds to the equilibrium value of the density in an infinite uncharged medium, while E^* is the nucleon binding energy in such a medium ($E^* < 0$). The quantity κ is related to the coefficient of volume compressibility of nuclear medium E_0'' by the relation[6]

$$\varkappa = E_0''/18\rho^{*2}A, \qquad (5)$$

where[6]

$$E_0'' = R^2(\partial^2 E_{vol}/\partial R^2)_{R=R_0}, \qquad (6)$$

E_{vol} is the volume energy of the nucleus, while R_0 is determined by the condition

$$\tfrac{4}{3}\pi R_0^3\rho^* = A. \qquad (7)$$

It is convenient to go over to dimensionless variables taking for the unit of energy the magnitude of $|E^*|$, for the unit of density ρ^*, and for the unit of spatial variables R_0. The Coulomb potential is measured in units of ZeR_0^{-1}. We introduce the function $\epsilon(\rho)$ defined by the relation

$$\mathscr{E}(\rho)/\rho = -1 + \varepsilon(\rho). \qquad (8)$$

For $\rho \to 1$

$$\varepsilon(\rho) \approx k(\rho - 1)^2, \qquad k = \varkappa\rho^{*2}/|E^*|. \qquad (9)$$

The energy of the nucleus (1) can be written in the form

$$E = E_{vol}^{(0)}\left\{1 - \frac{3}{4\pi}\int dv\left[\rho\varepsilon(\rho) + \frac{1}{4}\gamma^2\rho^{-1}(\nabla\rho)^2 + \frac{1}{2}X\rho\Phi\right]\right\}, \qquad (10)$$

$$E_{vol}^{(0)} = E^*A, \qquad \gamma^2 = \frac{\zeta\hbar^2}{2M|E^*|R_0^2}, \qquad X = \frac{Z^2e^2}{R_0A|E^*|}. \qquad (11)$$

The equilibrium density distribution corresponds to the energy minimum (10) subject to the additional condition that the number of particles is constant

$$\int \rho dv = \tfrac{4}{3}\pi. \qquad (12)$$

The parameter γ is a small quantity, and our problem consists of finding the corresponding expansion for ρ and for the energy of the nucleus. The equilibrium density distribution can be obtained from the equation

$$2\gamma^2\Delta w - dw^2\varepsilon(w^2)/dw - 2[\lambda + X\Phi]w = 0, \qquad (13)$$

where $w^2 = \rho$; λ is a Lagrangian multiplier which can be interpreted as a correction to the binding energy of the particle due to the finite dimensions of the nucleus and to the Coulomb interaction. At large distances outside the nucleus $\epsilon(w^2) \rightarrow 1$. For $\gamma \ll 1$ the main terms in (13) give in the exterior region

$$\rho \sim e^{-2r/\gamma}. \tag{14}$$

In the interior region under the condition of low compressibility the density differs little from unity. The correction can be obtained if we neglect in (10) the gradient terms which will give a contribution only of order γ^3. By utilizing (9) we obtain

$$\rho_{\text{int}} = w^2_{\text{int}} \approx 1 - \tfrac{1}{2} k^{-1} (\lambda + X\Phi). \tag{15}$$

In a narrow transition layer of thickness of order γ the nucleon density varies from the value (15) close to unity to an exponentially small quantity. This physically defines a certain effective nuclear surface. We shall discuss the case when there is an axis of symmetry which we shall assume to be taken as the z axis.

Let $y = y(z)$ be the generating curve for the effective surface. We introduce a system of coordinates associated with the surface. For one of the axes we choose the z axis, and we introduce in the usual manner the azimuthal angle φ. For the axis corresponding to the second variable u we take the direction of the external normal to the generating curve $y(z)$ drawn at the given point. Along the u axis we characterize each point by the distance from the point under consideration to the surface (the internal points correspond to $u < 0$). The element of volume in these variables is given by

$$dv = \sqrt{g}\, du\, dz\, d\varphi, \qquad \sqrt{g} = (R_1 + u)(1 + u/R_2), \tag{16}$$

where R_1 and R_2 are the principal radii of curvature. The system of coordinates introduced above is orthogonal.

For a reasonably chosen effective surface in a layer of order γ near it derivatives with respect to u must be of order γ^{-1}, while, as will be seen below, derivatives of the density with respect to z are of order γ. This enables us to simplify (13) in this region by writing it in the form

$$w^2 \varepsilon (w^2) + \lambda w^2 - \gamma^2 w'^2 - 2\gamma^2 \int_{+\infty}^{u} \frac{\partial \ln \sqrt{g}}{\partial u} w'^2\, du + X \int_{+\infty}^{u} \Phi \frac{\partial w^2}{\partial u}\, du = 0. \tag{17}$$

Here we have utilized the condition $w \rightarrow 0$ for $u > +\gamma$.

In zero order with respect to γ we obtain the equation

$$w_0^2 \varepsilon (w_0^2) = \gamma^2 w_0'^2. \tag{18}$$

It is, naturally, identical with the equation for the equilibrium density distribution for a semi-infinite nuclear medium ($R_1, R_2 \rightarrow \infty$), which, for example, has been discussed by Wilets [7]. Equation (18) has a solution satisfying the boundary conditions

$$w_0(-\infty) = 1, \qquad w_0(+\infty) = 0. \tag{19}$$

The argument of the function w_0 contains an arbitrary additive constant which cannot be found from conditions (19). It is determined from the additional condition $\alpha = 0$ (cf. Eq. (25) below). We shall not need the specific form of w_0. The only essential feature is that w_0 has derivatives with respect to u of order γ^{-1} in a narrow region of order γ near the surface, and differs by an exponentially small amount from unity and from zero respectively for $u < -\gamma$ and $u > +\gamma$.

The solution of (17) must go over into (15) for $u < -\gamma$. In the zero order approximation this requirement is satisfied. The situation is different in the case of terms of higher order. Equation (17) gives the correct asymptotic value of the density for $u < -\gamma$ determined by the expression (15) only for a particular choice of the effective surface since in order that (17) and (15) should be identical for $u < -\gamma$ it is necessary that the sum of the "extra" terms in (17) should vanish.

This can be achieved only by a proper choice of the effective surface, and, therefore, the condition for the existence of an equilibrium density distribution also determines the equilibrium shape of the nucleus. This condition has the form

$$2O_0 H + \lambda + X\Phi_S + 2O_1 (R_1 R_2)^{-1} - \tfrac{1}{4} k^{-1} [\lambda + X\Phi_S]^2 + [2H(\lambda + X\Phi_S) + X(\partial\Phi/\partial u)_S]\, \alpha = 0. \tag{20}$$

Here the constants are given by

$$O_0 = 2\gamma^2 \int_{-\infty}^{+\infty} w_0'^2\, du, \tag{21}$$

$$O_1 = 2\gamma^2 \int_{-\infty}^{+\infty} u w_0'^2\, du. \tag{22}$$

The subscript S denotes that the corresponding quantities are taken at the surface at the point $(z, y(z))$.

$$H = \tfrac{1}{2} (R_1^{-1} + R_2^{-1}) \tag{23}$$

is the average curvature of the surface. The potential Φ corresponds to a nucleus with a smeared out edge and with a density at the center corrected for finite compressibility in accordance with (15). The quantity λ is determined from the auxiliary condition (12) which in the approximation under consideration is

$$\frac{4\pi}{3} = \int dv_0 \rho_{\text{int}} + \alpha \int dS, \tag{24}$$

where ρ_{int} is determined by expression (15), the first integral is taken over the volume surrounded by the surface, and the second integral is taken over the surface

$$\alpha = \int_{-\infty}^{+\infty} [w^2 - \theta(-u)]\, du, \tag{25}$$

where $\theta(x)$ is the unit function.

The constant α in (20) and in the auxiliary condition (23) can be set equal to zero. As we have noted earlier, this fixes the additive constant in the argument of the function w_0 and determines the position of the effective surface within the limits of the diffuse layer.

In first order (the three first terms), Eq. (20) formally coincides with the condition for the equilibrium of the surface of a charged drop. Here the effect of the boundary being smeared out is also taken into account in the first order.

The density distribution w_0 is given by (18). The remaining terms represent a correction of the order of the expansion parameter. The density inside the nucleus is in this order determined by (15).

Without the term proportional to k^{-1} which takes into account the finite compressibility of nuclear matter, Eq. (20) with the auxiliary condition (24) is identical with the condition for the equilibrium of a charged liquid drop which was discussed in a paper by one of the authors [8]. It was obtained there by means of a direct variation of the energy of the nucleus regarded as a functional of the surface, where for the surface tension a correction was taken into account which was proportional to the average curvature of the surface:

$$E_S = O_0 \int dS\,(1 - \Gamma H), \tag{26}$$

$$\Gamma = -2O_1/O_0. \tag{27}$$

We now find an expression for the energy of the nucleus. In order to do this we must substitute in the integral (10) in the inner region $w = w_{\text{int}}$ [cf., (15)] and we must use equation (17) for w within the limits of the diffuse layer. As a result we obtain the following expression for the energy:

$$E = E^0_{\text{vol}} \Big\{1 - \frac{3}{4\pi} O_0 \int dS\,(1 - \Gamma H) - E^0_C - \frac{3}{4\pi}\frac{1}{4k}\int dv_0\,[\lambda^2 - X^2\Phi^2]\Big\}. \tag{28}$$

Here for λ and Φ we take their zero order values, the same as in the liquid drop model.

The term ΓH represents the change in the surface energy due to the deformation of the surface layer. The Coulomb energy $E_C^{(0)}$ is calculated as is usually done in the liquid drop model, i.e., for such a distribution of nucleons for which the density is equal to unity within the surface and to zero outside it. The sum of these terms can be interpreted as the energy of some effective uniformly charged drop with a sharp boundary, although actually the fact that the boundary is smeared out is taken into account up to terms of second order in γ. The term proportional to k^{-1} takes into account the compression of nuclear matter within the nucleus. As $k \to \infty$ only the "drop" terms remain in formulas for the energy and for the surface, and in the auxiliary equation.

For a spherical nucleus we have in the zero order approximation

$$\lambda = 2O_0 - X, \tag{29}$$

$$E = E^{(0)}_{\text{vol}}\{1 - 3O_0(1-\Gamma) - \tfrac{3}{5}X - k^{-1}(O_0^2 - \tfrac{1}{5}XO_0 + \tfrac{1}{70}X^2)\} \tag{30}$$

or, singling out the explicit dependence on A and Z we obtain

$$E = E^*A + aA^{2/3} + (b - a^2/c)A^{1/3} + dZ^2A^{-1/3}[1 + (a/cA^{1/3})(1 - \tfrac{5}{7}x)], \tag{31}$$

where x is defined by (37) (cf. below),

$$a = 3\,|E^*|\,O_0A^{1/3}, \qquad b = -a\Gamma A^{1/3}, \qquad d = 3e^2/5r_0, \tag{32}$$

with

$$r_0 = (3/4\pi\rho^*)^{1/3}, \tag{33}$$

$$c = E_0''/2A. \tag{34}$$

None of the constants in (32)—(34) depends on A and Z. Assuming a definite density distribution near the surface of the nucleus we can calculate with the aid of formulas (21) and (22) the quantities O_0 and O_1, and then estimate a and b. For this it is necessary that the condition $\alpha = 0$ be fulfilled. This occurs, in particular, for the Fermi density distribution:

$$w_0^2 = [1 + e^{(2u/\gamma)}]^{-1}.$$

For $\gamma \approx 1.0\,A^{-1/3}$ (see [9]) we obtain $O_0 = 0.5\,A^{-1/3}$ and $\Gamma = -1.0\,A^{-1/3}$. For $|E^*| = 16$ MeV[9] we have the values $a \approx 24$ MeV and $b \approx 24$ MeV. The quantity a does not differ appreciably from the coefficient of $A^{2/3}$ in the semiempirical formulas for nuclear masses, which is equal to 15—25 MeV depending on the specific form of the formula.

In modern semiempirical formulas for nuclear masses the compressibility of nuclear matter is in fact not taken into account (cf., for example, [10,11]). It can be estimated from the value of the nuclear radius by means of relation (24). In first approximation in terms of γ the radius of the effective surface is given by

$$R = 1 - ac^{-1} A^{-1/3} (1 - \tfrac{1}{2} da^{-1} Z^2 A^{-1}) = 1 - O_0/3k + X/30k. \tag{35}$$

A similar result was apparently for the first time obtained in the paper by Flügge and Woeste [12]. In its usual form formula (35) can be written as

$$R = r_0 A^{1/3} - dc^{-1} r_0 (1 - x), \tag{36}$$

where x is defined as usual in the liquid drop model:

$$x = \tfrac{1}{2} (Z^2/A)\, d/a = (Z^2/A)/(Z^2/A)_{\text{crit}}, \quad (Z^2/A)_{\text{crit}} \approx 50. \tag{37}$$

The approximation of order γ^2 discussed here corresponds, roughly speaking, to terms of order $A^{1/3}$ in the formula for the nuclear mass. In the same order there also arise corrections associated with the degeneracy of single particle quantum states (corrections due to shells). The single particle quantum corrections would correspond to terms of order A^0 (or γ^3). It is obvious that the quantum corrections are of an accidental nature and cannot be significant unless we are dealing with strictly defined quantum stationary states. Both models discussed here correspond to the quasiclassical approximation and because of this can be utilized only for the description of nuclear properties averaged over a group of levels of a single nucleus or generally over a group of nuclei. In this case the accuracy of the quasiclassical theory can be entirely adequate. In any case it is better than that given by the usual quantum conditions for the applicability of the quasiclassical approximation for the description of quantum states.

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