# FLUCTUATION OF NUMBER OF PARTICLES IN ELECTRON-PHOTON SHOWERS PRODUCED BY HIGH-ENERGY MUONS

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Simple expressions are obtained for the mean square of the number of particles in electron-photon showers produced by high energy $\mu$ mesons. The pattern of the fluctuations can be studied with these formulas.

As is well known, information on muon generation and interaction with matter can be gained from the muon spectrum. At the present time the spectrum of muons with energy $\sim 10^{12}$ eV is studied in experiments on bursts in ionization chambers [1,2]. It has been shown [3] that owing to fluctuations in the number of particles in an electron-photon shower produced by a muon and giving rise to bursts in an ionization chamber, the muon spectrum determined from the burst spectrum is not unique.

The fluctuations have been studied hitherto by using some model (Poisson, Furry, etc). The burst spectra determined on the basis of these models differ greatly from one another [4]. We solve the problem here, using a method developed by one of the authors [5], on the basis of the cross sections for the real processes that participate in the production of the shower. We assume that the electron-photon shower results from the emission of protons when the muon is slowed down and from direct production of pairs by the muon.

Following an earlier paper [6], we write the equation for the function $\Psi_\mu(E_\mu, E, t, N)$, defined as the probability that a particle with index $\mu$ (in our case, a muon) and energy $E_\mu$ produces N particles with energy larger than E after traversing a depth t:

$$\Psi_\mu(E_\mu, E, t+dt, N) = \Psi_\mu(E_\mu, E, t, N)\Big[1 - dt\int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_- \times W_p^\mu(E_\mu, E_+, E_-) - dt\int_0^{E_\mu} dE_\gamma W_b^\mu(E_\mu, E_\gamma)\Big]$$
$$+ dt \sum_{N_1, N_2} \delta_{N_1+N_2, N}\int_0^{E_\mu} dE_\gamma \times W_b^\mu(E_\mu, E_\gamma)\Psi_\gamma(E_\gamma, E, t, N_1)\Psi_\mu(E_\mu - E_\gamma, E, t, N_2)$$
$$+ dt \sum_{N_1, N_2, N_3} \delta_{N_1+N_2+N_3, N}\int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_- W_p^\mu(E_\mu, E_+, E_-) \times \Psi_e(E_+, E, t, N_1)\Psi_e(E_-, E, t, N_2) \times \Psi_\mu(E_\mu - E_+ - E_-, t, N_3). \quad (1)$$

Summation over each index is from 0 to $\infty$. $W_b^\mu(E_\mu, E_\gamma)dE_\gamma$ is the probability that a $\gamma$ quantum with energy $E_\gamma$ will be produced by bremsstrahlung from a muon with energy $E_\mu$; $W_p^\mu(E_\mu, E_+, E_-) \times dE_+ dE_-$ is the probability that an electron-positron pair with energies $E_-$ and $E_+$ will be produced by a muon with energy $E_\mu$.

From (1) we obtain, in analogy with [6], an equation for the average number $\overline{N}_\mu(E_\mu, E, t)$ of electrons with energy E generated by a muon with energy $E_\mu$, at a depth t, and the corresponding mean square number of particles $\overline{N_\mu^2}(E_\mu, E, t)$:

$$\frac{\partial}{\partial t}\overline{N}_\mu(E_\mu, E, t) = -\int_0^{E_\mu}[\overline{N}_\mu(E_\mu, E, t) - \overline{N}_\mu(E_\mu - E_\gamma, E, t)] \times W_b^\mu(E_\mu, E_\gamma)\,dE_\gamma - \int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_-\,[\overline{N}_\mu(E_\mu, E, t) - \overline{N}_\mu(E_\mu - E_+ - E_-, E, t)]\,W_p^\mu(E_\mu, E_+, E_-)$$
$$+ \int_0^{E_\mu} dE_\gamma W_b^\mu(E_\mu, E_\gamma)\,\overline{N}_\gamma(E_\gamma, E, t) + 2\int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_- W_p^\mu(E_\mu, E_+, E_-)\,\overline{N}_e(E_+, E, t), \quad (2)$$

$$\frac{\partial}{\partial t}\overline{N_\mu^2}(E_\mu, E, t) = -\int_0^{E_\mu} dE_\gamma W_b^\mu(E_\mu, E_\gamma)[\overline{N_\mu^2}(E_\mu, E, t) - \overline{N_\mu^2}(E_\mu - E_\gamma, E, t)]$$

$$-\int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_- W_p^\mu(E_\mu, E_+, E_-)[\overline{N_\mu^2}(E_\mu, E, t) - \overline{N_\mu^2}(E_\mu - E_-, E, t)] + \int_0^{E_\mu} dE_\gamma W_b^\mu(E_\mu, E_\gamma)\overline{N_\gamma^2}(E_\gamma, E, t)$$

$$+ 2\int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_- W_p^\mu(E_\mu, E_+, E_-)\overline{N_e^2}(E_+, E, t)$$

$$+ 2\int_0^{E_\mu} dE_\gamma W_b^\mu(E_\mu, E_\gamma)\overline{N}_\mu(E_\mu - E_\gamma, E, t)\overline{N}_\gamma(E_\gamma, E, t)$$

$$+ 4\int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_- W_p^\mu(E_\mu, E_+, E_-) \times \overline{N}_\mu(E_\mu - E_+ - E_-, E, t)\overline{N}_e(E_+, E, t)$$

$$+ 2\int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_- W_p^\mu(E_\mu, E_+, E_-) \times \overline{N}_e(E_+, E, t)\overline{N}_e(E_-, E, t);$$

$$\overline{N}_+(E_0, E, t) = \overline{N}_-(E_0, E, t) \equiv \overline{N}_e(E_0, E, t). \qquad (3)$$

We shall use for $\overline{N}_e$ and $\overline{N}_\gamma$ (the mean numbers of shower particles from the primary electron and the photon, respectively) the expressions derived [7] for light substances with allowance for the ionization losses; for the corresponding mean square values $\overline{N_e^2}$ and $\overline{N_\gamma^2}$ we shall use the expressions obtained under the same assumptions in our earlier paper [6]. The electron spectrum in a shower produced by a muon ($\overline{N}_\mu$) was already determined by one of the authors [8] in the usual formulation (source in the right-hand part of the cascade equations). We begin, however, with a solution of our equation (2) for $\overline{N}_\mu$, since it is possible to illustrate here in a simpler and physically more lucid form the character of the approximations also used in the calculation of $\overline{N_\mu^2}$.

Equation (2) can be readily simplified by recognizing that the muon loss in one radiation unit length [4] is very small; the total muon loss to bremsstrahlung and pair production is $\sim 4 \times 10^{-6}$ $E_\mu$. Inasmuch as the absorbers used in real installations do not exceed several dozen t-units [1], we neglect the change in muon energy after penetration through matter. We can then leave out the first two integrals in the right half of (2), which is rewritten

$$\frac{\partial}{\partial t}\overline{N}_\mu(E_\mu, E, t) = \int_0^{E_\mu} dE_\gamma \overline{N}_\gamma(E_\gamma, E, t) W_b^\mu(E_\mu, E_\gamma) + 2\int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_- \overline{N}_e(E_+, E, t) W_p^\mu(E_\mu, E_+, E_-). \qquad (2')$$

The terms remaining in the right half of (2) contain known quantities and represent the contribution made to $\overline{N}_\mu$ by the showers due to $\gamma$ quanta and electron pairs. We shall agree to measure t in electron radiation units $t_e$. Calculation of the first term yields

$$\int_0^{E_\mu} dE_\gamma W_b^\mu(E_\mu, E_\gamma)\overline{N}_\gamma(E_\gamma, E, t) \approx \frac{t_e}{t_\mu} c'(s_1)\overline{N}_\gamma(E_\mu, E, t), \qquad (4)$$

where $s_1$ is determined from

$$\lambda_1'(s_1)t + \ln(E_\mu/\beta) = 0, \qquad (5)$$

$$c'(s) = \frac{1}{s+2} + \frac{1.33 + b}{s(s+1)}, \qquad b = \frac{1}{9\ln(180mZ^{-1/3})}, \qquad (6)$$

m is the ratio of muon and electron masses, equal to 209, and $t_e/t_\mu$ is the ratio of the electron and muon radiation length, equal to $\ln(180mZ^{-1/3})/m^2\ln(180Z^{-1/3})$.

Since we shall be interested in muons of high energy, $E_\mu \gtrsim 10^{12}$ eV, we can use here the expression for the cross section in the case of total screening, which is valid within several percent for energies $E_\mu > 137\, m^2 m_e c^2$ [9]:

$$W_b^\mu(E_\mu, E_\gamma)\,dE_\gamma = \frac{t_e}{t_\mu}\frac{dE_\gamma}{E_\gamma}\left\{1 + \left(1 - \frac{E_\gamma}{E_\mu}\right)^2 - \left(\frac{2}{3} - b\right)\left(1 - \frac{E_\gamma}{E_\mu}\right)\right\}. \qquad (7)$$

In the integration with respect to $dE_\gamma$ we assume that $E_\gamma^s$ in $\overline{N}_\gamma(E_\gamma, E, t)$ has only a power-law dependence. The remaining factors which depend little on the energy (via s) have been taken outside the integral sign at the point $s_1(E_\mu)$, since the main integration region of $\int_0^{E_\mu} E_\gamma^s W_b^\mu(E_\mu, E_\gamma)\,dE_\gamma$ is at $E_\gamma \approx E_\mu$. An estimate of the next terms in the expansion near $E_\mu$ shows that this procedure introduces no additional error compared with the inaccuracy in the $\overline{N}_\gamma$ ($\sim t^{-1}$, see [7]). Finally, s is also assumed equal to its value at the point $E_\mu$. Since s enters as an exponent in the integrand, small deviations of s could result in appreciable errors. It is easy to verify, however, that this is prevented by condition (5).

We now turn to the calculation of the second integral in the equation for $\overline{N}_\mu$. We use the following expression from [10]:

$$W_{\mathrm{p}}^{\mu}(E_\mu, E_+, E_-)\,dE_+\,dE_-$$
$$= \frac{2\alpha^2 Z^2 r_e^2 n L t_e}{3\pi A (E_+ + E_-)^2}\left\{\left[1 + \left(1 - \frac{E_+ + E_-}{E_\mu}\right)^2\right]\right.$$
$$\times\left[A(x) + 2\frac{E_+^2 + E_-^2}{(E_+ + E_-)^2}B(x)\right]$$
$$+ \frac{(E_+ + E_-)^2}{E_\mu^2}\left[C(x) + 2\frac{E_+^2 + E_-^2}{E_\mu^2}D(x)\right]$$
$$\left. + 8\frac{E_+E_-(E_\mu - E_+ - E_-)}{(E_+ + E_-)^2 E_\mu (1 + x)}\right\}, \tag{8}$$

where

$$x = \frac{m^2 E_+ E_-}{E_\mu (E_\mu - E_+ - E_-)}, \tag{9}$$

$$A(x) = (1 + 2x)\ln\left(1 + \frac{1}{x}\right) - 2,\ B(x)$$
$$= (1 + x)\ln\left(1 + \frac{1}{x}\right) - 1, \tag{10}$$

$C(x)$ and $D(x)$ have a similar structure, and $L = \ln[l\sqrt{1+x}/\alpha Z^{1/3}]$ with $l \sim 1$.

This cross section has been obtained with logarithmic accuracy. In the henceforth significant region of x this results in an error $\sim 20\%$.

The function $W_{\mathrm{p}}^{\mu}(E_\mu, E_+, E_-)$ has a sharp maximum at $E_+, E_- \sim E_\mu/m$. Therefore, as in (5), we take all the factors with weak energy dependence outside the integral sign at the point $s_2(E_\mu/m)$. The equation for $s_2$ is of the form

$$\lambda_1'(s_2)\,t + \ln(E_\mu/m\beta) = 0. \tag{11}$$

In addition, we simplify (8) by using the fact that the main contribution to the integral is made by the region $E_+, E_- \sim E_\mu/m$. We put in the first square bracket $1 + [1 - (E_+ + E_-)/E_\mu]^2 \approx 2$, we neglect the second term in the curly bracket which is $\sim 1/m^{2-s_2}$ as large as the first in the principal region of integration, we set x equal to $m^2E_+E_-/E_\mu^2$, and we put $\ln(\sqrt{1+x}/Z^{1/3}\alpha) \approx \ln(\sqrt{2}/\alpha Z^{1/3})$.

We also introduce the notation

$$y = m(E_+ + E_-)/E_\mu, \qquad d = (2\alpha^2 Z^2/3\pi)\, r_e^2 L t_e n/A.$$

This simplification of the cross section enables us to integrate with respect to $E_+$ and $E_-$ from 0 to $\infty$. (As $E_+, E_- \to \infty$ the contribution to the integral from the asymptotic values of the simplified expressions for $W_{\mathrm{p}}^{\mu}$ is $\sim 1/m^{2-s_2}$ as large as the principal term. As will be shown below, the maximum value is $s_2 = 1$.) If we now change over to the variables x and y, then the corresponding integration will be from 0 to $\infty$ with respect to y and from 0 to $y^2/4$ with respect to x, or, what is the same, from 0 to $\infty$ with respect to x and from $2\sqrt{x}$ to $\infty$ with respect to y. Then

$$2\int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_-\, W_{\mathrm{p}}^{\mu}(E_\mu, E_+, E_-)\,\overline{N}_e(E_+, E, t)$$
$$= dA'(s_2)\,\overline{N}_e(E_\mu/m, E, t); \tag{12}$$

$$A'(s) = \left\{6\pi s^{-1}\operatorname{cosec}\frac{s\pi}{2} - 2^{(4-s)/2}\left[B\left(\frac{s}{2}, 1 - \frac{s}{2}\right)\right.\right.$$
$$\left.\left. - \Gamma\left(1 + \frac{s}{2}\right)\sum_{k=1}^{\infty}\frac{\Gamma(2k - s/2)}{\Gamma(2k+2)}\right]\right\}\left[1 + s\beta\left(1 - \frac{s}{2}\right)\right]$$
$$+ \frac{1}{3}\left\{\frac{2}{3}\Gamma\left(1 + \frac{s}{2}\right)\Gamma\left(1 - \frac{s}{2}\right) - \frac{2\pi}{s}\operatorname{cosec}\frac{s\pi}{2}\right.$$
$$\left. + 2^{-s/2}\left[B\left(\frac{s}{2}, 1 - \frac{s}{2}\right) - \Gamma\left(1 + \frac{s}{2}\right)\sum_{k=1}^{\infty}\frac{\Gamma(2k - s/2)}{\Gamma(2k+2)}\right]\right\}$$
$$\times\left[1 + \tfrac{1}{2}(s+1)(s+2) + \tfrac{1}{2}(s+2)\,s\,(s-2)\,\beta\,(2 - s/2)\right],$$
$$d = 2\alpha^2 Z^2 r_e^2 n L t_e/3\pi A. \tag{13}$$

We can now readily integrate with respect to t. The t-dependence of the right half of (13) is determined by $\exp[\lambda_1(s_1)t + s_1\ln(E_\mu/\beta)]$ in the first term and by $\exp[\lambda_1(s_2)t + s_2\ln(E_\mu/m\beta)]$ in the second. Each of these exponentials increases with t for all $s_1 < 1$ and $s_2 < 1$. We can therefore expand them about the upper limit in the integral with respect to t. Then the integration with respect to t for the regions $t < \ln(E_\mu/\beta)$ for the first term and $t < \ln(E_\mu/m\beta)$ for the second term reduces to a corresponding additional multiplication by $\lambda_1^{-1}(s_1)$ and $\lambda_1^{-1}(s_2)$. For $t \geq \ln(E_\mu/\beta)$ in the case of the first term and $t > \ln(E_\mu/m\beta)$ in the case of the second, the integrals

$$\int_0^t \exp[\lambda_1(s_{1,2})\,t' + y_{1,2}s_{1,2}]\,dt'$$

are calculated by the saddle point method, since the integrand has a resonant character for these values of t (maximum at $\lambda_1(s_{1,2}) = 0$). The mutual relations between these conditions give three variants of the solutions for different regions of t:

$$\overline{N}_\mu(E_\mu, E, t) = \frac{t_e}{t_\mu}\frac{c'(s_1)}{\lambda_1(s_1)}\overline{N}_\gamma(E_\mu, E, t) + \frac{dA'(s_2)}{\lambda_1(s_2)}\overline{N}_e\left(\frac{E_\mu}{m}, E, t\right)$$
$$\text{for } t < \ln(E_\mu/m\beta); \tag{14a}$$

$$\overline{N}_\mu(E_\mu, E, t) = \frac{t_e}{t_\mu}\frac{c'(s_1)}{\lambda_1(s_1)}\overline{N}_\gamma(E_\mu, E, t)$$
$$+ dA'(1)\left[3.12\pi\ln\frac{E_\mu}{m\beta}\right]^{1/2}\overline{N}_{e\,max}\left(\frac{E_\mu}{m\beta}, E, \ln\frac{E_\mu}{m\beta}\right) \tag{14b}$$

$$\text{for } \ln(E_\mu/m\beta) \leqslant t < \ln(E_\mu/\beta);$$

$$\bar{N}_\mu(E_\mu, E, t)$$
$$= t_e t_\mu^{-1} c'(1)\,[3,\,12\pi \ln(E_\mu/\beta)]^{1/2}\,\bar{N}_{\gamma\, max}(E_\mu, E, \ln(E_\mu/\beta))$$
$$+ dA'(1)\,\bar{N}_{e\, max}(E_\mu/m, E, \ln(E_\mu/m\beta))\sqrt{3{,}12\pi \ln(E_\mu/m\beta)}$$
$$\text{for } t \geqslant \ln(E_\mu/\beta). \qquad (14c)$$

It is easy to verify by calculation that in the integral with respect to dt the next higher terms of the expansion of the exponential in t are proportional to 1/t, i.e., (14) is obtained with the same accuracy as the initial $\bar{N}_e$ and $\bar{N}_\gamma$. The total error due to integration over the energy and over the depth does not exceed 10 per cent for $t \gtrsim 10$.

Let us explain the meaning of the transition from $t < t_{max}$ to $t > t_{max}$. This situation has already been encountered in the solution of the problem of the mean square number of particles [5] in showers due to photons and electrons. The solution of Eq. (2) is no longer determined by the free terms in the right halves, and goes over into the solution corresponding to the damping of the muon. In our approximation, where the terms that account for the muon energy loss are neglected, we have in this region $\partial\bar{N}_\mu(E_\mu, E, t)/\partial t = 0$, i.e., the solution yields a constant. Figure 1 shows the depth development of cascades due to muons with energies $10^{12}$, $10^{13}$, and $10^{14}$ eV in the ground and $E_\mu = 10^{13}$ eV in iron and lead. (Strictly speaking, the approximation used is not suitable for heavy substances, for no account is taken of particle scattering and of the energy dependence of the photon absorption coefficient. It is clear, however, that the character of the dependence of $\bar{N}_\mu$ on t remains the same even when these factors are taken into account. We have therefore presented for illustration the curves for heavy substances, too.)[1)]

Figure 2 shows the separate contribution made to $\bar{N}_\mu$ by showers generated by γ quanta ($\bar{N}_\mu^{I}$) and by pairs ($\bar{N}_\mu^{II}$) (the first and second terms in (14), respectively). The figure is drawn for the ground: $E_\mu = 10^{14}$ eV (Z = 10). It is seen from Fig. 2 that the cascades produced by pairs essentially with energies $\sim E_\mu/m$, which therefore have a smaller range than cascades from γ-quanta with energy $\sim E_\mu$, develop more rapidly and already are in equilibrium with the muon when $t > \ln(E_\mu/m\beta)$, whereas the cascades due to the γ-quanta reach equilibrium when $t \sim \ln(E_\mu/\beta)$. For large t, the contribution of both parts is approximately the same, for although the number of particles in showers due to pairs is two orders of magnitude smaller at the maximum, the probability of pair production in one t-unit is larger than the probability of emission of a γ quantum by approximately the same factor. The energy dependence at equilibrium is $\sim E_\mu G(1,\epsilon)/\beta$, where $\epsilon = Ef(\lambda_1)/\beta$ (see [7]).

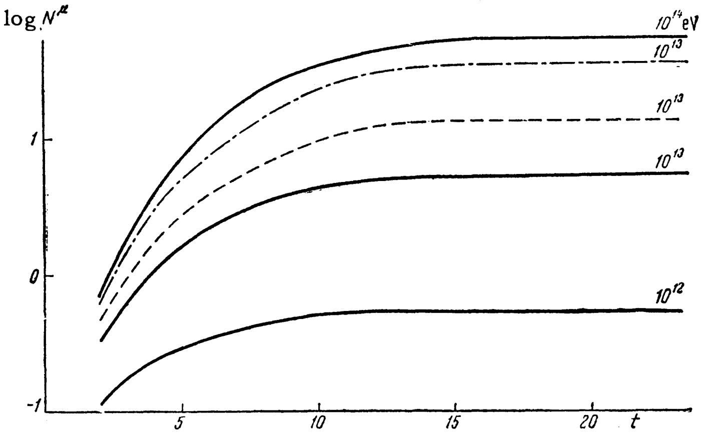


FIG. 1. Development of muon cascades with depth: solid line – in ground, dashed – in iron, dash-dot – in lead.

It must be noted that formulas (14) were obtained for $E \lesssim \beta$. However, formulas (14) can be used also when $E \gg \beta$, provided β is replaced everywhere by E and we put $D(s)G(s,\epsilon) \approx 1$. This can be easily verified by substituting in (2) the expressions for $\bar{N}_e$ and $\bar{N}_\gamma$ according to approximation A.

We now proceed to solve Eq. (3) for $\overline{N_\mu^2}$. Since it is assumed that the muon does not lose energy, we can neglect the first two terms of (3), too. We note further that the remaining terms are not of equal value. The integrals

$$2\int_0^{E_\mu} dE_\gamma\, W_b^\mu(E_\mu, E_\gamma)\,\bar{N}_\mu(E_\mu - E_\gamma, E, t)\,\bar{N}_\gamma(E_\gamma, E, t),$$

$$4\int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_-\, W_p^\mu(E_\mu, E_+, E_-)\,\bar{N}_\mu(E_\mu - E_+ - E_-, E, t)$$
$$\times\ \bar{N}_e(E_+, E, t)$$

are approximately two orders of magnitude smaller than the other terms. We can therefore neglect

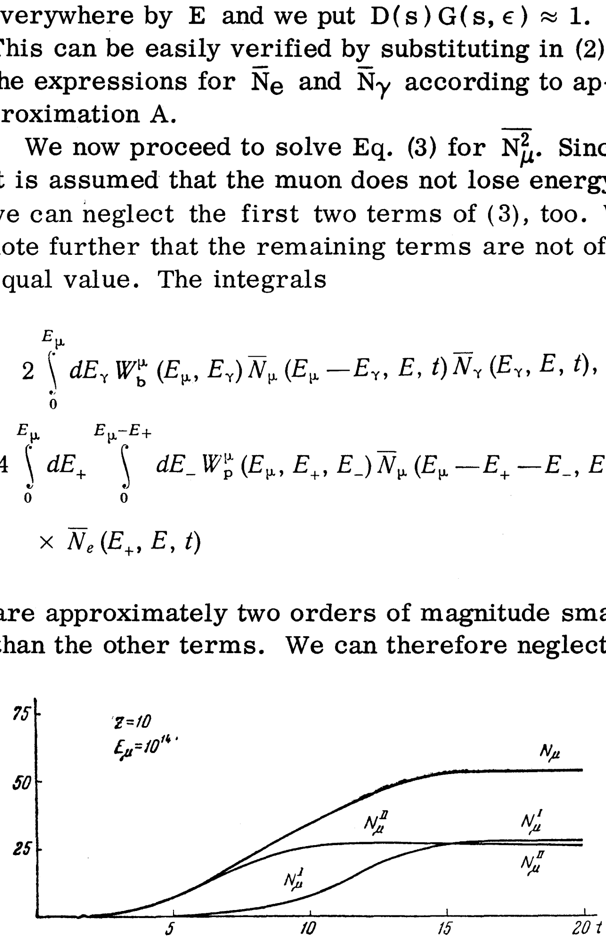


FIG. 2

[1)]In the calculation we used the critical energy calculated by Dovzhenko and Pomanskiĭ[11] ($\beta = 50.6 \times 10^6$ eV for ground, $20.7 \times 10^6$ eV for iron, and $7.4 \times 10^6$ eV for lead).

them, too. Equation (3) is then rewritten as

$$\frac{\partial}{\partial t}\overline{N_\mu^2}(E_\mu, E, t) = \int_0^{E_\mu} dE_\gamma W_b^\mu(E_\mu, E_\gamma)\overline{N_\gamma^2}(E_\gamma, E, t)$$

$$+ 2\int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_- W_p^\mu(E_\mu, E_+, E_-)[\overline{N_e^2}(E_+, E, t)$$

$$+ \overline{N}_e(E_+, E, t)\overline{N}_e(E_-, E, t)]. \qquad (3')$$

Let us calculate the right half of (3). We use for $\overline{N_\gamma^2}$ and $\overline{N_e^2}$ the expressions obtained in [6] for $s < 1.4$. It will be clear from what follows that the region of large s is not needed. As in the calculation of (4) and (2), we consider only a power-law dependence on the energy, $\sim E^{2s}$. For the first term in the right half of (3) we obtain in analogy with (4)

$$\int_0^{E_\mu} dE_\gamma \overline{N_\gamma^2}(E_\gamma, E, t) W_b^\mu(E_\mu, E_\gamma) = \frac{t_e}{t_\mu} c'(2s_1)\overline{N_\gamma^2}(E_\mu, E, t). \qquad (15)$$

In the calculation of the second term in the right half of (3) we can use the simplified expression for the pair production probability only if $s < 1$, and obtain an expression analogous to (12). As in (12), the main contribution to the integral is made by the region $E_+, E_- \sim E_\mu/m$. However, this no longer holds for $s \geq 1$. The integral is logarithmic and the essential region of integration for it will be $E_+, E_- \sim E_\mu$. We can therefore use for $s \geq 1$ the asymptotic pair production cross section [10] for high-energy secondary electrons and photons:

$$W_p^\mu(E_\mu, E_+, E_-)dE_+dE_- = \frac{4\alpha^2 Z^2 r_e^2 t_e n L_1}{3\pi m^2 A(E_+ + E_-)^2}\frac{E_\mu(E_\mu - E_+ - E_-)}{E_+E_-}$$

$$\times\left[\frac{E_+^2 + E_-^2}{(E_+ + E_-)^2}\left(1 - \frac{E_+ + E_-}{E_\mu} + \frac{(E_+ + E_-)^2}{E_\mu^2}\right)\right.$$

$$\left. + \frac{4E_+E_-}{(E_+ + E_-)^2}\left(1 - \frac{E_+ + E_-}{E_\mu}\right)\right] dE_+ dE_-,$$

$$L_1 = \ln(m/\alpha Z^{1/3}). \qquad (9')$$

We calculate first the second term in (3) for $s < 1$. The calculation of the first integral

$$2\int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_- \overline{N_e^2}(E_+, E, t) W_p^\mu(E_\mu, E_+, E_-)$$

is analogous to (12). We obtain

$$2\int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_- W_p^\mu(E_\mu, E_+, E_-)\overline{N_e^2}(E_+, E, t)$$

$$= dA'(2s_2)\overline{N_e^2}(E_\mu/m, E, t). \qquad (16)$$

In the calculation of the second integral

$$2\int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_- W_p^\mu(E_\mu, E_+, E_-)\overline{N}_e(E_+, E, t)\overline{N}_e(E_-, E, t)$$

we note that, like in (12), the integrand has a maximum $E_+, E_- \sim E_\mu/m$. We therefore use the simplified expression for the cross section and take all the weakly-varying functions outside the integral sign at the point $E_+, E_- \sim E_\mu/m$. We obtain

$$2\int_0^{E_\mu} dE_+ \int_0^{E_\mu - E_+} dE_- W_p^\mu(E_\mu, E_+, E_-)\overline{N}_e(E_+, E, t)\overline{N}_e(E_-, E, t)$$

$$\approx dF(2s_2)\overline{N}_e^2\left(\frac{E_\mu}{m}, E, t\right); \qquad (17)$$

$$F(2s) = \frac{1}{3}\left\{\frac{7\pi}{3}\operatorname{cosec} s\pi - 2^{1-s}s\left[B(s, 1-s)\right.\right.$$

$$\left.\left. - \frac{1}{3}\Gamma(1+s)\sum_{k=1}^{\infty}\frac{\Gamma(2k-s)}{\Gamma(2k+2)}\right] + 2B(s, 1-s)\right\}. \qquad (18)$$

The integration with respect to t for $s < 1$ determines, as in (14), the contribution from the first term up to $t \approx \ln(E_\mu/\beta)$, and from the second term up to $t \approx \ln(E_\mu/m\beta)$. However, as already noted, when $s \geq 1$ the essential region of integration for the second term of (3), which is due to pair production, is $E_+, E_- \sim E_\mu$. Therefore in order to join together smoothly the solutions for $t < \ln(E_\mu/\beta)$ and $t \geq \ln(E_\mu/m\beta)$, we introduce in Eq. (11) for $s_2$ and in the logarithm L a parameter $\chi$ in the following manner:

$$\lambda_1'(s_2')t + \ln(E_\mu\lambda/\beta m) = 0, \quad L' = \ln(\chi/\alpha Z^{1/3}). \qquad (19)$$

By stipulating that $\chi$ vary linearly with $s_2'$ from 1 to m as $s_2'$ varies from 0 to 1, we ensure smooth approximation in the region from $t = \ln(E_\mu/m\beta)$ to $t = \ln(E_\mu/\beta)$ of that part of $\overline{N_\mu^2}$ for which pair production is responsible. As will be verified later on, the additional inaccuracy introduced by this approximation is inessential, for $\overline{N_\mu^2}$ is determined primarily, accurate to $\sim 10$ per cent, by the muon bremsstrahlung. In addition, the approximation does not influence the solution for $t > \ln(E_\mu/\beta)$. After an integration with respect to t analogous to that in (14a), we obtain for the region $t < \ln(E_\mu/\beta)$:

$$\overline{N_\mu^2}(E_\mu, E, t) = \frac{t_e}{t_\mu}\frac{c'(2s_1)}{2\lambda_1(s_1)}\overline{N_\gamma^2}(E_\mu, E, t) + \frac{d'\chi^{-2s_2'}}{2\lambda_1(s_2')}$$

$$\times\left[A'(2s_2')\overline{N_e^2}\left(\frac{E_\mu\chi}{m}, E, t\right) + F'(2s_2')\overline{N}_e^2\left(\frac{E_\mu\chi}{m}, E, t\right)\right]. \qquad (20)$$

The functions $A'(2s)$ and $F'(2s)$ are shown in Fig. 3. Before we proceed to calculate the second

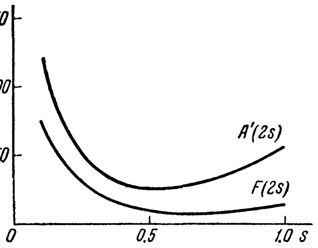


FIG. 3

term in (3) for $s \geq 1$, we note that integration with respect to t is perfectly similar here to that for $\overline{N}_\mu$, the only difference being that the integrand is no longer $\exp[\lambda_1(s)t + ys]$ but $\exp 2[\lambda_1(s)t + ys]$. Therefore the solution for $t > \ln(E_\mu/\beta)$ is also determined by the saddle point at the point $s = 1$. It follows therefore that we can put directly $s = 1$ in the integration of the second term in (3) with respect to the energy. Finally, we obtain with logarithmic accuracy for $t \gtrsim \ln(E_\mu/\beta)$:

$$\overline{N_\mu^2}(E_\mu, E, t)$$

$$= [1.56\pi \ln(E_\mu/\beta)]^{1/2}\, [t_e t_\mu^{-1} c'(2) \overline{N}^2_{\gamma\, max}(E_\mu, E, \ln(E_\mu/\beta))$$

$$+ d''A''(2)\, \overline{N}^2_{e\, max}(E_\mu, E, \ln(E_\mu/\beta))]$$

$$= \frac{G^2(1, \varepsilon)\, [(t_e/t_\mu)\, c'(2) + d''A''(2)]}{2\, [1.56\,\pi \ln(E_\mu/\beta)]^{1/2}} \left(\frac{E_\mu}{\beta}\right)^2, \qquad (21)$$

$$A''(2) = \frac{2\ln(m/\alpha Z^{1/3}) \ln^2 m}{m^2 \ln(2/\alpha Z^{1/3})}, \qquad d'' = \frac{\alpha \ln(m/\alpha Z^{1/3})}{6\pi \ln(180 Z^{-1/3})}. \qquad (22)$$

We thus obtain in our approximation, without account of the muon energy losses, a solution for the mean square that does not depend on t in the region of large t, i.e., the mean-square deviations are also constant. It must be noted, however, that for very large depths—on the order of or larger than the muon range—the muon losses and consequently the extinction of the entire cascade turn out to be significant. An account of this circumstance causes the mean square, and consequently also the variance, to increase with depth, i.e., the same pattern will be observed as in the extinction of a shower produced by an electron or a photon.[5]

Figure 4 shows the depth dependence of $\Delta^2 = \overline{N_\mu^2}/\overline{N}_\mu^2$ for a shower due to a muon with energy $E_\mu = 10^{14}$ eV in ground ($Z = 10$) and for two showers due to muons with energy $E_\mu = 10^{14}$ and $10^{13}$ eV, in iron ($Z = 26$).

At large depths $[t \gtrsim \ln(E_\mu/\beta)]$ the mean square deviations $\sqrt{\Delta^2 - 1}\, \overline{N}_\mu$ of the number of particles turn out to be very large ($\sim 10\overline{N}_\mu$). $\sqrt{\Delta^2 - 1}$ depends little on the energy and on the

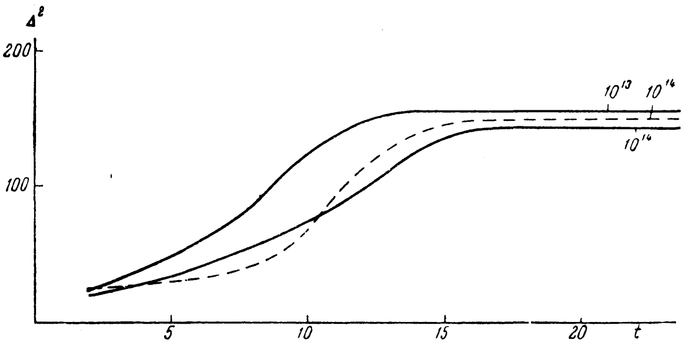


FIG. 4. Depth dependence of $\Delta^2$ for a muon shower: solid line – for iron, Z = 26; dashed – for ground, Z = 10.

substance $[\sim \ln^{-1/4}(E_\mu/\beta)]$. Large fluctuations are due essentially to the fact that the range of the photon- and pair-induced cascades generated by the muon up to their maximum, are of the same order as (in the case of pairs) or much smaller than (in the case of photons) the range of the muon with respect to the corresponding interactions. Thus, for $E_\mu = 10^{14}$ eV, the range is $\sim 10 t_e$ for pair production by a muon and $\sim 10^3 t_e$ for a photon. At the same time, the corresponding ranges of the cascade due to pairs and photons are respectively of order $10\, t_e$ and $15\, t_e$ up to the maxima.

Figures 5a and b illustrate the possible $\Delta^2$ for a muon with energy $E_\mu = 10^{14}$ eV in ground ($Z = 10$) if the multiplication of the cascade is (a) due only to the showers produced by bremsstrahlung photons emitted by the muon ($\Delta_I^2$, Fig. 5a) or (b) due only to showers produced by the pairs generated by the muon ($\Delta_{II}^2$, Fig. 5b).

We see that $\Delta_I^2$ is two orders of magnitude larger than $\Delta_{II}^2$ in the region of large t $[t \sim \ln(E_\mu/\beta)]$. This is the consequence of the considerably larger bremsstrahlung range compared to the pair production range. The increase

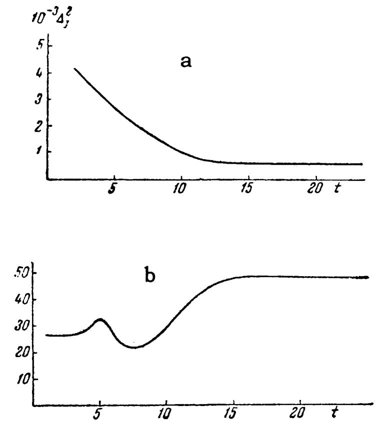


FIG. 5

in $\Delta_I^2$ towards smaller t signifies that there are large fluctuations at the start of the development of cascades due to bremsstrahlung photons (see[5]).

At the same time, $\Delta_{II}^2$ has a minimum in the region $t \sim \ln(E_\mu/m\beta)$. The presence of a minimum in the fluctuations of the cascades produced by the pairs is connected with the fact that the muon generates electrons which have essentially an energy $\sim E_\mu/m$. The showers produced by these electrons already reach their maximum number of particles, in the mean, when $t \sim \ln(E_\mu/m\beta)$ whereas the mean square of the number of particles increases more slowly and has a maximum at $t \gtrsim \ln(E_\mu/\beta)$; the maximum fluctuations are due to showers from electrons with energy $\sim E_\mu$, the number of which, however, is negligibly small.

This explains also the decrease in $\Delta^2$ for $t < \ln(E_\mu/\beta)$, for at small depths the main contribution to the average number of electrons is made by showers due to pairs. Therefore the best way to determine the muon energy is to register small bursts under a thin absorber, not thicker than $t \approx \ln(E_\mu/m\beta)$.

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