TRAJECTORIES OF REGGE VACUUM POLES

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Restrictions on the behavior of a vacuum pole $l_0(t)$ are imposed on the basis of the following properties of $l_0(t)$: 1) $l_0(t) = 1$; 2) $l_0(t)$ is an analytic function of t in the complex plane of t with a cut along the real axis from $4\mu_\pi^2$ to infinity.

In order to describe elastic scattering at high energies, the hypothesis was advanced by Gribov [1,2] that the singularity of the partial-wave amplitude $f_l(t)$ in the annihilation channel farthest to the right in the complex l-plane is a simple pole (called sometimes the vacuum pole or the Pomeranchuk pole). The position of this pole is a function $l_0(t)$ of the momentum transfer t and at high energies the elastic scattering amplitude is proportional to $s^{l_0(t)}$. Inasmuch as the function $l_0(t)$, the vacuum trajectory, plays a fundamental role in the explanation of processes at high energies, it is extremely important to find its properties and to determine its behavior as a function of t.

The function $l_0(t)$ possesses the following properties:

I. From the constancy of the total cross sections at high energies it follows [3,1] that $l_0(t) = 1$.

II. Gribov and Pomeranchuk [4] have shown that $l_0(t)$ is an analytic function of t in the complex t-plane with a cut on the real axis from $4\mu^2$ (μ is the pion mass) to infinity. On the real axis to the left of $t = 4\mu^2$ the function $l_0(t)$ is real.

III. In the same paper [4] it is shown that $l_0'(t) > 0$ in the interval $0 < t < 4\mu^2$.

IV. If it is assumed (Mandelstam [5], Froissart [6]) that the amplitude f(s, t) does not grow faster than a finite power $s^N t^M$ as $s \to \infty$ and (or) as $t \to \infty$, then $\mathrm{Re}\, l_0(t)$ is bounded, $\mathrm{Re}\, l_0(t) \leq N$.

In the present work we establish certain restrictions on the behavior of the function $l_0(t)$ on the basis of properties I and II (in some cases we also use properties III and IV). We introduce the quantity $x = t/4\mu^2$ and make the conformal transformation

$$z = -(\sqrt{x-1} - i)/(\sqrt{x-1} + i) \qquad (1)$$

of the two sides of the cut along the real axis from 1 to ∞ into the unit circle. This transforms the whole cut x-plane into the interior of the unit circle and the point x = 0 into the point z = 0. It follows that the function $l_0(z)$ will be analytic for $|z| < 1$.

1. Let us assume that condition IV is fulfilled, i.e., $\mathrm{Re}\, l_0(z) \leq N$. We consider the function $f(z) = l_0(z) - 1$ and use Carathéodory's theorem (cf. for example, [7]). According to this theorem a function f(z), analytic inside the unit circle, possessing inside the circle a bounded real part $\mathrm{Re}\, f(z) \leq A$ equal to zero at the point z = 0, obeys the inequality

$$|f(z)| \leqslant 2A|z|/(1-|z|). \qquad (2)$$

Using (2) and expressing f(z) in terms of $l_0(z)$ and z in terms of x by means of (1), we obtain for real $x < 0$

$$|l_0(x) - 1| \leqslant (N-1)(\sqrt{1-x} - 1). \qquad (3)$$

While N is unknown, it is expedient to use this inequality not as a restriction on the behavior of $l_0(x)$, but for the determination of a lower bound for N with the help of experimental data. From experiment it is well known that $l_0(t)$ vanishes for $t \approx -1\ \mathrm{BeV}^2$ ($x = x_0 \approx -13$). Substituting these values in (3) we find $N > 1.4$, i.e., $\max \mathrm{Re}\, l_0(t) > 1.4$.

2. Let us find lower bounds on the mean value of $|l_0(x)|^2$ on the cut

$$\overline{|l_0(x)|^2} = \int_1^\infty w(x)\,|l_0(x)|^2\,dx, \qquad (4)$$

where w(x) is some weight function, satisfying the conditions $w(x) > 0$ for $x > 1$, and $\int_1^\infty w(x)\,dx = 1$. The conformal transformation (1) transforms integral (4) to

$$\overline{|l_0(x)|^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,|l_0(z)|^2\,d\theta, \qquad z = e^{i\theta}, \qquad (5)$$

where

$$f(\theta) = \pi w(x)\,x\sqrt{x-1}, \qquad x = 1 + \tan^2\theta/2. \qquad (6)$$

The solution to the problem of finding the minimum of the integral (5) on the class of functions $l_0(z)$, analytic inside the unit circle and satisfying the condition $l_0(0) = 1$, is very well known in mathematics (cf., for example, [9,10,11]). The minimum can be shown to be [1)]

$$\overline{|l_0(x)|^2_{min}} = D^2(0), \quad (7)$$

where the function D(x) is expressed in terms of $f(\theta)$ as

$$D(z) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln f(\theta)\,\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\,d\theta\right\}. \quad (8)$$

The minimizing function is

$$l_{0min}(z) = D(0)/D(z). \quad (9)$$

If some other conditions, known to us from experiment, on $l_0(x)$ for $x < 0$ are imposed besides the condition $l_0(0) = 1$, then evidently this will lead to an increase in the minimum mean value $\overline{|l_0(x)|^2}$. For example, let $l_0(x_0) = a$. The minimum of integral (5), under the conditions $l_0(0) = 1$ and $l_0(z_0) = a$, is easy to find by expanding the function $l_0(z)$ into a complete system of orthogonal polynomials with weight function $f(\theta)$

$$l_0(z) = \sum c_n p_n(z). \quad (10)$$

(An analogous extremal problem was solved by us elsewhere [12]). For the minimum of $\overline{|l_0(x)|^2}$ we obtain

$$\overline{|l_0(x)|^2_{min}} = \frac{1}{z_0^2}\{D^2(0) + a(1-z_0^2)D(z_0)[aD(z_0) - 2D(0)]\}, \quad (11)$$

and the minimizing function is determined as

$$l_{0min}(z) = \frac{1}{z_0^2 D(z)}\left\{[D(0) - aD(z_0)(1-z_0^2)] + \frac{1-z_0^2}{1-zz_0}[D(z_0)a - D(0)]\right\}. \quad (12)$$

Let us consider some examples.

Let $w(x) = 1/\pi x\sqrt{x-1}$. From (7) - (9) we have $\overline{|l_0(x)|^2_{min}} = 1$ and $l_{0\,min} = 1$. We now take into consideration the experimental fact that $l_0(x)$ reaches zero at $x = x_0 \approx -13$ ($z_0 \approx -0.57$). From (11) we get $\overline{|l_0(x)|^2_{min}} = 1/z_0^2 \approx 3.1$. The minimizing function is then given by

$$l_{0min}(z) = (z_0 - z)/z_0(1 - zz_0). \quad (13)$$

[1)]Notice that conditions III and IV are not used here.

It is not difficult to confirm that for this function $l'_{0\,min}(x) > 0$ when $0 < x < 1$, i.e., the minimum found by us is in both cases a minimum defined on the class of functions satisfying condition III.

Although condition IV has not been used in our results, it is meaningless to consider functions $l_0(x)$ growing as $x^{1/4}$ or faster as $x \to \infty$ [the integral (4) diverges] if the weight function chosen above is employed. In order to obtain bounds on the mean value $\overline{|l_0(x)|^2}$ for a faster growth of $l_0(x)$, we take a weight function $w(x) = 16\sqrt{x-1}/\pi x^4$, which decreases more rapidly as $x \to \infty$. For this case we find $\overline{|l_0(x)|^2_{min}} = 1/2$ and $l_{0\,min}(z) = (1+z)^{-2}(1-z)^{-1}$. If we also take into account the vanishing of $l_0(x)$ when $x = x_0$, we have

$$\overline{|l_0(x)|^2_{min}} = {}^1/_4\, z_0^2 \approx 0.75,$$

$$l_{0min}(z) = (z_0 - z)/z_0(1+z)^2(1-z). \quad (14)$$

By direct differentiation we can verify that the function $l_{0\,min}(x)$ given by (14) satisfies the condition $l'_{0\,min}(x) > 0$ when $0 < x < 1$.

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[1]V. N. Gribov, JETP **41**, 667 (1961), Soviet Phys. JETP **14**, 478 (1962).

[2]V. N. Gribov, JETP **41**, 1962 (1961), Soviet Phys. JETP **14**, 1395 (1962).

[3]G. Chew and S. Frautschi, Phys. Rev. **123**, 1478 (1961).

[4]V. N. Gribov and I. Ya Pomeranchuk, JETP **43**, 308 (1962), Soviet Phys. JETP **16**, 220 (1963).

[5]S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[6]M. Froissart, Phys. Rev. **123**, 1053 (1961).

[7]S. Stoilow, Theory of Functions of a Complex Variable, (Russ. Transl.) IIL, 1962.

[8]Diddens, Lillethun, Manning, Taylor, Walker, and Wetherell, Phys. Rev. Lett. **9**, 111 (1962).

[9]V. I. Smirnov, Izv. AN SSSR ser. Fiz. 7, 337 (1932).

[10]G. Szego (Orthogonal Polynomials), Am. Math. Soc. N. Y., 1939.

[11]N. N. Meĭman, JETP **44**, 1228, (1963), Soviet Phys. JETP **17**.

[12]B. V. Geshkenbeĭn and B. L. Ioffe, JETP **45**, No. 9 (1963) (in press).

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