FIELD THEORY WITH NONLOCAL INTERACTION
II. THE DYNAMICAL APPARATUS OF THE THEORY

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The possibility is ascertained of a space-time (and in particular of a Hamiltonian) description of a system of fields which interact with each other in a nonlocal way. The basis taken for the dynamical apparatus of the theory is the renormalized Heisenberg field equations, modified in such a way that they automatically lead to a unitary scattering matrix. For this purpose use is made of the previously introduced[1] representation of the S matrix in the form of an exponential ordered with respect to the charge. The forms of the energy-momentum and charge operators are found, and also the forms of the field operators in the Schrödinger and interaction representations. It is shown that the nonlocal field theory does not give rise to any difficulties with negative energy for any choice of the form-factor.

1. INTRODUCTION

IN the preceding paper of this series[1] (hereafter cited as I) an expression was proposed for the scattering matrix of a nonlocal field theory (NFT) with a "hard" form-factor, which satisfies all the necessary requirements of unitarity and relativistic invariance and goes over in the local limit into the usual expression for the S matrix.

This expression could in principle be made the basis of the formal scattering apparatus in NFT, if we prescribed the recipe for getting the observable quantities from it. The question arises, however, whether it is possible to give a more detailed space-time description of a system of fields coupled to each other in a nonlocal manner. An affirmative answer to this question would make it possible to apply in NFT the well developed methods of the usual field theory, which would decidedly facilitate both the elucidation of general questions (in particular, problems of causality) and the construction of a compact and practically convenient formulation of NFT.

As was emphasized in I, the problem of the mathematical compatibility of the nonlocal Heisenberg field equations can always be solved in the affirmative sense: to any Lagrangian there corresponds a Hamiltonian which automatically satisfies the Bloch condition.[1)] Therefore in the construction of the dynamical apparatus of NFT the most suitable starting point are the Heisenberg field equations.

As in I, we here consider the example of a pseudoscalar neutral meson theory with pseudoscalar coupling. The treatment is conducted on the assumption that there are no bound states.

2. THE RENORMALIZATION OF THE NONLOCAL FIELD EQUATIONS

Although in NFT there is a hope of completely avoiding divergences (one of our subsequent papers will be devoted to this question), nevertheless from the very beginning we must carry out a renormalization of the field equations which leads to the elimination of unobservable quantities from the apparatus of the theory. The point is that the condition for the correctness of a number of the relations presented below is the equality of the masses of the particles which correspond to the operators for the Heisenberg fields and for the ingoing fields; from the point of view of the causality condition it is particularly essential to deal with the renormalized field equations.

[1)]There is, in addition, a wide class of NFT's whose Hamiltonians do not satisfy this condition. A particular case of such a Hamiltonian is

$$H = g\int d1'd1''d1'''F(1', 1'', 1''')\bar{\psi}(1')\gamma_5\varphi(1''')\psi(1''),$$

which has been the object of many investigations. No Lagrangian can be found to correspond to such a Hamiltonian, since the condition for the existence of a transition operator in the Heisenberg representation is precisely the Bloch condition. Theories with this sort of Hamiltonian are obviously to be excluded from the very beginning.

It is most convenient to make the renormalization by following the method of Gupta,[2] i.e., by transferring some of the terms (counterterms) from the free Lagrangian to the interaction Lagrangian. A convenience of this method is that it gives a closed expression for the renormalized interaction Lagrangian. We first renormalize the field operators, making the substitution

$$\varphi \to Z_3^{1/2}\varphi, \qquad \psi \to Z_2^{1/2}\psi. \tag{1}$$

Here and in what follows quantities in the Heisenberg representation are denoted by bold-face letters. The total Lagrangian[2)] in the new variables is of the form

$$\mathbf{L}_t = Z_2\mathbf{L}_f(M_0) + Z_3\,\mathbf{L}_b(\mu_0^2) + Z_2 Z_3^{1/2}\mathbf{L}(g_0),$$
$$\mathbf{L}_f = {}^1/_2\int d1\,[\bar{\psi}(1), (i\hat{\nabla} - M_0)\,\psi(1)],$$
$$\mathbf{L}_b = {}^1/_2\int d1\,\varphi(1)\,(\Box - \mu_0^2)\,\varphi(1),$$
$$\mathbf{L} = -\frac{g_0}{2}\int d1'\,d1''\,d1'''\,F(1', 1'', 1''')\,(\gamma_5)_{\alpha\beta}$$
$$\times\,[\bar{\psi}_\alpha(1')\,\varphi(1''')\,\psi_\beta(1'') - \psi_\beta(1'')\,\varphi(1''')\,\bar{\psi}_\alpha(1')], \tag{2}$$

M_0, μ_0, g_0 are the bare values of the masses and charge, Z is the renormalization constant, and F is the form-factor.

To eliminate the bare quantities from the theory we break up the expression (2) into a free Lagrangian $\mathbf{L}_0$ and an interaction Lagrangian $\mathbf{L}'$ in such a way that $\mathbf{L}_0$ involves only the true mass values $M = M_0 - \delta M$ and $\mu^2 = \mu_0^2 - \delta\mu^2$:

$$\mathbf{L}_0 = \mathbf{L}_f(M) + \mathbf{L}_b(\mu^2), \tag{3}$$

$$\mathbf{L}' = Z_1\mathbf{L}(g) + (Z_2 - 1)\,\mathbf{L}_f(M + \delta M) + \mathbf{L}_f(M + \delta M)$$
$$- \mathbf{L}_f(M) + (Z_3 - 1)\,\mathbf{L}_b(\mu^2 + \delta\mu^2)$$
$$+ \mathbf{L}_b(\mu^2 + \delta\mu^2) - \mathbf{L}_b(\mu^2). \tag{4}$$

Here $g = Z_1^{-1}Z_2Z_3^{1/2}g_0$ is the true value of the charge. We still have to add to the expression (4) a counterterm $\lambda\int d1\,[\varphi(1)]^4$, which describes the renormalization of the scattering of mesons by mesons.

In a local renormalizable field theory the constants Z, δM, $\delta\mu^2$ can be chosen in such a way that there is complete cancellation of all divergences; the values of these constants are then infinite. If the introduction of the form-factor is effective enough, then already in the unrenormalized NFT all of the integrals are "cut off" at momenta of the order Λ (the "cut-off" momentum involved in the form-factor). The constants Z, δM, and $\delta\mu^2$, which are now finite, can then be chosen by the condition that the terms of the integrals which increase with Λ are to cancel; this means that the integrals are "cut off" in the renormalized theory at momenta of the order of the masses of the particles. This fact is important from the point of view of the causality condition.

What has been said applies only to the integrals which appear in the "complete" S matrix $S = S(\infty)$. To eliminate the terms which increase with Λ from the S matrix $S(\sigma)$, which corresponds to a finite surface σ, we would have to choose the counterterms in Eq. (4) in a more complicated way, which would make them depend explicitly on σ. Together with their dependence on the gradients of the field operators this causes the appearance of peculiar "surface" divergences,[3] associated with the presence in the expression for $S(\sigma)$ of products of derivatives of the function $\theta(\sigma, 1)$[3)] taken at the same point 1.

We can obviously accomplish the liquidation of the surface divergences by introducing into the counterterms special form-factors which "shift apart" the arguments of the corresponding field operators. For example, when we write the quantity $\mathbf{L}_f(M)$ in Eq. (4) in the nonlocal form

$$\mathbf{L}_f(M) = \int d1'd1''\,\Phi(1', 1'')\,[\bar{\psi}(1'), \psi(1'')],$$
$$\Phi(1', 1'') = (i\hat{\nabla} - M)\,\delta(1' - 1''), \tag{5}$$

we must replace the quasilocal quantity Φ by a form-factor of general form, similar to the form-factor which occurs in the first term of Eq. (4), but in general quite unconnected with it. We shall not go into this matter, but shall choose the counterterms in the form (4) and assume that the integrals occurring in the S matrix $S(\sigma)$ converge in the NFT.

3. THE HEISENBERG FIELD EQUATIONS

Variation of the Lagrangian $\mathbf{L}_t$ with respect to φ, ψ, and $\bar{\psi}$ gives

$$(\Box - \mu^2)\,\varphi(1) = -\,\delta\mathbf{L}_t/\delta\varphi(1) \equiv \mathbf{j}(1),$$

where

$$\mathbf{j}(1) = -\frac{Z_1 g}{2}\int d1'\,d1''\,F(1', 1'', 1)\,[\bar{\psi}(1'), \gamma_5\psi(1'')]$$
$$+ (Z_3 - 1)\,(\Box - \mu^2 - \delta\mu^2)\,\varphi(1) - \delta\mu^2\varphi(1).$$

For brevity we shall omit in what follows the re-

[2)]We are actually concerned with the time integral of the Lagrangian, i.e., with the action.

[3)]The function $\theta(\sigma, 1)$ is equal to unity when the point 1 lies in the past from the surface σ, and is zero otherwise. This definition has an invariant meaning if the surface σ is spacelike.

lations corresponding to the fields ψ and $\bar{\psi}$. We reduce the equation we have obtained to integral form[4)]

$$\boldsymbol{\varphi}(1) = \varphi(1) + \int d2\,\theta\,(1-2)\,D\,(1-2)\,\mathbf{j}\,(2), \tag{6}$$

where $\varphi(1) = \varphi_{in}(1)$ is a solution of the equation $(\Box - \mu^2)\varphi = 0$ which satisfies the commutation rule $[\varphi(1), \varphi(2)] = iD(1-2)$.

The equation for the outgoing-wave operator is of the form

$$\varphi_{out}(1) = \varphi(1) + \int d2D\,(1-2)\,\mathbf{j}\,(2). \tag{7}$$

The connection between the operators φ_{out} and φ is given by a unitary matrix

$$\varphi_{out}(1) = S^+\varphi(1)\,S \tag{8}$$

only if a certain condition which the operator $\mathbf{j}$ must obey is satisfied. Writing Eq. (8) in the form $\varphi(1) + S^+[\varphi(1), S]$ and comparing it with Eq. (7), we easily see that this condition is that it be possible to represent $\mathbf{j}$ in the form

$$\mathbf{j}(1) = iS^+\,\delta S/\delta\varphi(1), \quad \mathbf{j}^+ = \mathbf{j}. \tag{9}$$

Varying this relation with respect to φ and using the symmetry of $\delta^2 S/\delta\varphi(1)\,\delta\varphi(2)$ under the interchange $1 \rightleftarrows 2$, we find finally

$$[\mathbf{j}\,(1), \mathbf{j}\,(2)] = i\,\{\delta\mathbf{j}\,(2)/\delta\varphi\,(1) - \delta\mathbf{j}\,(1)/\delta\varphi\,(2)\}. \tag{10}$$

We shall not write out the analogous conditions for the fermion currents $\mathbf{J} = iS^+\delta S/\delta\psi$ and the mixed condition for the currents $\mathbf{j}$ and $\mathbf{J}$.

These conditions are by no means always satisfied in a NFT. In particular it is known that they are violated in a theory with a charge-unsymmetrical Lagrangian. The question as to whether they hold in the NFT with the Lagrangian (4) is an open one (in this connection see I).

Therefore it is expedient to make a reformulation of the Heisenberg field equations such that the condition (10), and along with it Eq. (8), will be satisfied automatically. For this purpose we must base our work on some clearly unitary expression for the S matrix which, being completely relativistically invariant, would go over in the local limit into the usual expression for the S matrix. Then, using the relation (9), we can determine the "corrected" expression for the current

$$\tilde{\mathbf{j}}(1) = iS^+\,\delta S/\delta\varphi\,(1) \tag{9'}$$

[4)]Strictly speaking the choice of the free term here [and also in Eqs. (7) and (6′)] in the form of simply $\varphi(1)$ requires justification, since in going over to the renormalized equations we have had to introduce factors $Z_3^{1/2}$, and so on [cf. Eq. (1)]. On this matter see Appendix I.

and the "corrected" field equation

$$\tilde{\boldsymbol{\varphi}}(1) = \varphi(1) + \int d2\theta\,(1-2)\,D\,(1-2)\,\tilde{\mathbf{j}}\,(2). \tag{6'}$$

The new outgoing-wave operator

$$\tilde{\varphi}_{out}(1) = \varphi(1) + \int d2D\,(1-2)\,\tilde{\mathbf{j}}\,(2)$$

is connected with $\varphi(1)$ by the relation (8).

It is convenient to choose the starting expression for the S matrix in the form of an exponential ordered with respect to charge (see I). In our present case, however, unlike the unrenormalized theory considered in I, in which the quantity L obtained from **L** by the replacements $\boldsymbol{\varphi} \rightarrow \varphi$, and so on, depends linearly on the charge, this dependence is now more complicated. Therefore we must introduce a special index λ, in terms of which the ordering will be done.

For this purpose we replace the expression (4) for **L** by λ**L**. The solutions of the Heisenberg field equations

$$\boldsymbol{\varphi}(1) = \varphi(1) + \lambda \int d2\,\theta\,(1-2)\,D(1-2)\mathbf{j}(2)$$

and so on will then be functionals of φ, ψ, and $\bar{\psi}$, and functions of λ. Substitution of these solutions in the expression (4) makes it into a functional $\mathbf{L}(\varphi, \psi, \bar{\psi}, \lambda)$; then the desired expression for the S matrix takes the form[5)]

$$S = \tilde{T}_\lambda \exp\left\{i\int_0^1 d\lambda\mathbf{L}\right\}, \tag{11}$$

where the symbol $\tilde{T}_\lambda$ means the exponential ordered so that the values of λ increase from left to right.

The unitarity of the expression (11) follows directly from the Hermiticity of **L**. The relativistic invariance follows from the analogous properties of the solutions of the field equations; in particular, the functions $\theta(x)$ and so on are always accompanied by functions which vanish outside the light cone.

Finally, let us verify that in the local limit, when $\mathbf{L} \rightarrow \mathbf{L}_{loc}$, Eq. (11) goes over into the usual expression for the S matrix. In fact, it follows from Eq. (11) that

$$\partial S/\partial\lambda = iS\mathbf{L} \rightarrow iS\mathbf{L}_{loc} = iT\,(L_{loc}\,S),$$

where L_{loc} is the expression into which the quantity L defined above goes over in the local limit. Integration with respect to λ gives the well known expression (cf. [3,5])

[5)]In the analogous expression in I we had instead of **L** a sum of retarded commutators which is equal to **L**.

$$S = T \exp (i\lambda L_{\text{loc}}).$$

Thus the procedure for constructing the equation (6′) is as follows. First one solves the starting equations (6) with **L** replaced by λ**L**. From the solutions one forms the quantity $\mathbf{L}(\varphi, \psi, \bar{\psi}, \lambda)$ and constructs the S matrix (11). Finally, the equations (6′) are obtained by means of Eq. (9′). We note that even if the condition (10) was satisfied in the original scheme, this in general still does not guarantee equality of the quantities $\tilde{\mathbf{j}}$ and **j**, which can differ by terms which contain cyclic products of retarded (or advanced) commutators and thus vanish in the local limit (in this connection see [6]).[6)]

This is a sufficiently general procedure, and can be useful in cases in which the question of the existence of a unitary S matrix is either unclear, or is known to have the negative answer.

4. THE ENERGY-MOMENTUM TENSOR

The dynamical apparatus of NFT has as its basis the field equations (6′) obtained above; its development requires the introduction of a matrix $S(\sigma)$ which corresponds to a finite surface σ and goes over into the expression (11) for $\sigma \to \infty$. By analogy with this expression and from the requirement of correspondence with the local theory we can write (see footnote [3)])

$$S(\sigma) = \tilde{T}_\lambda \exp \left\{ i \int_0^1 d\lambda \int d1\, \theta(\sigma, 1)\, \mathcal{L}_H(1) \right\}, \qquad (12)$$

where $\mathcal{L}_H$ is the interaction Lagrangian density in the Heisenberg representation.

As was already emphasized in I, the connection between the operators $\tilde{\varphi}$ and φ is by no means a unitary one,[7)] but is of the form

$$\tilde{\varphi}(1) = S^+(\sigma)\, \varphi(1)\, S(\sigma) + \chi(1/\sigma), \qquad (13)$$

where the point 1 lies on σ and the operator χ, which vanishes in the local limit, can be written in the form

$$\chi(1/\sigma) = -i \int d2 D(1-2) \left[S^+(\sigma) \frac{\delta S(\sigma)}{\delta \varphi(2)} - \theta(1-2) S^+ \frac{\delta S}{\delta \varphi(2)} \right].$$

This operator makes the two operators $\tilde{\varphi}$ and φ simultaneously independent of σ. Because it is not zero we have the inequality

$$[\tilde{\varphi}(1), \tilde{\varphi}(2)] \neq 0 \qquad (14)$$

outside the light cone, which is a reflection of the acausality of NFT. Questions relating to this will be treated in one of the following papers.

From now on, wherever we do not state otherwise, we shall use a flat surface σ, writing $S(\sigma)$ in the form $S(t)$. Let us consider the in-operator $\varphi(1)$, in terms of which all of the quantities in our apparatus can be expressed. This operator is by no means the same as the free operator

$$\varphi_0(1) = e^{iH_0 t} \varphi_0(\mathbf{x}_1)\, e^{-iH_0 t},$$

$$\varphi_0(\mathbf{x}_1) = \sum_k (2V\omega_k)^{-1/2} (a_k e^{i\mathbf{k}\mathbf{x}_1} + \text{adjoint}), \qquad (15)$$

but is connected with it by a transformation independent of the time[8]

$$\varphi(1) = S(0)\varphi_0(1)\, S^+(0).$$

The evolution in time of the operator $\varphi(1)$

$$\partial\varphi(1)/\partial t_1 = i[\varphi(1), H_{0\,(in)}] \qquad (16)$$

is determined by the free Hamiltonian H_0, in which the operators φ_0 are replaced by φ, i.e.,

$$H_{0\,(in)} = S(0)\, H_0 S^+(0).$$

Let us go on to the determination of the energy-momentum operator in NFT, $\mathbf{P}_\mu = (\mathbf{H}, \mathbf{P}_i)$, for which a number of extremely complicated expressions have been given in the literature. It is actually not hard to show that on the assumption that there are no bound states we have

$$\mathbf{P}_\mu = P_{0\,(in)\,\mu}, \qquad (17)$$

i.e., the desired operator is obtained from the free operator by the replacements $\varphi_0 \to \varphi$, and so on. In particular,

$$\partial\tilde{\varphi}(1)/\partial t_1 = i\,[\tilde{\varphi}(1), H_{0\,(in)}]. \qquad (18)$$

In fact, let us use the expansion

$$\tilde{\varphi}(1) = \sum_{n,m} \int d2 \ldots d(n+2m+1)\, \Phi(1, \ldots n+2m+1)$$
$$\times \varphi(2) \ldots \varphi(n+1)\, \bar{\psi}(n+2) \ldots \psi(n+2m+1),$$

where the function Φ, which is a combination of form-factors and functions D^R, S^R, and so on (see I), depends only on the differences of its arguments. When we use Eq. (16) and integrate by

[6)]Another question which is not clear is that of the existence of a "corrected" Lagrangian $\tilde{\mathbf{L}}$ for which $\tilde{\mathbf{j}} = -\delta\tilde{\mathbf{L}}/\delta\tilde{\varphi}$, $\tilde{\mathbf{J}} = -\delta\tilde{\mathbf{L}}/\delta\tilde{\psi}$, and so on. In principle there may be no such operator in NFT; this, of course, is no objection to the theory.

[7)]We point out that because of the presence of higher derivatives in the counterterms a local renormalized field theory is closely similar to a NFT which has an extremely complicated Hamiltonian[4] and leads to a nonzero value of χ. The resultant nonunitarity of the connection between $\tilde{\varphi}$ and φ has been noted in the literature (cf. [7]).

parts, we can easily reduce the commutator $i[\tilde{\varphi}(1), H_{0(in)}]$ to $\partial\tilde{\varphi}/\partial t_1$. and thus prove Eq. (18).[8] The other equations in Eq. (17) can be proved analogously, and so also can the equation $Q = Q_{0(in)}$ for the charge operator.

Introducing the eigenfunctions of the operator $P_\mu[(P_\mu - P_\mu)\Psi_p = 0]$, we can easily show that

$$\langle \Psi_p | \tilde{\varphi}(x) | \Psi_{p'} \rangle = \exp(i(P - P')x) \langle \Psi_p | \tilde{\varphi}(0) | \Psi_{p'} \rangle.$$

As in the local theory, the wave function Ψ_0 of the vacuum state corresponds to the lowest (zero) value of P_μ. The wave functions of excited states of the field, in particular the function Ψ_k of a one-particle state, are constructed in the usual way by the action of in operators on the vacuum.

On our assumptions the energy spectrum of the system is the same as in the absence of the interaction, i.e., $P_\mu^2 > 0$, $P_0 > 0$. It can be shown that up to a phase factor

$$S\Psi_0 = \Psi_0.$$

The analogous condition of stability of a one-particle state,

$$S\Psi_k = \Psi_k,$$

which is equivalent to the conditions $\langle \Psi_0 | \tilde{j} | \Psi_k \rangle = 0$ and $\langle \Psi_0 | \partial\tilde{j}/\partial\varphi | \Psi_0 \rangle = 0$ (on the mass shell), is a consequence of the renormalization (see Appendix I).

In concluding this section let us briefly consider the one-particle Green's function of the boson field (a full treatment of the Green's functions will be the subject of a special investigation):

$$iD'_F(1-2) = \langle \Psi_0 | T(\tilde{\varphi}(1), \tilde{\varphi}(2)) | \Psi_0 \rangle.$$

This expression is relativistically invariant in form, since for the vacuum average [but not for the operators, see Eq. (14)] we have the equation[9]

$$iD'(1-2) = \langle \Psi_0 | [\tilde{\varphi}(1), \tilde{\varphi}(2)] | \Psi_0 \rangle = 0 \qquad (14')$$

outside the light cone.

For the proof we need only note that the left member of Eq. (14′) is an odd function of the difference $1-2$ and therefore must be proportional to the sign function $\epsilon(1-2)$ (there are no other odd functions at our disposal). Because of the relativistic invariance of the field equations for $\tilde{\varphi}$, this function must appear in combination with a function which vanishes outside the light cone.

We can construct the Källen-Lehmann representation for the function D'_F in the usual way:

$$D'_F(x) = D_F(x) + \int d\kappa^2 \rho(\kappa^2) D_F(x, \kappa^2),$$

where $\rho(\kappa^2) \geq 0$. The possibility of such a representation for NFT has already been pointed out by Lehmann.[9] We must note, however, that actually there is no need of a special assumption that in the intermediate one-meson states $P_\mu^2 > 0$. If this were not so there would be a violation of the condition (14), since the commutator function with an imaginary mass does not vanish outside the light cone.

5. THE INTERACTION AND SCHRÖDINGER REPRESENTATIONS

The apparatus of NFT can also be developed in the interaction and Schrödinger representations. Let us temporarily introduce an arbitrary surface σ and assume that the wave function $\Psi_{int}(\sigma)$ in the interaction representation evolves according to the law

$$\Psi_{int}(\sigma) = S(\sigma)\Psi_{int}(-\infty), \qquad (19)$$

where $S(\sigma)$ is given by the expression (12). We thus arrive at the usual Tomonaga-Schwinger equation:

$$i\delta\Psi_{int}(\sigma)/\delta\sigma(1) = \mathcal{H}(1/\sigma)\Psi_{int}(\sigma), \qquad (20)$$

where $\mathcal{H}(1/\sigma) = [i\delta S(\sigma)/\delta\sigma(1)]S^+(\sigma)$, or, as can be shown without difficulty,

$$\mathcal{H}(1/\sigma) = -\int_0^1 d\lambda S(\sigma)\mathcal{L}_H(1)S^+(\sigma). \qquad (21)$$

In local theory, where the function χ in Eq. (13) is equal to zero, we would simply get $\mathcal{H} = -\mathcal{L}$. In NFT the operator $\mathcal{H}$ has an extremely complicated form.

The expression (21) satisfies the Bloch condition simply because of the very fact that $S(\sigma)$ exists (cf. I). By definition we can write

$$S(\sigma) = T_\sigma \exp\left[-i\int d1\, \theta(\sigma, 1)\mathcal{H}(1/\sigma_1)\right], \qquad (22)$$

where the symbol T_σ corresponds to ordering with respect to the index of the surface σ_1 which passes through the point 1. According to the definition (12) $S(\sigma)$ does not depend on the choice of the shape of these intermediate surfaces, as can also be directly verified by means of the Bloch condition.

Returning to the flat surface σ and making the usual requirement that the quantities in the various representations coincide for $t = 0$, we find $\Psi = S(0)\Psi_{int}(-\infty)$, from which we have

[8]A direct proof of the identity of the operators H and $H_{0(in)}$ is contained in Appendix II.

[9]This important fact has been emphasized previously by A. D. Galanin.

$$\Psi_{\text{int}}(t) = S(t, 0)\,\mathbf{\Psi}, \quad \varphi_{\text{int}}(1) = S(t_1, 0)\,\tilde{\varphi}(1)\,S^+(t_1, 0), \tag{23}$$

where $S(t, 0) = S(t)S^+(0)$. An important point is that the operator $\varphi_{\text{int}}(1)$ of the interaction representation by no means coincides with the free operator $\varphi_0(1)$ [cf. Ef. (15)]. If they did coincide the inequality (14) would be violated.

By using Eqs. (18) and (20) we can write the Hamiltonian which determines the evolution of $\varphi_{\text{int}}(1)$ in the form

$$H_{\text{int}} = S(t, 0)\,H_{0\,(in)}\,S^+(t, 0) - H'(t),$$

where $H'(t)$ is the interaction Hamiltonian which corresponds to $\mathcal{H}(1/\sigma)$ for the case of a plane surface. Using the results of Appendix II we have

$$H_{\text{int}} = S(t)\,H_0\,S^+(t) - H'(t) = H_0.$$

Thus we have

$$\varphi_{\text{int}}(1) = e^{iH_0t_1}\varphi_S(\mathbf{x}_1)\,e^{-iH_0t_1}. \tag{24}$$

Here φ_S is the field operator in the Schrödinger representation, which does not depend on the time, but in NFT is not the same as the free operator $\varphi_0(\mathbf{x}_1)$. It can be shown in the usual way that

$$S(t, 0) = e^{iH_0t}e^{-i\mathbf{H}t},$$

from which, using Eqs. (23) and (24), we find

$$\tilde{\varphi}(1) = e^{i\mathbf{H}t_1}\varphi_S(\mathbf{x}_1)\,e^{-i\mathbf{H}t_1}. \tag{25}$$

Thus in NFT the connection between the representations is made in just the same way as in local theory. The only difference is that the operators φ_S and φ_{int} do not coincide with the free operators. We can find the connection between them by choosing in Eq. (13) the plane surface σ which corresponds to $t_1 = 0$, setting $t_1 = 0$, and using the relation between φ and φ_0 (see Section 4). This gives

$$\varphi_S(\mathbf{x}_1) = \varphi_0(\mathbf{x}_1) + \chi|_{t_1=0}. \tag{26}$$

6. THE PROBLEM OF THE INITIAL CONDITIONS

In conclusion we consider a number of questions relating to the statement of the initial-value problem in NFT (cf. [10,11]).

First there is the question as to whether in NFT one must really prescribe a larger number of initial conditions than in local theory. The answer to this question is usually related to the shape of the form-factor, and is taken to be negative if the Fourier transform of the form-factor contains no poles. In the opposite case it is supposed to be necessary to introduce additional degrees of freedom of the field, which correspond to the poles of the form-factor and which lead to well known difficulties with negative energy, indefinite metric, and so on.[11]

It can be shown, however, that in ordinary acausal NFT, independently of the form of the form-factor, there do not have to be any additional degrees of freedom; accordingly there is no need for additional initial conditions. Speaking more exactly, the NFT and the theory with additional degrees of freedom are two mutually exclusive theories, which correspond to different choices of the solutions of the field equations.

This assertion can be most simply illustrated with the example of an ordinary local field theory:

$$(i\hat{\nabla} - M - g\gamma_5\varphi(1))\,\psi(1) = 0,$$
$$(\Box - \mu^2)\,\varphi(1) = g\bar{\psi}(1)\,\gamma_5\,\psi(1) = \mathrm{j}(1).$$

The second of these equations can be written in the form

$$\varphi(1) = \varphi(1) + \int d2\,\theta\,(1-2)\,D\,(1-2)\,\mathrm{j}\,(2),$$

where $\varphi(1)$ is an arbitrary solution of the Klein-Gordon equation. Setting

$$\varphi(1) = 0, \tag{27}$$

we arrive at a typical nonlocal theory of a single field ψ:

$$\left(i\hat{\nabla} - M - g\gamma_5\int d2\,F\,(1-2)\,\mathrm{j}\,(2)\right)\psi(1) = 0$$

with the form-factor $F = \theta D$. It is easy to show that the solutions of this equation have an anticommutator which is different from zero outside the light cone. The usual choice, making $\varphi(1)$ obey the condition

$$[\varphi(1),\ \varphi(2)] = iD\,(1-2) \tag{28}$$

immediately returns us to the local theory, leading to the appearance of an additional degree of freedom of the field (particles with the mass μ) and owing to this assuring the correct causal properties of the theory.

An analogous dilemma—either additional degrees of freedom, or acausality—occurs also in the general case. Thus for the equation

$$(\Box - \mu^2)(\Box - \mu'^2)\,\varphi = \mathrm{j}(\varphi)$$

or

$$(\Box - \mu'^2)\,\varphi(1) = \varphi(1) + \int d2\theta\,(1-2)\,D\,(1-2)\,\mathrm{j}\,(2)$$

the choice (27) leads to a nonlocal (and therefore noncausal) theory which does not suffer from difficulties with negative energy, whereas subjecting

φ to the condition (28) gives a theory with additional degrees of freedom which in return is free from difficulties with causality.[10] It is obvious that the choice of solution made above [see Eqs. (6), (7), (6′)] corresponds to a genuine NFT free from difficulties with negative energy and not requiring the use of additional initial conditions with any choice of the form-factor. Some restrictions on the nature of the singularities of this function do follow, by the way, from the condition of macrocausality.

The next question is, in just what way must one prescribe the initial state of a system in NFT? From an examination of the relations (19)—(22) it follows at once that in this respect the situation in NFT is the same as in local theory. The initial state of the system can be fixed by giving an arbitrary function $\Psi_{int}(\sigma_0)$. The subsequent evolution of the system, which can be followed from surface to surface, is determined by the matrix $S(\sigma, \sigma_0) = S(\sigma) S^+(\sigma_0)$. As in local theory, the prescription of the function $\Psi_{int}(\sigma_0)$ is on one hand arbitrary, and on the other hand uniquely determines the evolution of the system. This is evidence that it is possible to have a Hamiltonian description of a system of fields in NFT.

It must be specially emphasized that this possibility is not limited at all by the acausal nature of NFT. As is clear from the relevant discussion,[3] the causality condition restricts neither the form of the function Ψ_{int} nor the determinism of the usual description, but does restrict the form of the scattering matrix $S(\sigma)$ [or, what is the same thing, the Hamiltonian $\mathcal{H}(1/\sigma)$ [11]]. The causality condition for this latter quantity is

$$\frac{\delta}{\delta\sigma(2)}\left(\frac{\delta S(\sigma)}{\delta\sigma(1)} S^+(\sigma)\right) \sim \frac{\delta\mathcal{H}(1/\sigma)}{\delta\sigma(2)} = 0, \tag{29}$$

and is indeed violated in NFT. In this connection we point out the obvious difference between the mathematical condition for consistency

$$[\mathcal{H}(1/\sigma), \mathcal{H}(2/\sigma)] = i\left\{\frac{\delta\mathcal{H}(2/\sigma)}{\delta\sigma(1)} - \frac{\delta\mathcal{H}(1/\sigma)}{\delta\sigma(2)}\right\}$$

and the causality condition which follows from it and Eq. (29),

$$[\mathcal{H}(1/\sigma), \quad \mathcal{H}(2/\sigma)] = 0.$$

The fact that these conditions are not the same is of fundamental importance in NFT (cf. I).

A closely related question is that of the existence of a system of canonical variables in NFT.[10,11] Although, according to Eq. (14), we cannot choose as such variables the field operators themselves, nevertheless the desired system of operators exists, being connected with the field operators in a unique, though not unitary, way. In particular we can make the following choice:

$$\varphi(1, \sigma) = S^+(\sigma)\varphi(1)S(\sigma) \tag{30}$$

and so on, where the point 1 lies on σ. The prescription of the operator $\varphi(1, \sigma)$ determines the evolution of the field variable with respect to σ. For $\sigma \to -\infty$ we have to do with the quantity $\varphi(1) = \varphi_{in}(1)$, and for $\sigma \to \infty$, with the quantity $\varphi_{out}(1)$. In particular, a scattering process is described in this way.

The possibility (30) has been considered by Hayashi,[10] who, however, did not succeed in overcoming the difficulties connected with the possibility that there is no unitary correspondence between the operators φ_{out} and φ_{in}. This difficulty is absent in the scheme developed here.

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APPENDIX I

We shall show that choosing the free term in Eq. (6′), as also in Eqs. (6) and (7), in the simple form $\varphi(1)$ (with coefficient unity) actually leads to the correctly normalized renormalized operator $\tilde{\varphi}(1)$. With this choice the condition for stability of the one-particle state, $\langle \Psi_k | S | \Psi_k \rangle = 1$, and the equivalent condition

$$\langle \Psi_0 | \tilde{j} | \Psi_k \rangle = 0 \tag{A.I.1}$$

lead, when we use Eq. (6′), to the equation $\langle \Psi_0 | \tilde{\varphi} | \Psi_k \rangle = \langle \Psi_0 | \varphi | \Psi_k \rangle$. This assures that the expression for the spectral density of the Lehmann representation will coincide on the mass shell with the analogous expression in the absence of interactions, and this leads to the correct normalization

$$(k^2 - \mu^2) D_F'(k)|_{k^2=\mu^2} = 1.$$

Corresponding to this, the relation $k^2 D_F'|_{k^2\to\infty} = Z_3^{-1}$ holds. The fact that the condition (A.I.1) holds can be assured by a suitable choice of the constants Z, δM, $\delta\mu^2$.

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[10] The statement applies to the theory in which j is a local function of φ. Otherwise causality will be violated with this choice also.

[11] The situation in NFT differs from that in local theory in that the quantity $\mathcal{H}(1/\sigma)$ depends on the values of the field operators not only on the surface σ, but also in the whole space.

APPENDIX II

Starting from the relation (22) and using the method of a paper by Gell-Mann and Low,[12] we can write

$$[H_0, S(t)] = -H'(t)\, S(t) + i\delta\lambda\, \partial S(t)/\partial\lambda\,|_{\lambda=1},$$

$$[H_0, S^+(t)] = S^+(t)\, H'(t) + i\delta\lambda\, \partial S^+(t)/\partial\lambda\,|_{\lambda=1}. \quad \text{(A.II.1)}$$

Here δ is a positive infinitesimal quantity which has its origin in a factor $e^{-\delta|t|}$ which assures the meaningfulness of the infinite integrals which occur in $S(t)$; $H' \to \lambda H'$, where λ is set equal to unity after the calculations have been done. From Eq. (A.II.1) it indeed follows that

$$S(t) H_0 S^+(t) = H_0 + H'(t) + \text{const}, \quad \text{(A.II.2)}$$

since for $\delta \to 0$ the quantity $i\delta\lambda(\partial S/\partial\lambda)S^+$ has a non-operator character, being composed only of diagrams for the vacuum-vacuum transition.[14] This constant corresponds to a shift of the vacuum-state energy which is immaterial. It follows from Eq. (A.II.2) that up to a constant

$$H_{0\,(in)} = S(0)\, H_0 S^+(0) = H_0 + H'(0),$$

where $H'(0)$ is identical with the interaction Hamiltonian in the Schrödinger representation.

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