TRANSVERSE AND LONGITUDINAL STATES OF BOSON FIELDS AND DIBARIC PARTICLES

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The physical nature of the restrictions imposed on the solutions of the Klein equation by the Kemmer equations is elucidated. In this connection, second-order equations for bosons seem to be preferable to the Kemmer equations. The extension of Kemmer algebra to the field of matrices possessing inverses is investigated and equations are written down for particles with various masses in longitudinal and transverse field states. The recently proposed wave equations for the intermediate boson in the theory of weak interaction are similar to the equations of the vector part of the $\beta_\mu(\sigma)$ representation.

THE generalization of the concept of four-dimensional transversality and longitudinality to include products of Dirac bispinors and second-rank undors leads to a new physical interpretation of the boson wave equations v + ps and pv + s and allows us to introduce not a relative but an absolute distinction between the orthogonal representations $\beta_\mu^{(+)}$ and $\beta_\mu^{(-)}$ of the Kemmer algebra.

It is possible to introduce reducible representations that depend on the continuous real parameter σ, forming algebras of more general type than Kemmer algebras and going over into the latter in the limit as $\sigma^2 \to 1$. In these $\beta_\mu(\sigma)$ representations there exist $\beta_\mu^{-1}(\sigma)$ and their use in relativistically invariant wave equations yields a theory of boson fields with two mass states without direct introduction of the higher derivatives in the wave equations.[1)]

1. TRANSVERSE AND LONGITUDINAL PROJECTION OPERATORS

In the reducible four-dimensional representations of the Kemmer algebra

$$\beta_\mu^{(\pm)} = {}^1/_2\,(\gamma_\mu \pm \bar{\gamma}_\mu), \tag{1.1}$$

satisfying the orthogonality conditions [1]

$$\beta_\mu^{(\pm)}\beta_\mu^{(\mp)} = \beta_\mu^{(\pm)2} + \beta_\mu^{(\mp)2} - I = 0 \tag{1.2}$$

(without summation), there exists a complete system of covariant projection operators

$$P_T\,(c) = (\beta_\lambda^{(+)}c_\lambda)^2/c_\sigma^2, \qquad P_L\,(c) = (\beta_\lambda^{(-)}c_\lambda)^2/c_\sigma^2, \tag{1.3}$$

where c_μ is an arbitrary non-zero c-vector ($[c_\mu c_\nu] = 0$). The system (1.3) has the usual properties of projection operators:

$$P_T\,(c)\,P_L\,(c) = P_L\,(c)\,P_T\,(c) = P_T\,(c) + P_L\,(c) - I = 0. \tag{1.4}$$

The Larmor transformations [2,3] in the space of the 16-component $\psi(x)$ act on P_T and P_L in the following fashion:

$$\gamma_5 P_T \gamma_5 = \bar{\gamma}_5 P_T \bar{\gamma}_5 = R_5 P_L R_5 = P_L, \tag{1.5}$$

$$\gamma_5 P_L \gamma_5 = \bar{\gamma}_5 P_L \bar{\gamma}_5 = R_5 P_T R_5 = P_T. \tag{1.6}$$

The meaning of these operators consists in separating from the undor $\psi(x)$ the parts that are transverse and longitudinal with respect to the vector c_μ:

$$\psi_T\,(x) = P_T\,(c)\,\psi\,(x), \qquad \psi_L\,(x) = P_L\,(c)\,\psi\,(x). \tag{1.7}$$

The covariant projection operators can be used to resolve the components of the undor $\psi(x)$ along mutually perpendicular directions in four-dimensional space. In the frame of the vectors $t_\mu^{(1)}$, $t_\mu^{(2)}$, $t_\mu^{(3)}$, and n_μ, having the properties

$$t_\lambda^{(i)}t_\lambda^{(k)} - \delta_{ik} = n_\lambda^2 + I = t_\lambda^{(i)}n_\lambda = 0, \tag{1.8}$$

the parts of $\psi(x)$ which are polarized in the $t_\mu^{(i)}$ direction are separated by the operators

$$P_L\,(t^{(i)}) = {}^1/_2\,(I + t_\lambda^{(i)}t_\sigma^{(i)}\gamma_\lambda\bar{\gamma}_\sigma)\,P_T\,(n). \tag{1.9}$$

If one takes for c the propagation vector p_μ, then $P_T(p)$ and $P_L(p)$ will separate, in the Hilbert space of the Klein equation with real mass, the subspaces ψ_T and ψ_L with vectors that are

1)The main results of this work were reported at the Third Uzhgorod Conference on the Theory of Elementary Particles and Quantized Fields, in October 1961.

either transverse or longitudinal to p_μ in the coordinate space[2]. Thus, two types of Kemmer fields are possible, satisfying the Klein equation in vacuum:

a) field transverse to p_μ,

$$((\beta_\lambda^{(+)}\partial_\lambda)^2 - m^2)\psi = 0 \text{ or } (\beta_\lambda^{(+)}\partial_\lambda + m)\psi = \beta_\lambda^{(-)}\partial_\lambda\psi = 0, \tag{1.10}$$

—mixture of vector and pseudoscalar fields;

b) field longitudinal to p_μ,

$$((\beta_\lambda^{(-)}\partial_\lambda)^2 - m^2)\psi = 0 \text{ or } (\beta_\lambda^{(-)}\partial_\lambda + m)\psi = \beta_\lambda^{(+)}\partial_\lambda\psi = 0 \tag{1.11}$$

—mixture of pseudovector and scalar fields.

The proposed new interpretation of the Kemmer equations in reducible representations makes it possible to write down the field equations with different masses in transverse and longitudinal states:

$$(\beta_\lambda(\sigma)\partial_\lambda + m)\psi = 0, \tag{1.12}$$

where

$$\beta_\mu(\sigma) = {}^1/_2(\gamma_\mu + \sigma\bar{\gamma}_\mu) \tag{1.13}$$

is the representation of an algebra containing the Kemmer algebra as a particular case when $\sigma^2 \to 1$.

2. ALGEBRA OF NONORTHOGONAL REPRESENTATIONS AND MASS STATES OF THE FIELD

The relativistic invariance of the wave equations (1.12) is ensured by the relations

$$[\beta_\mu(\sigma')\, s_{\nu\rho}(\sigma)] = -i(\delta_{\mu\nu}\beta_\rho(\sigma'\sigma^2) - \delta_{\mu\rho}\beta_\nu(\sigma'\sigma^2)), \tag{2.1}$$

$$s_{\mu\nu}(\sigma) = -i[\beta_\mu(\sigma)\beta_\nu(\sigma)] = -i{}^1/_4[\gamma_\mu\gamma_\nu] - i{}^1/_4\sigma^2[\bar{\gamma}_\mu\bar{\gamma}_\nu], \tag{2.2}$$

$$[R_\mu\beta_\mu(\sigma)] = \{R_\mu\beta_\nu(\sigma)\} = 0 \text{ (without summing, } \mu \neq \nu), \tag{2.3}$$

where the reflection matrices R_μ are defined in terms of γ_μ, $\bar{\gamma}_\mu$, or $\beta_\mu(\sigma)$:

$$R_\mu = \gamma_\mu\bar{\gamma}_\mu = \sigma^{-1}(2\beta_\mu^2(\sigma) - (1+\sigma^2)/2). \tag{2.4}$$

Thus, (1.12) are relativistically invariant if $\sigma^2 = 1$ in the operator of the Lorentz transformations $S_{\mu\nu}(\sigma)$ of the undor $\psi(x)$; $s_{\mu\nu} = -i[\beta_\mu^{(\pm)}\beta_\nu^{(\pm)}]$ is a universal operator of the transformation of arbitrary boson fields with wave function in the form of a second-rank undor.

The algebra of the nonorthogonal representations is specified by the commutation rules

$$\beta_\mu(\sigma)\beta_\nu(\sigma)\beta_\rho(\sigma) + \beta_\rho(\sigma)\beta_\nu(\sigma)\beta_\mu(\sigma) - {}^1/_2(1+\sigma^2)(\delta_{\mu\nu}\beta_\rho(\sigma) + \delta_{\nu\rho}\beta_\mu(\sigma)) + {}^1/_2(1-\sigma^2)\delta_{\mu\rho}\beta_\nu(-\sigma) = 0; \tag{2.5}$$

the matrices R_μ for $\sigma \neq 1$ will not be generalized units of $\beta_\mu(\sigma)$, since

$$R_\mu\beta_\mu(\sigma) = \sigma\beta_\mu(\sigma^{-1}) \text{ (without summing)}, \tag{2.6}$$

but now there appear in $G(\beta_\mu(\sigma))$ the inverse matrices

$$\beta_\mu^{-1}(\sigma) = 4(1-\sigma^2)^{-1}\beta_\mu(-\sigma), \qquad \beta_\mu^{-1}\beta_\mu = I \text{ (б. c.)}, \tag{2.7}$$

$$\beta_{\mu R}^{-1}(\sigma) = -4\sigma(1-\sigma^2)^{-1}\beta_\mu(-\sigma^{-1}), \qquad \beta_{\mu R}^{-1}\beta_\mu = R_\mu \tag{2.8}$$

with simple commutation rules

$$ {}^1/_2(\beta_\mu^{-1}(\sigma))\beta_\nu(\sigma) + \beta_\nu^{-1}(\sigma)\beta_\mu(\sigma)) = {}^1/_2(\beta_\mu(\sigma)\beta_\nu^{-1}(\sigma) + \beta_\nu(\sigma)\beta_\mu^{-1}(\sigma)) = \delta_{\mu\nu}I. \tag{2.9}$$

Equations (1.12) lead to additional conditions of a more complicated type than usual, owing to the nonorthogonality of the representations $\beta_\mu(\sigma)$ and $\beta_\mu(-\sigma)$:

$$(\beta_\lambda(-\sigma)\partial_\lambda + (4m)^{-1}(1-\sigma^2)\Box^2)\psi = 0. \tag{2.10}$$

Elimination of the matrices from (1.12) and (2.10) yields an equation with two masses

$$(\Box^2 - m_1^2)(\Box^2 - m_2^2)\psi = 0, \qquad m_1 = 2m/(1+\sigma), \quad m_2 = 2m/(1-\sigma), \tag{2.11}$$

and it can be shown that m_1 corresponds to $\psi_T(x)$ and m_2 to the function $\psi_L(x)$. This follows from the fact that $(P_\sigma(c) = (\beta_\lambda(\sigma)c_\lambda)^2/c_\sigma^2)$

$$P_{\sigma'}(c)\beta_\lambda(\sigma'')c_\lambda = {}^1/_2(1+\sigma'\sigma'')\beta_\lambda(\sigma'')c_\lambda, \tag{2.12}$$

that is, in the space $\beta_\lambda(\sigma)c_\lambda$ the operators $P_{\sigma'}(c)$ play the role of a generalized Kronecker tensor.

With the aid of these operators it is also easy to obtain a symmetrical formulation of the theory, which is convenient in several cases:

$$(\beta_\lambda^{(+)}\partial_\lambda + m_1)\psi_T = (\beta_\lambda^{(+)}\partial_\lambda + m_1^{-1}\Box^2)\psi_T = 0, \tag{2.13}$$

$$(\beta_\lambda^{(-)}\partial_\lambda + m_2)\psi_L = (\beta_\lambda^{(-)}\partial_\lambda + m_2^{-1}\Box^2)\psi_L = 0. \tag{2.14}$$

The Green's function of (1.12) is of the form

$$\bar{S}(x,\sigma) = (\beta_\lambda(\sigma)\partial_\lambda - m + m^{-1}(\beta_\lambda(-\sigma)\partial_\lambda)^2 - (4m^2)^{-1}(1-\sigma^2)\Box^2\beta_\lambda(-\sigma)\partial_\lambda) \times ({}^1/_2(1+\sigma^2)\Box^2 - m^2 - (4m^2)^{-1}(1-\sigma^2)^2(\Box^2)^2)^{-1}\delta(x) \tag{2.15}$$

and is connected with the Green's function of (2.11) by

$$\bar{\Delta}(x,\sigma) = (2\pi)^{-4}P\int(dk)(k_\lambda^2 + m_1^2(\sigma))^{-1}(k_\lambda^2 + m_2^2(\sigma))^{-1}\exp(ik_\lambda x_\lambda). \tag{2.16}$$

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[2] Quite general classes of projection operators in Hilbert space were considered from a somewhat different point of view in a paper by Fedorov.[4]

A similar relation (unlike the ordinary Kemmer theory[5]) connects $S(x,\sigma)$ with $\Delta(x,\sigma)$, where $\Delta(x,m)$ are Klein functions:

$$\Delta(x,\sigma) = -i(2\pi)^{-3}\int (dk)\,\varepsilon(k)\,\delta((k_\lambda^2 + m_1^2(\sigma)) \times (k_\lambda^2 + m_2^2(\sigma)))\exp(ik_\lambda x_\lambda) = (2\,|m_1^2 - m_2^2|)^{-1}(\Delta(x,m_1) - \Delta(x,m_2)). \quad (2.17)$$

3. INTERACTION WITH ELECTROMAGNETIC FIELD

In Eq. (1.12) the electromagnetic interaction is usually introduced by the substitution

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu, \qquad \partial_\mu \to D_\mu^+ = \partial_\mu - ieA_\mu$$

in the equation for the left factor $\psi^+(x) = \psi^* R_4$. The additional conditions assume the form

$$(\beta_\lambda(-\sigma)D_\lambda + (4m)^{-1}(1-\sigma^2)D_\lambda^2 + (ie/2m)\,\beta_\mu(-\sigma)\,\beta_\nu(\sigma)\,F_{\mu\nu})\,\psi = 0, \quad (3.1)$$

$$(\beta_\lambda(-\sigma)D_\lambda^+ + (4m)^{-1}(1-\sigma^2)D_\lambda^{+2} + (ie/2m)\,\beta_\mu(-\sigma)\,\beta_\nu(\sigma)\,F_{\mu\nu})^T\psi^+ = 0. \quad (3.2)$$

A very convenient symmetrical notation is $(\alpha = 4\sigma(1-\sigma^2)^{-1})$

$$(\alpha\beta_\lambda^{(+)}D_\lambda + m_1 - m_1^{-1}D_\lambda^2)\,\psi = (-\alpha\beta_\lambda^{(-)}D_\lambda + m_2 - m_2^{-1}D_\lambda^2)\,\psi = 0, \quad (3.3)$$

etc., which is impossible in the limit of the Kemmer theory.

The current and energy-momentum conservation laws remains the same as before, but the following quantities are modified.

$$j_\mu(\sigma) = e\psi^+\beta_\mu(\sigma)\psi,$$

$$T_{\mu\nu}(\sigma) = T_{\nu\mu}(\sigma) = 2m(1-\sigma^2)^{-1}(\psi^+(\{\beta_\mu(\sigma)\,\beta_\nu(\sigma)\} - \delta_{\mu\nu}I)\,\psi - {}^1/_4(1-\sigma^2)(\psi^+\beta_\mu(\sigma)D_\nu\psi - D_\nu^+\psi^+\beta_\mu(\sigma)\psi)). \quad (3.4)$$

In addition, in the Kemmer theory for arbitrary interactions that are compatible with charge conservation there are two conservation laws

$$\partial_\lambda(\psi^+(\beta_\mu^{(\mp)}\beta_\lambda^{(\pm)} + \beta_\lambda^{(\pm)}\beta_\mu^{(\mp)})\,\psi) = \partial_\lambda(\psi^+\beta_\lambda^{(\mp)}\psi) = 0. \quad (3.5)$$

On going over to the nonorthogonal representations, (3.5) are no longer trivial:

$$\partial_\lambda\sigma_{\mu\lambda}(\sigma) - (2m\sigma)^{-1}(1-\sigma^2)\,j_\lambda(-\sigma)\,F_{\mu\lambda} = \partial_\lambda J_\lambda(-\sigma) + (ie/2m)\,\psi^+(\beta_\mu(-\sigma)\,\beta_\nu(\sigma) + \beta_\nu(\sigma)\,\beta_\mu(-\sigma))\,\psi F_{\mu\nu} = 0, \quad (3.6)$$

where

$$\sigma_{\mu\nu}(\sigma) = \sigma^{-1}(\psi^+(\beta_\mu(-\sigma)\,\beta_\nu(\sigma) + \beta_\nu(\sigma)\,\beta_\mu(-\sigma) - ({}^1/_2)(1-\sigma^2)\,\delta_{\mu\nu}I)\,\psi - (4m)^{-1}(1-\sigma^2)(\psi^+\beta_\mu(-\sigma)D_\nu\psi - D_\nu^+\psi^+\beta_\mu(-\sigma)\psi)), \quad (3.7)$$

$$J_\mu(\sigma) = e\psi^+\beta_\mu(-\sigma)\psi + (e/4m)(1-\sigma^2)(\psi^+D_\mu\psi - D_\mu^+\psi^+\psi). \quad (3.8)$$

In conclusion we must make the following remarks. The groups of matrices γ_μ and $\bar{\gamma}_\mu$, on which the representations $\beta_\mu(\sigma)$ are constructed, are two commuting subgroups of rotations in the group $G_{256}(\gamma,\bar{\gamma})$ of an eight-dimensional irreducible representation of a Dirac algebra. A transition to equations with more than two mass states calls for introduction of a corresponding number of commuting Dirac algebras and is connected with a transition to a representation of even higher rank, but in this case the equations which determine the different σ become incompatible with the hermiticity of the representations. The generalization of the obtained results to isotopic space makes it possible to construct a consistent theory of mass isodoublets without resorting to the introduction of the mass operator.

[1] A. A. Borgardt, JETP **24**, 24 (1953).

[2] A. A. Borgardt, DAN SSSR **78**, 1114 (1951).

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