RENORMALIZATION CONSTANTS AND BOUND STATES IN A NONRELATIVISTIC MODEL OF FIELD THEORY

M. A. BRAUN

Leningrad State University

Submitted to JETP editor January 14, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) 45, 936-947 (October, 1963)

A nonrelativistic model is studied which is a generalization of the quantum mechanics of two particles interacting with a potential. A new point interaction of the particles is introduced which involves their conversion into a third particle, which has the same quantum numbers as a bound state of the primary particles. In the theory so obtained the mass of the new particle and its interaction constant are arbitrary. In the special case in which the mass of the new particle is chosen exactly equal to the mass of the bound state of the original particles the entire theory is identical with ordinary quantum mechanics. The renormalization constants Z and Z_1 then become zero.

ZIMMERMANN has shown that a compound particle can be described in quantum field theory by a local field.[1] In the present paper we shall use a model, which will be described, to discuss the question of whether one can describe a compound particle by an independent local field and thus achieve complete external symmetry among the compound and elementary particles. This is an interesting question in connection with the suggested possibility of constructing the S matrix on the basis of its analytic properties and unitarity. As is well known, with this approach compound and elementary particles appear on a completely equal footing.

We consider in the Lagrangian formalism two theories: one in which besides the elementary particles described by the fields there are compound particles, and another in which instead of compound particles we introduce independent particles (fields) with the same quantum numbers. These two theories lead to identical analytic properties and unitarity conditions for the S matrix. Therefore we can assume that the scattering amplitudes will be the same in these theories, i.e., the two theories are physically indistinguishable. It is then still not understood how in the second case, with the compound particles described by independent fields, the characteristics—masses and interaction constants—of these particles are determined.

A paper by Salam [2] puts forward without proof, on the basis of intuitive arguments, two requirements which are to determine the mass and the coupling constant of a compound particle when it is described by an independent field: the renormalized charge is to be equal to zero and there is to be no mass renormalization. Salam concluded from this that if Z_1 and δM are the constants in the charge and mass renormalizations of the compound particle, then $Z_1 = 0$ and $\delta M = 0$. A number of other papers [3-5] have discussed the condition $Z = 0$ (Z is the wave-function renormalization constant) and reached rather indefinite conclusions that when this is so the particle behaves in some sense like a compound particle.

In this paper we shall start from a theory in which it is exactly known what the bound states (i.e., compound particles) are, and what their properties are—from the nonrelativistic quantum mechanics of two particles interacting with a certain potential. These particles, which are assumed to have no spin, will be conventionally called "nucleons" or m-particles (m is their mass). We complete the physical scheme with one more interaction—a point interaction of the nucleons with conversion into another particle of mass M, the M-particle, which has the same quantum numbers as the bound state of the nucleons, the deuteron. We are interested in seeing to what extent and under what conditions the M-particle so introduced imitates the deuteron.

For the reader's convenience we list the main results.

1. After the interaction of the nucleons with the M-particle is introduced the resulting theory is in general different from the previous theory, and is physically entirely reasonable for arbitrary values of the mass of the M-particle and its coupling constant with the nucleons, g (unrenormalized). It is curious that for certain values of

the mass of the M-particle and the constant g the bound state disappears and only the elementary particles remain in the theory. There is an upper bound on the renormalized coupling constant of the M-particle with the nucleons. This restriction is somewhat more complicated than that found by Landau.[6] The renormalization constants Z and Z_1 are in general finite, and $\delta M = \infty$.

2. When the mass of the M-particle is chosen equal to the mass of the deuteron, the entire theory is the same as the initial theory (without the M-particle). Then $Z = Z_1 = 0$, in agreement with both [2] and [3-5]. It is important that the renormalized coupling constant is then entirely independent of the unrenormalized constant. Accordingly, the latter is arbitrary, and is not zero as in [2]. The renormalized coupling constant is just equal to the residue at the pole of the deuteron wave function in momentum space, as it should be according to dispersion theory.

3. In the theory with the M-particle, when one of the nucleons and the M-particle are physical particles the vertex part is in general a nondecreasing function. This explains the fact that in the general case the dispersion relations for this function do not give any restrictions on the mass of the M-particle. When $Z = Z_1 = 0$ the vertex part becomes a decreasing function, and the dispersion equation for it is then a nontrivial equation which allows us to determine the mass of the M-particle (cf.[7]).

In this connection we note further that in our model the anomalous thresholds that appear in the vertex part for definite ratios of the masses of the particles are not related to the elementary or nonelementary nature of the particles.

Thus the answer to the question stated at the beginning—whether one can describe a compound particle by an independent field—is in the affirmative in quantum mechanics. The mass of the compound particle is given uniquely by the conditions $Z = 0$, $Z_1 = 0$, or by the dispersion relation for the vertex part. It is remarkable that these conditions do not arise from the theory itself, but must be imposed on it from outside.

1. THE MODEL

We consider two identical scalar particles of mass m which interact with each other with a potential V. Besides this, these particles can undergo conversion into a third particle of mass M (the M-particle). In the second-quantization formalism the Lagrangian of the system is

$$L = L_m + L_M + L_{mm} + L_{mM}, \tag{1}$$

L_m and L_M describe free m- and M-particles.

The entire system is assumed nonrelativistic and Galilean-invariant. Therefore we have, for example, for the nucleons

$$L_m = \int d^3x \left(i\psi^+\dot{\psi} - \frac{1}{2m} \nabla\psi^+ \nabla\psi - m\psi^+\psi \right), \tag{2}$$

$\psi(x_\alpha)$ is the operator for the nucleon field; x_α is the coordinate-time four-vector ($\alpha = 0, 1, 2, 3$); we keep the notation x for the coordinate three-vector. In what follows we use the notation $x_\alpha y_\alpha = x_0 y_0 - \mathbf{xy}$. L_M differs from L_m by the replacement of ψ by the operator φ of the M field, and of the mass m by M.

L_{mm} and L_{mM} relate to the interaction between m-particles and the interaction between m- and M-particles:

$$L_{mm} = -\frac{1}{2}\int d^3x\, d^3y\, \psi^+(x_\alpha)\, \psi^+(y_\alpha)\, V(x-y)\, \psi(x_\alpha)\, \psi(y_\alpha) \tag{3}$$

(with $x_0 = y_0$),

$$L_{mM} = -g\int d^3x\, (\varphi^+\psi\psi + \psi^+\psi^+\varphi). \tag{4}$$

The commutation rules for equal times are the usual ones:

$$[\psi(x), \psi^+(y)] = [\varphi(x), \varphi^+(y)] = \delta(x-y). \tag{5}$$

The other commutators are zero.

We note further that for Galilean invariance it is necessary to regard the difference $2m - M \equiv \delta > 0$ as small. Then the energy of the motion of the center of mass (without any constant term), which is all that changes under a Galilean transformation, can be given approximately the same value for a pair of m-particles and for the M-particle into which this pair has been converted: $p^2/4m \approx p^2/2M$. It is essential here that we regard p^2 as a quantity small in comparison with m^2 or M^2.

When $g = 0$ the system reduces to ordinary quantum mechanics and consists of free M-particles and nucleons interacting with the potential V. In the two-nucleon sector we get exhaustive information about the system from the Green's function:

$$G(x_\alpha, y_\alpha, x'_\alpha, y'_\alpha) = -\langle 0 \,|\, T(\psi(x_\alpha)\, \psi(y_\alpha)\, \psi^+(x'_\alpha)\, \psi^+(y'_\alpha))\, |0\rangle,$$

where $|0\rangle$ is the vacuum state. For equal initial and final times ($x_0 = y_0$, $x'_0 = y'_0$) this Green's function reduces to the Green's function of the Schrödinger equation for two nucleons and can be expressed in a simple way in terms of the solution of this equation.

We denote by $G(k_1 k_2 E, k_1' k_2' E')$ the Fourier transform of the Green's function G for $x_0 = y_0$ and $x_0' = y_0'$:

$$G(k_1k_2E, k_1'k_2'E') = \int d^4x d^4y d^4x' d^4y' \, \delta(x_0 - y_0) \, \delta(x_0' - y_0') \times G(x_\alpha, y_\alpha, x_\alpha', y_\alpha') \times \exp(ik_{1\alpha}x_\alpha + ik_{2\alpha}y_\alpha - ik_{1\alpha}'x_\alpha' - ik_{2\alpha}'y_\alpha'),$$
$$k_{10} + k_{20} = E, \quad k_{10}' + k_{20}' = E'. \tag{6}$$

It is an easy calculation to show that

$$G(k_1k_2E, k_1'k_2'E') = (2\pi)^7 \, \delta(E - E') \, \delta^3(K - K') \, G(\varepsilon, k, k'), \tag{7}$$

where

$$K = k_1 + k_2, \quad K' = k_1' + k_2', \quad k = {}^1/_2 (k_1 - k_2),$$
$$k' = {}^1/_2 (k_1' - k_2'), \quad \varepsilon = E - K^2/4m - 2m;$$
$$G(\varepsilon, k, k') = -i \int d^3p \, \frac{f_p^{(\pm)}(k) f_p^{(\pm)*}(k')}{\varepsilon - p^2/m + i0} - i \, \frac{f_0(k) f_0(k')}{\varepsilon - \varepsilon_0}. \tag{8}$$

Here $f_p^{(\pm)}(k)$ and $f_0(k)$ are solutions of the stationary Schrödinger equation in momentum space for the scattering states (the signs ± refer to converging or diverging waves) and the bound state (deuteron) with the binding energy ϵ_0:

$$(p^2/m - k^2/m) \, f_p^{(\pm)}(k) = \frac{1}{(2\pi)^3} \int d^3k' v(k - k') \, f_p^{(\pm)}(k') \tag{9}$$

and an analogous equation for $f_0(k)$; v is the Fourier transform of the potential V. We assume that the nucleons form only one bound state. This restriction is unimportant for the results.

Let us see what changes arise from the turning-on of the interaction between m-particles and M-particles. The two-nucleon sector is now characterized by two Green's functions:

$$G_1 = -\langle 0 | T(\psi\psi\psi^+\psi^+) | 0 \rangle, \quad G_2 = i \langle 0 | T(\varphi\varphi^+) | 0 \rangle.$$

They can both be found explicitly without difficulty, in terms of G, the coupling constant g, and the masses m and M.

Let us begin with G_2. Obviously the proper mass of the M-particle is (see Fig. 1)

$$M(x_\alpha - y_\alpha) = ig^2 G(x_\alpha, x_\alpha, y_\alpha, y_\alpha). \tag{10}$$

From this we get as the expression for the function G_2 in momentum space

$$G_2(p_\alpha) = \Delta_M(p_\alpha) \left[1 + ig^2 \Delta_M^*(p_\alpha) \frac{1}{(2\pi)^3} \int d^3k \, d^3k' G(\varepsilon, k, k') \right]^{-1},$$
$$\varepsilon = p_0 - p^2/4m - 2m. \tag{11}$$

The function

$$\Delta_M(p_\alpha) = -[p_0 - p^2/2M - M + i0]^{-1}$$

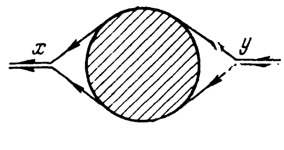

FIG. 1. Single lines correspond to nucleons, and double lines to M-particles.

is the Green's function of the free M-particle. As we have already mentioned, the difference $2m - M = \delta$ is assumed to be small. Therefore $G_2(p_\alpha)$ is actually a function of the one argument ϵ.

Before considering G_1 we shall find the expressions for the vertex parts $\Gamma(x_\alpha, y_\alpha, z_\alpha)$ and $\Gamma^+(y_\alpha, x_\alpha, z_\alpha) = \Gamma(-x_\alpha, -y_\alpha, -z_\alpha)$, which correspond to Feynman diagrams of the types of Fig. 2, a, b, which do not contain any internal M lines. It is not hard to see that

$$\Gamma^+(y_\alpha, z_\alpha, x_\alpha) = g\delta^4(y - x) \, \delta^4(z - x) + igV(y - z) \, \delta(y_0 - z_0) \, G(y_\alpha, z_\alpha, x_\alpha, x_\alpha). \tag{12}$$

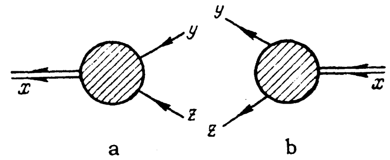

FIG. 2

In momentum space this gives

$$\Gamma^+(k_{1\alpha}, k_{2\alpha}, p_\alpha) = (2\pi)^4 \, \delta^4(k_1 + k_2 - p) \, g\gamma(k^2, \varepsilon), \tag{13}$$

where $k = {}^1/_2(k_1 - k_2)$, $\epsilon = p_0 - p^2/4m - 2m$, and

$$\gamma(k^2, \varepsilon) = i(\varepsilon - k^2/m) \int d^3k' G(\varepsilon, k, k'). \tag{14}$$

The contributions to the Green's function G are: first, that from diagrams that have no M lines—this contribution is equal to G—and second, the contribution from the diagram with M lines shown schematically in Fig. 3. This contribution is given by

$$-i \int \Delta_m(x_\alpha - u_\alpha) \, \Delta_m(y_\alpha - v_\alpha) \, d^4u \, d^4v \times \Gamma^+(u_\alpha, v_\alpha, z_\alpha) \, d^4z G_2(z_\alpha - z_\alpha') \times d^4z' \, \Gamma(z_\alpha', u_\alpha', v_\alpha') \, d^4u' \, d^4v' \, \Delta_m(u_\alpha' - x_\alpha') \, \Delta_m(v_\alpha' - y_\alpha').$$

Setting $x_0 = y_0$ and $x_0' = y_0'$, we find in momentum space [dropping a delta function of the total momentum, multiplied by $(2\pi)^7$]

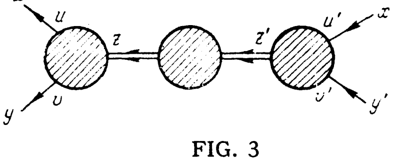

FIG. 3

$$G_1(\varepsilon, k, k') = G(\varepsilon, k, k') + i\frac{g^2}{(2\pi)^3}\frac{\gamma(k^2, \varepsilon)\gamma(k'^2, \varepsilon)G_2(\varepsilon)}{(\varepsilon - k^2/m + i0)(\varepsilon - k'^2/m + i0)}. \quad (15)$$

For the physical interpretation of the model it is necessary to carry out a renormalization of the mass and the "charge" g. This will be done in the next section.

2. RENORMALIZATION AND THE PROPERTIES OF THE MODEL

The observed mass of the M-particle is determined from the equation $G_2^{-1} = 0$. Inserting a counterterm of the form $-\delta M\varphi^+\varphi$ into the Lagrangian, we can arrange matters so that M will itself be the observed mass. We then have

$$\delta M = -ig^2(2\pi)^{-3}\int d^3k d^3k' G(\varepsilon, k, k'), \quad (16)$$

when $p_0 - p^2/2M - M = 0$, i.e., when $\epsilon = -\delta$.

After the mass renormalization

$$G_2^{-1}(\varepsilon) = -\varepsilon - \delta + ig^2(2\pi)^{-3}\int d^3k d^3k' \, (G(\varepsilon, k, k') - G(-\delta, k, k')). \quad (17)$$

We note that δM is infinite. Because of the integrations over k and k′ the expressions δM and G_2 depend on the value of the wave function $F_p^{(\pm)}(x)$ in ordinary space at the origin of coordinates: $F_p^{(\pm)}(0) \equiv a^{(\pm)}(p)$. As can be verified easily from Eq. (9), for $p \to \infty$ the amplitude $a^{(\pm)}(p)$ goes to $(2\pi)^{-3/2}$, provided that the decrease of $v(k)$ as $k^2 \to \infty$ is not slower than $1/k^2$ (that the singularity of $V(x)$ at the origin is not stronger than $|x|^{-1}$). Therefore the integral

$$\int d^3p \, |a^{(\pm)}(p)|^2/(\delta + p^2/m)$$

contained in δM does not exist. After the mass renormalization G_2^{-1} becomes finite.

The function G_2^{-1} goes to zero not only at the point $\epsilon = -\delta$ which corresponds to the physical M-particle, but also at a point $\epsilon = \epsilon_1$ which gives the bound state in the model. The equation for the determination of ϵ_1 is

$$1 = g^2\int d^3p \frac{|a^{(\pm)}(p)|^2}{(\varepsilon_1 - p^2/m)(\delta + p^2/m)} + \frac{g^2 a_0^2}{(\varepsilon_1 - \varepsilon_0)(\delta + \varepsilon_0)}. \quad (18)$$

If $\delta < |\epsilon_0|$, then $\epsilon_1 < \epsilon_0$, and $\epsilon_1 \to \epsilon_0$ from below for $g \to 0$. If, on the other hand, $\delta > |\epsilon_0|$, then $\epsilon_1 > \epsilon_0$. Moreover, $\epsilon_1 < 0$, since otherwise the right member is complex.

Let us denote the value of the integral in Eq. (18) for $\epsilon_1 = 0$ by $-c^2$. Then it is clear that for

$$(\delta + \varepsilon_0)(c^2 + 1/g^2) < a_0^2/|\varepsilon_0| \quad (19)$$

there is no bound state. When the inequality (19) is replaced by equality, a bound state appears with $\epsilon_1 = 0$. With further increase of the left member the level goes deeper, and for $g \to 0$ we again have $\epsilon_1 \to \epsilon_0$.

In our model the charge renormalization is not associated with the removal of infinities, and both renormalization constants, Z for the wave-function renormalization for the M-particle and Z_1 for the vertex part, are in general finite.

Z is determined in the usual way:

$$Z = \lim_{\varepsilon \to -\delta} \Delta_M^{-1}(\varepsilon) G_2(\varepsilon) \quad (20)$$

(the value $\epsilon = -\delta$ corresponds to the physical M-particle). It follows that

$$Z^{-1} = 1 + g^2 b^2;$$

$$b^2 = \int d^3p \frac{|a^{(\pm)}(p)|^2}{(p^2/m + \delta)^2} + \frac{a_0^2}{(\varepsilon_0 + \delta)^2}, \quad (21)$$

where a_0 is the value of the deuteron wave function at the origin (in ordinary space). We have $0 \le Z \le 1$, as must be the case.

We define the constant Z_1 in terms of the value of $\gamma(k^2, \epsilon)$ when all three particles are physical. We then have $\epsilon = -\delta$, and also, by the law of conservation of energy and momentum,

$$2m + k^2/m + p^2/4m = p^2/2M + M,$$

which gives $k^2 = -m\delta$. We therefore set

$$Z_1^{-1} = \gamma(-m\delta, -\delta)$$

or in explicit form

$$Z_1^{-1} = 1 - \frac{1}{(2\pi)^{3/2}}\int d^3p \, d^3k' \frac{v(k-k') f_p^{(\pm)}(k') a^{(\pm)*}(p)}{\delta + p^2/m} - \frac{a_0}{\delta + \varepsilon_0}\frac{1}{(2\pi)^{3/2}}\int d^3k' v(k-k') f_0(k') \quad (22)$$

(with $k^2 = -m\delta$).

We now define the renormalized charge g^r, vertex part γ^r, and M-particle Green's function G_2^r:

$$g^r = Z_1^{-1} Z^{1/2} g; \quad G_2^r = Z^{-1} G_2; \quad \gamma^r = Z_1 \gamma. \quad (23)$$

All of the matrix elements of the scattering matrix (not only in the two-nucleon sector, but in any sector) can be expressed in terms of these renormalized quantities alone.

It must be emphasized once more that in our model for $\delta \neq -\epsilon_0$ the constants Z and Z_1 are finite, and the quantity g^r is a finite function of g. Therefore in this case the renormalized charge

g^r has no particular advantages as compared with the bare charge g, in contrast with the case of relativistic theory (with spin effects included), in which the bare charge may be infinite, and at any rate has no meaning in perturbation theory. In our present model either of the constants g or g^r can with equal justification be regarded as a physical parameter of the theory. For our purposes it is reasonable to regard g as the parameter, since we start from the Lagrangian formulation of the theory and ascribe meanings to the Hamiltonian, the interaction, and the Green's functions. If we fix on g^r as the parameter, it can happen that as the second parameter of the theory—the mass of the M-particle—is varied these quantities lose their meanings (on this point see also the end of Section 3).

The theory we have constructed is physically entirely reasonable for arbitrary values of the renormalized mass M and the bare charge g. It is obvious that the M-particle is an independent particle. Its characteristics are in no way connected with the interaction between the nucleons. Let us note some features of the system of interacting m- and M-particles.

First, the bound state which existed for the nucleons before the introduction of the interaction with the M-particles is shifted to the point ϵ_1 determined by Eq. (18) when this interaction is introduced, and when the condition (19) holds the bound state disappears altogether. To convince ourselves of this, let us consider the poles of $G_1(\epsilon, k, k')$ as a function of ϵ. They arise from the poles of G and G_2 [the vertex part γ can be expressed in terms of G, see Eq. (14)]. The poles at the points $\epsilon = -\delta$ and $\epsilon = \epsilon_1$, which come from G_2, correspond to the physical M-particle and the new bound state. The latter is absent when the inequality (19) is satisfied. Corresponding to the old bound state (the deuteron) there is a pole of G at the point $\epsilon = \epsilon_0$. The function G_1, however, has no pole at this point, since the second term in Eq. (15) also has a pole at $\epsilon = \epsilon_0$, which exactly cancels the pole of the first term.

A second feature, which can be seen directly from Eq. (23), is that the magnitude of the renormalized charge is bounded (unlike the bare charge, which can be arbitrary). It is clear that

$$(g^r)^2 \leqslant 1/Z_1^2 b^2, \qquad (24)$$

where Z_1 and b do not depend on the charge g. If we neglect the interaction between the nucleons (V = 0), then $Z_1 = 1$, $a(p) = (2\pi)^{-3/2}$, and we find

$$(g^r)^2 \leqslant 8\pi \sqrt{\delta}\, m^{-3/2}. \qquad (25)$$

This relation (with the equals sign) was obtained by Landau [6] from a consideration of the scattering amplitude of the particles in the general theory (in comparing the relations we must remember the difference between our definition of the coupling constant and Landau's).

We would like to emphasize that in our model the Landau relation follows from two assumptions: 1) that the interaction between the nucleus is infinitely small, and consequently there are no bound states of the nucleons before the introduction of the interaction with the M-particles; 2) the bare interaction of the nucleons with the M-particles is infinitely large, $g \to \infty$.[1)]

Let us further consider the vertex part $\gamma(k^2, \epsilon)$ when the M-particle and one of the nucleons are physical particles. We then have $\epsilon = -\delta$. The quantity analogous to $\gamma(k^2, -\delta) \equiv \gamma(k^2)$ has been considered in the relativistic theory.[7] On the assumption that $\gamma(k^2)$ is analytic and falls off for $k^2 \to \infty$ it is possible to express $\gamma(k^2)$ approximately in terms of quantities that have no relation to the M-particle, and the result is that we get an equation for the determination of the mass of the M-particle.

We can try to carry out an analogous procedure in our model also. The function $\gamma(k^2)$ will have analytic properties similar to those in the relativistic case if we take for the potential V(x) a superposition of Yukawa potentials $e^{-\kappa|x|}/|x|$ with different constants $\kappa \geq \kappa_0$. $\gamma(k^2)$ satisfies the obvious equations:

$$\gamma(k^2) = (\delta + k^2/m)\tilde{\gamma}(k^2),$$

$$(\delta + k^2/m)\tilde{\gamma}(k^2) + (2\pi)^{-3}\int d^3k' v(k - k')\tilde{\gamma}(k'^2) = 1, \qquad (26)$$

from which it follows that with this choice of the potential γ is an analytic function of k^2 in the entire complex plane, except for a cut along the negative real axis to the left of the point $[\kappa_0 + (m\delta)^{1/2}]^2$.

This can be verified by using, for example, a method which we have suggested for the study of the wave function of the bound state.[9] With this method we can also find without difficulty an explicit expression for the discontinuity across the cut. It corresponds to the contribution from anomalous thresholds in the relativistic theory. An essential difference from [7] appears when we

[1)]See also a paper by Gribov et al.[8]

write the dispersion relation for $\gamma(k^2)$: $\gamma(k^2)$ does not fall off for $k^2 \to \infty$. This causes an important change in the result: there is no restriction on the mass of the M-particle.

We shall see later that there is an exceptional case in which $\gamma(k^2)$ does fall off—this is precisely the case in which the M-particle can be regarded as a bound state of the nucleons. Therefore in our model the assumption that $\gamma(k^2)$ falls off is equivalent to the assumption we made earlier, that the M-particle is a compound, not an elementary, particle.

We note that $\gamma(k^2)$ always has a threshold which corresponds to the anomalous threshold in relativistic theory. This shows that the anomaly is not related to the elementary or nonelementary nature of the particle, but is a property determined solely by the values of the particle masses.

3. THE CASE IN WHICH THE M-PARTICLE IS COMPOUND

In the preceding sections we have constructed a model of the interaction of two types of elementary particles—nucleons and M-particles. The masses of the particles and the quantities that characterize their interaction—the charge g and the potential V—have remained arbitrary. This is essentially what we wish to say when we call the two particles elementary and independent. Let us now consider a special case of our model—that in which the mass of the M-particle is equal to the mass of the bound state of the nucleons, the deuteron, which existed before the introduction of the interaction between nucleons and M-particles, i.e., the case in which $\delta = 2m - M = -\epsilon_0$. We shall at once find that in this case our model is exactly equivalent in its physical content to the ordinary quantum mechanics of the nucleons alone, as if there were no M-particles at all. The M-particle itself then precisely duplicates the behavior of the deuteron.

$\delta = -\epsilon_0$ is a singular point for all of the renormalization constants. We see from Eqs. (16), (21), and (22) that for $\delta = -\epsilon_0$ the values are $Z = Z_1 = 0$, $\delta M = \infty$. Let us consider the behavior of all the quantities that characterize the model for values of δ close to $-\epsilon_0$. For simplicity we shall suppose that the condition (19) is satisfied. Then G_2 has only one pole. This simplifies the arguments, but the results are the same for the opposite case.

It can be seen from Eq. (21) that for $\delta + \epsilon_0 \to 0$

$$Z \sim (\delta + \varepsilon_0)^2/g^2 a_0^2, \tag{27}$$

and it follows from Eq. (22) that

$$Z_1 \sim m(\delta + \varepsilon_0)/(2\pi)^{3/2} a_0 N, \tag{28}$$

where N is the residue of the deuteron wave function $f_0(k^2)$ at the point $k^2 = m\epsilon_0$.

It is remarkable that for $\delta = -\epsilon_0$ the renormalized charge is finite and independent of the bare charge g:

$$(g^r)^2_{\delta=-\varepsilon_0} = (2\pi)^3 N^2/m^2. \tag{29}$$

This value of the renormalized charge corresponds precisely to what one gets in the theory without the M-particles, with the deuteron regarded as a bound state of the nucleons.

According to Eq. (17), for $\delta + \epsilon_0 \to 0$ the unrenormalized propagation function G_2 goes linearly to zero at all points. The renormalized propagation function increases linearly at all points. The unrenormalized vertex part does not depend on δ at all, and the renormalized vertex part goes to zero everywhere except at the point which corresponds to the physical M-particle ($\epsilon = -\delta$), at which it goes to the finite value $(k^2 - m\epsilon_0)f_0(k)/N$.

Let us now see what sort of scattering of the particles the model will give for $\delta + \epsilon_0 \to 0$. We first consider the scattering of two nucleons. It is described by the Green's function $G_1(\epsilon, k, k')$ with $\epsilon = k^2/m = k'^2/m$. In Eq. (15) the contribution to G_1 from the second term will be zero for $\delta + \epsilon_0 \to 0$, since we have just seen that there we have $G_2 \to 0$. Therefore the scattering of the nucleons will be the same as is given by the Green's function G, i.e., the same as before the introduction of the interaction with the M-particles.

Let us take a general matrix element of the S matrix which describes the interaction between nucleons and M-particles. For $\delta \to -\epsilon_0$ it goes over into the corresponding matrix element of the S matrix for the scattering of nucleons by deuterons without the introduction of any M-particles.

To convince ourselves of this, let us consider the corresponding Feynman diagrams. We divide all of the diagrams into two groups: those that do not contain any internal M lines, and those that do contain such lines. For $\delta \to -\epsilon_0$ the diagrams that contain internal M lines go to zero, since $G_2 \to 0$. There remain the diagrams without internal M lines. If there are also no external M lines, it is clear at once that the result will be the same as when the scattering of nucleons is calculated in the absence of M-particles.

Suppose there are external M lines. Let us see what the part of a diagram connected with one such line gives (Fig. 4). Apart from a factor we get for the part in question the expression

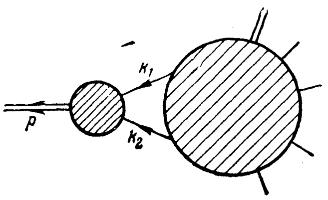

FIG. 4

$$g^r\gamma^r\,({}^1/_4\,(k_1-k_2)^2,\,-\delta)\,\Delta_m(k_{1\alpha})\,\Delta_m(k_{2\alpha}),$$

where Δ_m is the Green's function of the free nucleon, $k_{1\alpha}+k_{2\alpha}=p_\alpha$, and $p_0=p^2/2M+M$. This expression is of course to be multiplied by a function of $k_{1\alpha}$ and $k_{2\alpha}$ which corresponds to the rest of the diagram, and integrated over k_1 and k_2. For $\delta\to-\epsilon_0$ the quantity $g^r\gamma^r(k^2,-\delta)$ goes over into $(2\pi)^{3/2}(k^2/m-\epsilon_0)f_0(k)$. We would get exactly the same expression if we looked for the scattering of the deuteron in the theory without M-particles. This can be verified by an elementary calculation, using, for example, the method suggested by Zimmermann.[1]

Thus for $\delta\to-\epsilon_0$ the entire theory goes over into the theory as it was before the introduction of the M-particles, into a theory containing only one elementary particle, the nucleon. The M-particle goes over into the deuteron. This transition can be seen directly if we consider the state vector Φ_p of a physical M-particle with the momentum p. Obviously

$$\Phi_p=Z^{1/2}\Phi_p^0+\int d^3k_1\,d^3k_2\,\delta^3(p-k_1-k_2)\,\chi(k_1k_2)\,\Psi^0_{k_1k_2}, \tag{30}$$

where Φ_p^0 is the state of the "bare" M-particle and $\Psi^0_{k_1k_2}$ is the state of two noninteracting nucleons.

The expansion (30) is possible because the states Φ_p^0 and $\Psi^0_{k_1k_2}$ form a complete system of states in the two-nucleon sector, since they are the eigenstates of the Hamiltonian of the noninteracting particles. For $\delta\to-\epsilon_0$ the constant $Z\to0$. In the limit

$$\Phi_p=\int d^3k_1\,d^3k_2\,\delta^3(p-k_1-k_2)\,\chi(k_1,k_2)\,\Psi^0_{k_1k_2}.$$

Thus Φ_p is orthogonal to the Φ_p^0 and can be expressed in terms of the two-nucleon states $\Psi^0_{k_1k_2}$.

Some concluding remarks: When one of the nucleons and the M-particle are physical particles the vertex part $\gamma(k^2)$ goes over for $\delta\to-\epsilon_0$ into the quantity $(k^2-m\epsilon_0)f_0(k)/N$, and consequently falls off for $k^2\to\infty$. We have already mentioned that in this case one can determine the energy of the bound state by means of the dispersion relation for $\gamma(k^2)$.

For $\delta\to-\epsilon_0$ the renormalized coupling constant g^r is uniquely determined in terms of N and does not depend on g. This is perhaps the most characteristic way the case $\delta=-\epsilon_0$ differs from other cases. Owing to this feature, for $\delta\to-\epsilon_0$ the entire theory is uniquely determined, with no room for any arbitrariness. In the case $\delta=-\epsilon_0$ the restriction on g^r of the Landau type, obtained from the condition (24), is replaced by the well known relation connecting N with the effective radius ρ. For g^r this gives

$$(g^r)^2=8\pi m^{-3/2}|\varepsilon_0|^{1/2}/(1-\rho\sqrt{m|\varepsilon_0|}). \tag{31}$$

This equation goes over into the Landau condition in the limit $\rho\to0$ (point interaction of the nucleons).

One final remark: In this section we have regarded g as fixed and have varied δ, letting it go to $-\epsilon_0$. One could try to fix g^r and let $\delta\to-\epsilon_0$.[2)] Then it is easy to see that Eq. (28) holds for Z_1 as before, $Z=1-(g^r)^2m^2/(2\pi)^3N^2$ (giving a finite value), and $g=Z_1Z^{-1/2}g^r\to0$. In the limit the scattering of nucleons would again agree with what one gets in the theory without M-particles. On the other hand the scattering of the M-particles would go over into the scattering of deuterons multiplied by a factor arising from replacement of the value of g^r from Eq. (29) by the fixed value of g^r.

The theory obtained in the limit does not correspond to any physical interaction, since $g\to0$, and therefore it is incorrect, lacking in balance. For example, in the limit G_2 goes over into the free Green's function, so that we should have $Z=1$, but this is not the case. The most unpleasant thing is that in the limit the S matrix is not unitary: it differs by a numerical factor from the scattering matrix for the deuterons in the theory without M-particles, which is certainly unitary. Therefore the passage to the limit $\delta\to-\epsilon_0$ with fixed g^r has no physical meaning.

4. CONCLUSION

The model which we have constructed is a generalization of a quantum-mechanical system of interacting particles. In this model all particles are regarded as independent and elementary [when the condition (19) is satisfied]. The ordinary quantum-mechanical system is obtained from the model in a special case, with a special choice of the mass of the newly introduced particle. This special case is characterized by a number of dis-

2)I express my gratitude to V. Ya. Faĭnberg, who called my attention to this possibility.

tinctive features: the vanishing of the renormalization constants Z, Z_1, and δM^{-1}, and as a consequence of this, the falling off of the vertex part $\gamma(k^2)$.

We leave open the question as to whether or not any such effects are actually observable. Clearly it could be expected that they would show up in the asymptotic behavior of observable quantities (electro-magnetic form-factors, say) at high energies. But the asymptotic behavior at high energies is not described by quantum mechanics, and therefore investigations in this direction require bringing in the relativistic theory of particles.

In the low-energy region in quantum mechanics the model shows that there is no sharp distinction between a bound state ("deuteron") and an independent M-particle. As the mass of the M-particle approaches that of the deuteron all physical quantities go over in a continuous way into the corresponding quantum-mechanical quantities for nucleons and deuterons. This means that, strictly speaking, it is impossible to judge from low-energy experimental data whether the deuteron is a bound state or an independent particle. Either of the two alternatives is possible in principle.

This conclusion is of course not to be taken too literally. In the case of the hydrogen atom, say, there is a whole series of S states. Here one could speak of a whole set of independent particles and ask whether the observed particles are the S states of the electron in the field of the proton or independent particles. But such a statement of the problem is very artificial and uninteresting.

The ordinary quantum theory of the hydrogen atom is simpler and much richer than a theory with independent particles. It enables us to find all of the levels, which would have to be given from the beginning in a theory with independent particles. The actual deuteron is another matter; there is only one level, and that is poorly defined in the framework of quantum mechanics operating with potentials. In this case the question of elementary or nonelementary nature of the particle is very much in place. As we have already said, however, it cannot be solved if we remain in the low-energy region.

We shall not deal here with the most interesting question, which is that of the degree to which the results obtained in the model are valid for relativistic field theory. We intend to devote a separate paper to this. We remark only that in our opinion there is no change as to matters of principle. A possible complication is that in the relativistic theory the renormalized constants are always equal to zero, independently of the value of the mass of the M-particle. In this case one can consider a theory in which a cut-off momentum L is introduced, and study the behavior of all quantities for fixed L, taking the limit $L \to \infty$ at the end of the calculations. The criterion for "nonelementary" character of the particles is, we believe, our previous condition $Z = Z_1 = 0$, which is now to be written for a fixed finite L and solved for the mass of the M-particle, after which we are to take $L \to \infty$.

The writer is deeply grateful to the members of the seminar of the theoretical section of the Physics Institute (NIFI) of the Leningrad State University for interesting and helpful discussions.

[1] W. Zimmermann, Nuovo cimento **10**, 597 (1958).

[2] A. Salam, Nuovo cimento **25**, 224 (1962).

[3] J. C. Houard and B. Jouvet, Nuovo cimento **18**, 466 (1960).

[4] R. Acharya, Nuovo cimento **24**, 870 (1962).

[5] J. Dowker, Nuovo cimento **25**, 1135 (1962).

[6] L. D. Landau, JETP **39**, 1856 (1960), Soviet Phys. JETP **12**, 1294 (1961).

[7] R. Blankenbecler and L. F. Cook, Jr., Phys. Rev. **119**, 1745 (1960).

[8] Gribov, Perelomov, and Zel'dovich, JETP **40**, 1190 (1961), Soviet Phys. JETP **13**, 836 (1961).

[9] M. A. Braun, JETP **42**, 1106 (1962), Soviet Phys. JETP **15**, 764 (1962).

Translated by W. H. Furry
163