INTENSITY OF CERENKOV RADIATION WITH DISPERSION TAKEN INTO ACCOUNT

V. P. ZRELOV

Joint Institute for Nuclear Research

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It is shown that significant errors can arise if the dispersion of the medium is neglected in calculating Cerenkov radiation close to threshold. The correct expressions are given.

THE Frank-Tamm formula [1] for the Cerenkov energy radiated per unit length of path for a singly charged particle is

$$\frac{dW}{dl}=\frac{e^2}{c^2}\int_{\omega_1}^{\omega_2}\left(1-\frac{1}{\beta^2 n^2(\omega)}\right)\omega d\omega=\frac{e^2}{c^2}\int_{\omega_1}^{\omega_2}\sin^2\theta(\omega)\,\omega d\omega. \quad (1)$$

In the calculation of Cerenkov energy losses of charged particles it is standard practice to neglect dispersion $n(\omega)$ and to take the bracketed term $[1-\beta^{-2}n^{-2}(\omega)]$ outside the integral (in which case n is taken for some mean wavelength $\bar{\lambda}$); the following expression is used in these calculations:

$$dW/dl\approx 2\pi^2 e^2(1-\beta^{-2}n^{-2}(\bar{\lambda}))(\lambda_2^2-\lambda_1^2)/\lambda_1^2\lambda_2^2. \quad (2)$$

At large Cerenkov angles this procedure does not introduce any significant error in the determination of dW/dl; however, neglect of dispersion in the medium does have an important effect on the calculation at small radiation angles (close to threshold).

If one assumes than $n(\lambda)$ is given by the two-term Cauchy formula, which is a rather good description of the function $n(\lambda)$ in the region of normal dispersion,

$$n(\lambda)=a+b\lambda^{-2}, \quad (3)$$

then the change of radiation angle θ with wavelength is given by the derivative

$$d\theta/d\lambda=-2b/\lambda^3 n\,\mathrm{tg}\,\theta. \quad (4)^*$$

At threshold ($\theta\sim 0$) the derivative in (4) is large (especially in the blue region of the spectrum). Thus, near threshold the radiation angle will be a strong function of λ; it then follows from (1) that the energy loss is also a strong function of λ.

Using the function $n(\lambda)$ given in (3) one can obtain a simple expression for the radiated energy per unit length of path in the wavelength region between λ_1 and λ_2:

$$dW/dl=2\pi^2e^2(1-1/\beta^2 n(\lambda_1)\,n(\lambda_2))(\lambda_2^2-\lambda_1^2)/\lambda_1^2\lambda_2^2. \quad (5)$$

It is evident from a comparison of (2) and (5) that an account of the dispersion $n(\lambda)$ in the expression for the radiated energy leads only to the simple replacement of $n^2(\bar{\lambda})$ by $n(\lambda_1)\,n(\lambda_2)$.

Account of the dispersion in the form given in (3) in the expression for the number of Cerenkov photons dN/dl emitted by a particle per unit length of path leads to a more complicated expression:

$$\frac{dN}{dl}=2\pi\alpha\left\{\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2}-\frac{1}{\beta^2}\left[\frac{\lambda_2 n(\lambda_2)-\lambda_1 n(\lambda_1)}{2a\lambda_1\lambda_2 n(\lambda_1)\,n(\lambda_2)}+\frac{1}{2a\sqrt{ab}}\left(\mathrm{arctg}\frac{a\lambda_2}{\sqrt{ab}}-\mathrm{arctg}\frac{a\lambda_1}{\sqrt{ab}}\right)\right]\right\}, \quad (6)^*$$

Here $\alpha=e^2/\hbar c=1/137$. The second term in the rectangular brackets in (6) is equal to the first to within several percent (approximately 5%). Hence, (6) can be written approximately as

$$\frac{dN}{dl}\approx 2\pi\alpha\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2}\left\{1-\frac{1}{\beta^2}\frac{\lambda_2 n(\lambda_2)-\lambda_1 n(\lambda_1)}{(\lambda_2-\lambda_1)\,n(\lambda_1)\,n(\lambda_2)\,a}\right\}, \quad (7)$$

where $2\pi\alpha=0.04585$ and $\lambda_2>\lambda_1$.

When $n(\lambda_2)=n(\lambda_1)=n(\bar{\lambda})$ and $a\approx n(\bar{\lambda})$, (7) assumes the usual form

$$dN/dl\approx 2\pi\alpha(1-\beta^{-2}n^{-2}(\bar{\lambda}))(\lambda_2-\lambda_1)/\lambda_1\lambda_2. \quad (8)$$

In order to see the difference in the results obtained with (6) and (8) we calculate the number of photons obtained from these expressions for several values of β and an actual dispersion relation $n(\lambda)$ corresponding to K7 glass ($n_D=1.5142$):

$$n(\lambda)=1.5020+0.42085\cdot 10^{-10}\lambda^{-2},$$

where λ is given in centimeters. In the wavelength region between $\lambda_1=3000$ Å and $\lambda_2=6563$ Å the calculation based on (6) with $\beta=0.66$ gives 13.4 photons/cm whereas the calculation according to (8) with $\bar{\lambda}=(\lambda_1+\lambda_2)/2$ gives 5.7 photons/cm, that is to say, a result differing by 2.4 times. At high velocities the difference in the number of photons

*tg = tan.

*arctg = $\tan^{-1}$.

β	$\theta(\bar{\lambda})$ with $\bar{\lambda}$ from (9)	dN/dl,	photons/cm	
		Eq. (6)	Eq. (8) with $\bar{\lambda}=(\lambda_1+\lambda_2)/2$	Eq. (8) with $\bar{\lambda}$ from (9)
1.0	49°0′	474	471	473
0.90	43°11′	391	387	389
0.80	34°56′	274	269	272
0.70	20°26′	104	97.2	101
0.68	15°18′	60.7	53.2	57.6
0.66	6°24′	13.4	5.7	10.1

as computed by the two expressions is smaller, as is shown graphically in the table.

The calculations show that the error in the determination of dN/dl by the simple formula (8) can be reduced if the refractive index is taken for a mean wavelength defined by the following expression:

$$\bar{\lambda} = \frac{\lambda_2\lambda_1}{\lambda_2 - \lambda_1}\ln\frac{\lambda_2}{\lambda_1}. \quad (9)$$

The value of dN/dl for $\bar{\lambda}$ given by (9) is given in the last column of the table.

A correct calculation of the Cerenkov radiation intensity is especially important in the design of differential (angle) Cerenkov counters operating at angles $\theta = 5° - 10°$, where an error in dN/dl can lead either to an incorrect increase in the length of the counter or to a low value of the efficiency for particle detection as compared with the computed value.

[1] I. E. Tamm and I. M. Frank, DAN SSSR **14**, 107 (1937).

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