CLASSICAL MODEL OF A CHARGED PARTICLE WITH ANGULAR MOMENTUM

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On the basis of equations proposed by the author in a previous paper [1] it is shown that by compensating the electrostatic repulsive field by the field of virtual vector mesons it is possible to construct a stable classical model of a charged particle having mechanical angular momentum and a magnetic moment.

IT was shown previously [1] that it is possible to construct a stable extended classical model of a charged particle by balancing the electrostatic repulsive forces by "meson" attractive forces. The corresponding vector mesons must have an infinite imaginary charge and an infinite imaginary mass. If their mass is chosen to be real, then the equilibrium turns out to be unstable; an imaginary finite mass would lead to velocities exceeding the velocity of light. Therefore, in order that the Lagrangian should satisfy the necessary requirements of relativistic invariance not only formally, it is necessary to choose the mass of the particles to be imaginary infinite. Then from the equations it follows that the corresponding charge is also imaginary infinite.

Outside the region occupied by the electric charge the "meson" field has an infinitely small spatial period, and, therefore, is not directly observable. Thus, no new physical constants appear in the theory in addition to the radius (or the mass) which is of a purely field-theoretical nature and turns out to be finite as a result of the subtraction of the energy of the "meson" field from the energy of the electrostatic field.

In this article we consider the problem of a certain extension of the proposed model. We can assume that it has not only a static charge density, but also a stationary current distribution. Such a model can exist without radiating provided the electromagnetic force is everywhere balanced by the "meson" force.

In accordance with [1], the basic equations can be written in the rest system of the charge as a whole in the following form

$$\Delta\varphi = -4\pi\rho, \tag{1}$$

$$\Delta\psi = -4\pi\rho\lambda - \varkappa^2\psi, \tag{2}$$

$$\mathrm{rot\,rot}\,\mathbf{A} = -4\pi c^{-1}\mathbf{j}, \tag{3*}$$

$$\mathrm{rot\,rot}\,\mathbf{B} = -4\pi c^{-1}\lambda\mathbf{j} - \varkappa^2\mathbf{B}, \tag{4}$$

$$\mathbf{E} - \lambda\mathbf{F} + c^{-1}[\mathbf{j}, \mathbf{H} - \lambda\mathbf{G}] = 0. \tag{5*}$$

Here φ, **A**, **E**, **H**, are the electromagnetic and ψ, **B**, **F**, **G** are the "meson" potentials and fields, ρ, **j** are the densities of the electric charge and current. The constant λ which is invariant by definition characterizes the charge as the source of "meson" field, and the sign of κ^2 is chosen in accordance with the fact that this field is virtual.

In future we shall assume that the current **j** and the potentials **A** and **B** have only azimuthal components with respect to a certain symmetry axis and do not themselves depend on the azimuth. Thus, the law of conservation of charge and the Lorentz condition for the electromagnetic potential will both be satisfied.

Just as in [1], we set $\lambda^2 \to -\infty$ and transform Eqs. (1)—(4) in accordance with this. The nature of the transformation can be seen sufficiently well from dealing with Eqs. (1)—(2). By combining these equations we obtain

$$\Delta\lambda\psi = \frac{\Delta\lambda^2(\varphi - \lambda\psi)}{1-\lambda^2} - \frac{\varkappa^2}{1-\lambda^2}\lambda\psi. \tag{6}$$

We introduce the notation

$$\lambda\psi \equiv x, \quad \varphi - \lambda\psi \equiv -y, \quad (1-\lambda^2)^{-1} \equiv \nu^2, \tag{7}$$

and then in the limit obtain

$$\Delta(x - y) = -\varkappa^2\nu^2 x, \tag{8}$$

$$\rho = \varkappa^2\nu^2 x/4\pi. \tag{9}$$

The product $\kappa\nu$ is assumed to be finite. The magnetic quantities are subjected to analogous transformations.

*rot = curl; $[\mathbf{j}, \mathbf{H}] = \mathbf{j} \times \mathbf{H}$.

The system (1)—(5) is nonlinear. It does not admit the usual separation of variables in finite form by means of expansion in terms of Legendre polynomials. Physically this is understandable: it is impossible to balance the forces of some one magnetic multipole, in particular a dipole, in the second term of (5) by a finite number of electric multipoles in the first term. Therefore, the charge as a whole will have not only a magnetic dipole moment but also all magnetic moments of odd order and all electric moments of even order. Possibly this unattractive property of the classical model will not go over into the quantum theory where the rigid condition of equilibrium is replaced by the much less restrictive condition that the solutions should be stationary.

For the electric quantities we seek an expansion of the form

$$x = \sum_{k=0}^{\infty} x_{2k}(r) P_{2k}(\cos\theta), \quad y = \sum_{k=0}^{\infty} y_{2k}(r) P_{2k}(\cos\theta) \quad (10)$$

and for the magnetic quantities of the form

$$x = \sum_{k=0}^{\infty} x_{2k+1}(r) P^1_{2k+1}(\cos\theta), \quad y = \sum_{k=0}^{\infty} y_{2k+1}(r) P^1_{2k+1}(\cos\theta). \quad (11)$$

Substitution into (8) yields

$$\frac{d^2}{dr^2} r(x_s - y_s) - \frac{s(s+1)}{r^2} r(x_s - y_s) = -x^2 v^2 r x_s, \quad (12)$$

where s can be either odd or even.

On substituting into the nonlinear equations (5) we have to re-expand the products of Legendre polynomials in terms of the first powers of these polynomials. For this we have the following formulas:

$$P_{2k} P_{2l} = \sum_{n=|k-l|}^{k+l} L^{2n}_{2k,2l} P_{2n}, \quad (13)$$

$$P^1_{2k+1} P^1_{2l+1} = \sum_{n=|k-l|}^{k+l+1} M^{2n}_{2k+1,\,2l+1} P_{2n}, \quad (14)$$

$$P_s P^1_t = \sum_{n=|s-t|/2}^{(s+t)/2} N^{2n}_{s,t} P_{2n}; \quad (15)$$

with s, t in (15) being either odd or even, but both of the same parity. The coefficient $L^{2n}_{2k,2l}$ can be obtained, in accordance with [2], in the following manner ("Heron's formula"):

$$L^{2n}_{2k,2l} = \frac{4n+1}{4(k+l+n)+1} \frac{[2(k+l-n)-1]!!}{[2(k+l-n)]!!} \times \frac{[2(k+n-l)-1]!!}{[2(k+n-l)]!!} \times \frac{[2(n+l-k)-1]!!}{[2(n+l-k)]!!} \frac{[2(k+l+n)]!!}{[2(k+l+n)-1]!!}. \quad (16)$$

Further we have

$$M^{2n}_{2k+1,\,2l+1} = \tfrac{1}{2}[(2k+1)(2k+2) + (2l+1)(2l+2) - 2n(2n+1)] L^{2n}_{2k+1,\,2l+1}, \quad (17)$$

$$N^{2n}_{s,t} = \frac{s(s+1) + 2n(2n+1) - t(t+1)}{4n(n+1)} L^{2n}_{s,t}, \quad (18)$$

where the order of the subscripts in the last formula is, evidently, not immaterial.

Now, comparing coefficients of the same polynomials we obtain the condition for the equilibrium of the radial component of the force:

$$\sum_{l=0}^{\infty} \sum_{k=|n-l|}^{n+l} \left[L^{2n}_{2k,2l}\, x_{2k} \frac{dy_{2l}}{dr} - M^{2n}_{2k+1,\,2l+1}\, x_{2k+1} \frac{1}{r} \frac{d}{dr}(r y_{2l+1}) \right] = 0, \quad (19)$$

where $n \geq 0$. For the component of the force which is perpendicular to the radius we obtain (the azimuthal component is equal to zero because of symmetry considerations)

$$\sum_{l=1}^{\infty} \sum_{k=|n-l|}^{n+l} N^{2n}_{2k,\,2l} x_{2k} y_{2l} + \left(\sum_{l=0}^{n-1} \sum_{k=n-l-1}^{n+l} + \sum_{l=n}^{\infty} \sum_{k=l-n}^{l+n} \right) \times N^{2n}_{2k+1,\,2l+1} (2l+1)(2l+2) x_{2k+1} y_{2l+1} = 0, \quad (20)$$

with $n \geq 1$.

Equations (12), (19), and (20) form a complete system which must be solved within the region occupied by the charge $r \leq r_0$. At the origin of coordinates $r = 0$ the condition of regularity must be satisfied: the expansion for a quantity with index s starts with r^s. At $r = r_0$ the components of the field and the static potential of the charge are continuous. Because the "meson" field oscillates in space we do not have to impose conditions on it at $r = \infty$. Therefore, the conditions for it being continuous at $r = r_0$ can always be satisfied. Then the conditions for the continuity of the electromagnetic field can be expressed only with the aid of the quantities x, y, determined from the interior problem. In accordance with the expansion (10) each term of even order corresponds to an electric multipole, and each term of odd order corresponds to a magnetic multipole.

In accordance with (7) the electromagnetic potentials are simply defined as the differences $x - y$. Consequently, in zero order we have

$$(x_0 - y_0)_{r=r_0} = \frac{e}{r_0}, \quad \left[r \frac{d}{dr}(x_0 - y_0) \right]_{r=r_0} = -\frac{e}{r_0}. \quad (21)$$

In all higher orders the multipole moments are excluded since the corresponding boundary conditions are homogeneous. Thus,

$$\frac{d}{dr} r(x_1 - y_1)\Big|_{r=r_0} = -(x_1 - y_1)\Big|_{r=r_0}, \quad r\frac{d}{dr}(x_2 - y_2)\Big|_{r=r_0} = -3(x_2 - y_2)\Big|_{r=r_0} \quad \text{etc.} \tag{22}$$

The somewhat different form in which the even and odd multipoles are recorded is associated with the fact that the former are electric multipoles, and the latter are magnetic multipoles Together with conditions (21) and (22), and also the condition at the origin, the system of equations given above is complete and can be solved numerically if we restrict ourselves to several terms in the expansions (10)—(11).

The energy and the angular momentum can be easily expressed in terms of integrals taken only over the interior region. Thus, the energy is equal to

$$\mathcal{E} = \frac{1}{2}\int [\rho(\varphi - \lambda\psi) + \frac{\mathbf{j}}{c}(\mathbf{A} - \lambda\mathbf{B})]\, dV = \frac{e^2}{r_0}\int_0^{\varkappa\nu r_0} z^2\, dz \sum_{k=0}^{\infty}\left[\frac{x_{2k}y_{2k}}{4k+1} + \frac{(2k+1)(2k+2)}{4k+3} x_{2k+1}\, y_{2k+1}\right]. \tag{23}$$

The parameter $\kappa\nu r_0$ must be chosen from the condition that $\mathcal{E}$ is minimum. In the zero order approximation[1] the energy calculated in accordance with (23) did not have any minimum. However, it was possible to change somewhat the definition of the quantity ν associated with the "meson" charge λ. In particular, by replacing ν by $\pi/2\kappa r_0 + \epsilon$ we found the only positive minimum value of $\mathcal{E}$. The choice of the coefficient $\pi/2$ gives the smallest minimum. The transition from ν to ϵ establishes a certain connection between the mass and the charge of the "mesons." In future, possibly, we should minimize $\mathcal{E}$ also with respect to the shape of the charge by letting it deviate from spherical.

By utilizing the definition of angular momentum[3] it is possible to express it also in terms of an integral taken over only the interior region, with the surface integral vanishing identically. We obtain

$$|\mathbf{M}| = \frac{1}{c}\left|\int [\mathbf{r}, (\mathbf{A} - \lambda\mathbf{B})]\, \rho dV\right| = \frac{e^2}{c}\int_0^{\nu\varkappa r_0} z^3 dz \left\{\frac{2}{3} x_0 y_1 + \sum_{k=1}^{\infty} x_{2k}\left[\frac{2k(1-2k)}{16k^2-1} y_{2k-1} + \frac{2(k+1)(2k+1)}{(4k+1)(4k+3)} y_{2k+1}\right]\right\}. \tag{24}$$

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[1] A. S. Kompaneets, JETP **43**, 2185 (1962), Soviet Phys. JETP 16, 1544 (1963).

[2] E. W. Hobson, Theory of Spherical and Ellipsoidal Harmonics (Russ. Transl., IIL, 1942, p. 89).

[3] G. Wentzel, Quantum Theory of Fields, Interscience, 1949 (Russ. Transl. of German original, Gostekhizdat, 1947, pp. 105-106).

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