THE EFFECT OF COLLISIONS ON THE EXCITATION SPECTRUM OF A SYSTEM OF ELECTRONS

Yu. Ya. POLYAK and V. M. ELEONSKIĬ

Ural Branch, Academy of Sciences, U.S.S.R.

Submitted to JETP editor December 13, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **45**, 159-163 (August, 1963)

An expression for the longitudinal dielectric constant of a system of electrons is obtained in the hydrodynamic approximation in which the effect of spatial dispersion on electron-ion and electron-electron collisions is taken into account. It is shown that when spatial dispersion is taken into account electron-ion as well as electron-electron collisions become important and that in the presence of spatial dispersion collision damping causes a small shift in the characteristic frequencies of the electron oscillations.

IN the present communication we consider the effect of electron-ion and electron-electron collisions on the excitation spectrum of a classical system of interacting electrons. If the effect of spatial dispersion on the process of pair collisions is neglected, then the inclusion of collisions provides, as is known, a collision mechanism for the damping of longitudinal excitations in a system of interacting electrons.[1] In the region of small wave numbers collision damping may exceed the Landau damping[1] which is associated with the Cerenkov dissipation mechanism. For a more detailed analysis of the effect of collisions on the excitation spectrum of an electron system (especially for an evaluation of the effect of electron-electron collisions) one must take into account the effect of spatial dispersion on the two particle collision process. The relations that determine the dependence of collision damping and frequency shift on the wave number of the perturbation in the long-wave spectral region are obtained below in the hydrodynamic approximation.

To describe the system of interacting electrons we use a kinetic equation with a longitudinal self-consistent field and the Landau form of the collision integral

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{r}} - \frac{e}{m}\mathbf{E}\frac{\partial f}{\partial \mathbf{v}} = \frac{\partial j_\alpha}{\partial v_\alpha},$$

$$\operatorname{div}\mathbf{E} = 4\pi e\left(n_i - \int d\mathbf{v} f\right), \tag{1}$$

where n_i is the mean ion density, and

$$j_\alpha = m^{-1}\int d\mathbf{v}' I_{ei}^{\alpha\beta}(\mathbf{v}-\mathbf{v}')\left[\frac{f_i'}{m}\frac{\partial f}{\partial v_\beta} - \frac{f}{M}\frac{\partial f_i'}{\partial v_\beta'}\right] + m^{-2}\int d\mathbf{v}' I_{ee}^{\alpha\beta}(\mathbf{v}-\mathbf{v}')\left[f'\frac{\partial f}{\partial v_\beta} - f\frac{\partial f'}{\partial v_\beta'}\right] \tag{2}$$

is the collision flux due to electron scattering by ions and electrons. In Eq. (2) $f_i = (M/2\pi T)^{3/2} \times \exp(-Mv^2/2T)$, M is the ion mass, and I_{ei}, I_{ee} are Landau collision kernels given explicitly in [1] and [2].

Linearization of Eq. (1) about a homogeneous Maxwell distribution for the electrons with a temperature equal to that of the ions allows one to show that in an $(\omega, \mathbf{k})$-representation the perturbed distribution $f(\mathbf{v}, \mathbf{r}, t) - f_0(\mathbf{v})$ satisfies the equation

$$F_{k\omega}(\mathbf{v}) = -\left(\frac{\omega_0}{k}\right)^2 \frac{\mathbf{k}\partial f_0/\partial \mathbf{v}}{\omega - \mathbf{kv}} n_{k\omega} + i\frac{\partial(\delta j_\alpha)/\partial v_\alpha}{\omega - \mathbf{kv}},$$

$$F_{k\omega}(\mathbf{v}) \equiv \iint dt\, d\mathbf{r} e^{-i\omega t + i\mathbf{kr}}(f - f_0). \tag{3}$$

Here ω_0 is the electron plasma frequency, $f_0 = (m/2\pi T)^{3/2}\exp(-mv^2/2T)$ the unperturbed distribution function, $n_{k\omega} = \int d\mathbf{v}\, F_{k\omega}(\mathbf{v})$ the density fluctuation, while the linearized collision flux is

$$\delta j_\alpha = \left\{D_{ei}^{\alpha\beta}\frac{\partial}{\partial v_\beta} + A_{ei}^\alpha + D_{ee}^{\alpha\beta}\frac{\partial}{\partial v_\beta} + A_{ee}^\alpha\right\}F_{k\omega}(\mathbf{v}) - m^{-2}\int d\mathbf{v}' I_{ee}^{\alpha\beta}(\mathbf{v}-\mathbf{v}')\left[F_{k\omega}'\frac{\partial f_0}{\partial v_\beta} - f_0\frac{\partial F_{k\omega}'}{\partial v_\beta'}\right]. \tag{4}$$

The Fokker-Planck diffusion and dynamic-friction coefficients are determined by the usual relations, and, on the basis of the results in [2] and [3], we find that the diffusion coefficients have the form

$$D_{ei}^{\alpha\beta} = \frac{1}{2}\nu_{ei}(v)\left\{\left[1 - \frac{m}{M}\frac{T}{mv^2}\right]v^2\delta_{\alpha\beta} - \left[1 - 3\frac{m}{M}\frac{T}{mv^2}\right]v_\alpha v_\beta\right\},$$

$$D_{ee}^{\alpha\beta} = \frac{1}{2}\nu_{ee}(v)\left\{\left[\left(1 - \frac{T}{mv^2}\right)\Phi\left(v\sqrt{\frac{m}{2T}}\right) + \frac{1}{v}\sqrt{\frac{2T}{\pi m}}e^{-mv^2/2T}\right]v^2\delta_{\alpha\beta} - \left[\left(1 - 3\frac{T}{mv^2}\right)\Phi\left(v\sqrt{\frac{m}{2T}}\right) + \frac{2}{v}\sqrt{\frac{3T}{\pi m}}e^{-mv^2/2T}\right]v_\alpha v_\beta\right\}. \tag{5}$$

Here ν_{ei}, $\nu_{ee} \sim v^{-3}$ are the collision frequencies for momentum transfer, and $\Phi(z)$ the probability integral.

We shall use the hydrodynamic approximation to find the longitudinal dielectric constant for a system of interacting electrons. In the hydrodynamic approximation the state of the system is determined by the first few moments of the distribution function, which represent the density, mean velocity, stress tensor, and energy flux. The perturbed distribution function can be written in the form of an asymptotic expansion in terms of the above hydrodynamic variables:

$$F_{k\omega}(\mathbf{v}) \sim \Big\{ n_{k\omega} + \frac{m}{T} u^{\alpha}_{k\omega} v_{\alpha} + \frac{1}{2T}\left[\Pi^{\alpha\beta}_{k\omega} - T n_{k\omega}\delta_{\alpha\beta}\right] \times \left[\frac{m}{T} v_{\alpha} v_{\beta} - \delta_{\alpha\beta}\right] + \ldots \Big\} f_0(v). \tag{6}$$

Here

$$u^{\alpha}_{k\omega} = \int d\mathbf{v}\, v_{\alpha} F_{k\omega}(\mathbf{v}), \quad \Pi^{\alpha\beta}_{k\omega} = m \int d\mathbf{v}\, v_{\alpha} v_{\beta} F_{k\omega}(\mathbf{v}).$$

Expansion (6) together with Eq. (3) provides a system of hydrodynamic equations for the moments of the distribution function $n_{k\omega}$, $u^{\alpha}_{k\omega}$, $\Pi^{\alpha\beta}_{k\omega}, \ldots$. Calculations show that if only the first three terms are retained in expansion (6), then the system of equations for the moments will have the form

$$\begin{aligned}
&[1 + P(\omega, \mathbf{k})]\, n_{k\omega} + iS_{\alpha}(\omega, \mathbf{k})\, u^{\alpha}_{k\omega} - iK_{\alpha\beta}(\omega, \mathbf{k})\left[\Pi^{\alpha\beta}_{k\omega}/T - n_{k\omega}\delta_{\alpha\beta}\right] = 0,\\
&P^{\gamma}(\omega, \mathbf{k})\, n_{k\omega} + u^{\gamma}_{k\omega} + iS^{\gamma}_{\alpha}(\omega, \mathbf{k})\, u^{\alpha}_{k\omega} - iK^{\gamma}_{\alpha\beta}(\omega, \mathbf{k})\left[\Pi^{\alpha\beta}_{k\omega}/T - n_{k\omega}\delta_{\alpha\beta}\right] = 0,\\
&P^{\mu\nu}(\omega, \mathbf{k})\, n_{k\omega} + iS^{\mu\nu}_{\alpha}(\omega, \mathbf{k})\, u^{\alpha}_{k\omega} + \Pi^{\mu\nu}_{k\omega}/m - iK^{\mu\nu}_{\alpha\beta}(\omega, \mathbf{k})\left[\Pi^{\alpha\beta}_{k\omega}/T - n_{k\omega}\delta_{\alpha\beta}\right] = 0
\end{aligned} \tag{7}$$

Here

$$P^{\mu_1 \cdots \mu_l}(\omega, \mathbf{k}) = \left(\frac{m}{T}\right)^{l/2} \left(\frac{\omega_0}{k}\right)^2 \int d\mathbf{v}\, v_{\mu_1} \cdots v_{\mu_l} \frac{\mathbf{k}\, \partial f_0/\partial \mathbf{v}}{(\omega - \mathbf{k}\mathbf{v})} \tag{8}$$

is the collisionless polarization tensor for the electron system while the quantities

$$S^{\mu_1 \cdots \mu_l}_{\alpha}(\omega, \mathbf{k}) = \left(\frac{m}{T}\right)^{l/2} \frac{M}{T}\left(1 + \frac{m}{M}\right) \int d\mathbf{v}\, v_{\mu_1} \cdots v_{\mu_l} \frac{A^{\alpha}_{ei}(\mathbf{v})}{\omega - \mathbf{k}\mathbf{v}} f_0, \tag{9}$$

$$\begin{aligned}
K^{\mu_1 \cdots \mu_l}_{\alpha\beta}(\omega, \mathbf{k}) = \left(\frac{m}{T}\right)^{l/2} \frac{m}{2T} \Big\{ &\int d\mathbf{v} \frac{v_{\mu_1} \cdots v_{\mu_l}}{\omega - \mathbf{k}\mathbf{v}} \Big[2D^{\alpha\beta}_{ei} - \left(1 + \frac{m}{M}\right)\left(A^{\alpha}_{ei} v_{\beta} + A^{\beta}_{ei} v_{\alpha}\right)\Big] f_0 \\
&+ 2\int d\mathbf{v} \frac{v_{\mu_1} \cdots v_{\mu_l}}{\omega - \mathbf{k}\mathbf{v}} \Big\{ 2D^{\alpha\beta}_{ee} - A^{\alpha}_{ee} v_{\beta} - A^{\beta}_{ee} v_{\alpha} - \frac{T}{m}\left(\frac{\partial A^{\alpha}_{ee}}{\partial v_{\beta}} + \frac{\partial A^{\beta}_{ee}}{\partial v_{\alpha}}\right)\Big\} f_0 \Big\}
\end{aligned} \tag{10}$$

are the moments of the Landau collision integral in which the effect of spatial dispersion of the medium on particle collision is taken into account. This last circumstance is due to the fact that Eq. (4) was used in deriving the hydrodynamic system of equations and it was solved for in terms of the sought distribution function. In other words, the moment with respect to the collision integral is obtained with the free electron propagator taken into account.

Because of the homogeneity and isotropy of the medium $u^{\alpha}_{k\omega} = k_{\alpha} u_{k\omega}$, $T\Pi^{\alpha\beta}_{k\omega} = \Pi_{k\omega}\delta_{\alpha\beta} + \bar{\Pi}_{k\omega} k_{\alpha} \times k_{\beta}/k^2$, and the system (7) can be reduced to a system of equations for the scalar hydrodynamic variables $n_{k\omega}$, $u_{k\omega}$, $\Pi_{k\omega}$, and $\bar{\Pi}_{k\omega}$. A consequence of the first two equations in (7) is the obvious relation $\omega n_{k\omega} = k^2 u_{k\omega}$.

Let us consider the approximation in which one neglects the stress tensor in expansion (6). In this case system (7) yields the following for the longitudinal dielectric constant

$$\varepsilon(\omega, \mathbf{k}) = \varepsilon_0(\omega, \mathbf{k}) + i\left(1 + \frac{m}{M}\right) \frac{\omega^2}{k^2} \frac{m}{T} \int d\mathbf{v} \frac{\nu_{ei}(v)}{\omega} \frac{(\mathbf{k}\mathbf{v}) f_0(v)}{\omega - \mathbf{k}\mathbf{v}}, \tag{11}$$

where $\epsilon_0(\omega, \mathbf{k})$ is the collisionless dielectric constant.[1] We note that in this approximation the electron-electron collision integral vanishes.

We now present the long-wave asymptotic form for the dielectric constant from Eq. (11), and for the stress tensor, which can be computed in the approximation under consideration. It is not difficult to see that when $k \ll k_0 = \omega_0 \sqrt{m/T}$

$$\begin{aligned}
\operatorname{Re} \varepsilon(\omega, \mathbf{k}) &= \operatorname{Re} \varepsilon_0(\omega, \mathbf{k}) - \sqrt{\frac{\pi}{2}} \left(1 + \frac{m}{M}\right) \frac{\nu_{ei}(T)}{\omega} \exp\left\{-\frac{m}{2T}\left(\frac{\omega}{k}\right)^2\right\},\\
\operatorname{Im} \varepsilon(\omega, \mathbf{k}) &= \operatorname{Im} \varepsilon_0(\omega, \mathbf{k}) + \sqrt{\frac{2}{\pi}} \left(1 + \frac{m}{M}\right) \frac{\nu_{ei}(T)}{\omega} \left[\frac{1}{3} + \frac{2}{5}\frac{T}{m}\left(\frac{k}{\omega}\right)^2\right],
\end{aligned} \tag{12}$$

$$\begin{aligned}
\operatorname{Re} \Pi^{\alpha\beta}(\omega, \mathbf{k}) &= \operatorname{Re} \Pi^{\alpha\beta}_0(\omega, \mathbf{k}) + \sqrt{\frac{\pi}{8}} \left(1 + \frac{m}{M}\right) \frac{\nu_{ei}(T)}{\omega} m \left(\frac{\omega}{k}\right)^2 \times \exp\left\{-\frac{m}{2T}\left(\frac{\omega}{k}\right)^2\right\} \left[\delta_{\alpha\beta} - 2\frac{k_{\alpha} k_{\beta}}{k^2}\right] n_{k\omega},\\
\operatorname{Im} \Pi^{\alpha\beta}(\omega, \mathbf{k}) &= \operatorname{Im} \Pi^{\alpha\beta}_0(\omega, \mathbf{k}) - \frac{5}{\sqrt{4\pi}} \left(1 + \frac{m}{M}\right) \frac{\nu_{ei}(T)}{\omega} \times \left\{\left[\frac{1}{3} + \frac{4}{7}\frac{T}{m}\left(\frac{k}{\omega}\right)^2\right]\delta_{\alpha\beta} + 2\left[\frac{1}{3} + \frac{8}{7}\frac{T}{m}\left(\frac{k}{\omega}\right)^2\right]\frac{k_{\alpha} k_{\beta}}{k^2}\right\} T n_{k\omega}.
\end{aligned} \tag{13}$$

In the above expressions $\Pi^{\alpha\beta}_0(\omega, \mathbf{k})$ is the collisionless stress tensor,[1] while $\nu_{ei}(T)$ is determined by the relation $\nu_{ei}(v) \equiv \nu_{ei}(T)[T/mv^2]^{3/2}$.

In the approximation being examined the shift of the characteristic frequencies of the electron oscillations vanishes exponentially for $k < k_0$, whereas the collision damping effect may exceed Landau damping for sufficiently small wave numbers.

Now let us consider the solution of (7). Our calculations show that the longitudinal dielectric constant has the form

$$\begin{aligned}\varepsilon(\omega, \mathbf{k}) = {} & \varepsilon_0(\omega, \mathbf{k}) + i\left(\frac{\omega}{k}\right)^2 \frac{k_\alpha S_\alpha}{\omega} + iK_{\alpha\alpha} \\ & + iG^{-1}(\omega, \mathbf{k})\Big\{(K_{11} - K_{\alpha\alpha})\left(P^{\beta\beta} + i\left(\frac{\omega}{k}\right)^2 \frac{k_\alpha S_\alpha}{\omega} + iK_{\alpha\alpha}^{\beta\beta}\right) \\ & + \frac{m}{T}\left(\frac{\omega}{k}\right)^2 [(1 - iK_{11}^{\beta\beta}) K_{\alpha\alpha} \\ & - (3 - iK_{\alpha\alpha}^{\beta\beta}) K_{11}]\Big[P + i\left(\frac{\omega}{k}\right)^2 \frac{k_\alpha S_\alpha}{\omega} + \left(\frac{\omega_0}{\omega}\right)^2 \\ & - i\frac{1}{3}\sqrt{\frac{2}{\pi}}\frac{\nu_{ei}(T)}{\omega}\left(1 + \frac{m}{M}\right)\Big]\Big\}. \end{aligned} \tag{14}$$

Here we have introduced the designations

$$K_{11} = k^{-2}k_\alpha k_\beta K_{\alpha\beta}, \quad K_{11}^{\alpha\alpha} = k^{-2}k_\mu k_\nu K_{\mu\nu}^{\alpha\alpha},$$

$$\begin{aligned} G(\omega, \mathbf{k}) = {} & (1 - iK_{11}^{\alpha\alpha})\left[1 - i\left(\frac{\omega}{k}\right)^2 \frac{m}{T} K_{\alpha\alpha}\right] \\ & - (3 - iK_{\alpha\alpha}^{\mu\mu})\left[1 - i\left(\frac{\omega}{k}\right)^2 \frac{m}{T} K_{11}\right]. \end{aligned}$$

Let us take into account the fact that in the examined hydrodynamic approximation the parameters $\nu_{ei}(T)/\omega$, $\nu_{ee}(T)/\omega$, and $(k/\omega)\sqrt{T/m}$ are small. Then performing the corresponding expansions in Eq. (14) and retaining terms up to the second order in each of the above mentioned parameters, we find that for $m/M \sim 0$,

$$\begin{aligned} \operatorname{Im} \varepsilon(\omega, \mathbf{k}) = {} & \operatorname{Im} \varepsilon_0(\omega, \mathbf{k}) + \frac{1}{3}\sqrt{\frac{2}{\pi}}\frac{\nu_{ei}(T)}{\omega} \\ & + \frac{2}{5}\sqrt{\frac{2}{\pi}}\left[1 + \frac{4}{3}\left(\frac{\omega_0}{\omega}\right)^2\right]\frac{T}{m}\left(\frac{k}{\omega}\right)^2 \frac{\nu_{ei}(T)}{\omega} \\ & + \frac{1}{6\sqrt{\pi}}\left[1 + \left(\frac{\omega_0}{\omega}\right)^2\right]\frac{T}{m}\left(\frac{k}{\omega}\right)^2 \frac{\nu_{ee}(T)}{\omega}, \end{aligned} \tag{15}$$

$$\begin{aligned} \operatorname{Re} \varepsilon(\omega, \mathbf{k}) \sim {} & \operatorname{Re} \varepsilon_0(\omega, \mathbf{k}) + \frac{8}{15\pi}\left[\frac{4}{15} + \left(\frac{\omega_0}{\omega}\right)^2\right]\frac{T}{m}\left(\frac{k}{\omega}\right)^2 \left(\frac{\nu_{ei}}{\omega}\right)^2 \\ & + \frac{1}{36\pi}\left[1 + 32\left(\frac{\omega_0}{\omega}\right)^2\right]\frac{T}{m}\left(\frac{k}{\omega}\right)^2 \left(\frac{\nu_{ee}}{\omega}\right)^2 \\ & + \frac{\sqrt{2}}{90\pi}\left[121 + 2\left(\frac{\omega_0}{\omega}\right)^2\right]\frac{T}{m}\left(\frac{k}{\omega}\right)^2 \left(\frac{\sqrt{\nu_{ei}\nu_{ee}}}{\omega}\right)^2. \end{aligned} \tag{16}$$

In this manner we find that the expansion of the imaginary part of the dielectric constant arises from the odd powers of the small parameters ν_{ei}/ω and ν_{ee}/ω, while the expansion of the real part contains the even powers of these parameters. Expressions (15) and (16) for the dielectric constant show that when spatial dispersion is taken into account not only electron-ion but electron-electron collisions are important. Further, we see that, when spatial dispersion is taken into account, there arise in Eq. (16) for the real part of the dielectric constant terms that are quadratic in the parameters ν_{ei}/ω and ν_{ee}/ω. The latter determine the slight **shift** of the characteristic electron oscillation frequencies due to the presence of collision damping.

In conclusion we wish to point out that the hydrodynamic approximation examined above is essentially the kind of generalization of the method of moments of the distribution function in which the moments with respect to the collision integral take into account the effect of the free movement of the particles on the pair collision process.

[1] V. P. Silin and A. A. Rukhadze, Elektronnye svoĭstva plazmy i plazmopodobnykh sred (Electronic Properties of Plasma and Plasma-like Media), Atomizdat, 1961.

[2] S. I. Braginskiĭ, JETP **33**, 459 (1957), Soviet Phys. JETP **6**, 358 (1958).

[3] B. A. Trubnikov, JETP **34**, 1341 (1958), Soviet Phys. JETP **7**, 926 (1958).

Translated by A. Skumanich
29