ASYMPTOTIC BEHAVIOR OF SCATTERING AMPLITUDES AND THE "GHOST" PROBLEM OF THE VACUUM REGGE TRAJECTORY

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Asymptotic expressions are obtained for elastic $\pi\pi$, πN, and NN scattering amplitudes in the region where the principal Regge pole passes through the value $j = 0$. The Gell-Mann hypothesis on the vanishing of the residues of physical partial wave amplitudes is in agreement with their analytic properties near $j = 0$ and ensures the finite nature of all scattering amplitudes.

1. INTRODUCTION

ACCORDING to recently developed ideas,[1] the asymptotic behavior of amplitudes for the elastic scattering of strongly interacting particles through small angles at large energies is governed by a single Regge pole with quantum numbers of the vacuum. This pole is generally referred to as the principal vacuum pole, the Pomeranchuk pole, or the P-pole. It is a pole of a function of two complex variables $f(j,t)$, which is the analytic continuation of the partial wave amplitude $f^j(t)$ in the crossed channel, corresponding to the annihilation of a particle-antiparticle pair into another pair (the annihilation or t channel). The function $f(j,t)$ coincides with $f^j(t)$ when $j = 2n$ —an even value of the quantum number corresponding to the total angular momentum, and $\sqrt{t}$ is equal to the energy in the barycentric frame (c.m.s.). In the elastic scattering channel (s-channel) t has the meaning of the square of the momentum transfer.

The position of the P-pole in the j plane is a function of t: $j_P = \alpha(t)$, and in the region of interest to us t lies to the right of other singularities of $f(j,t)$. For $t = 0$ (scattering through zero angle) $\alpha(0) = 1$, which ensures the constancy of the total scattering cross section at large energies. For smaller t (scattering through small angles) the position of the pole should move to the left along the real axis in the j plane. This gives rise to the falling off of the elastic scattering cross section with increasing energy and to the narrowing of the effective cone of elastic scattering as compared with diffraction,[1] as was already observed experimentally.[2] For certain $t = t_0$ the pole trajectory may pass the point $j = \alpha(t_0) = 0$, as apparently is the case in actuality.[3]

But the value $j = 0$ corresponds to a physical point in the complex angular momentum plane. To it corresponds a stationary state of the system in the t channel, i.e., a particle of spin 0 and mass $\sqrt{t_0}$. Since $t_0 < 0$, we have a physically meaningless state—a "ghost." In the scattering channel this manifests itself in the scattering amplitude becoming infinite at $t = t_0$.

Gell-Mann has advanced a hypothesis,[4] which in a very natural way eliminates this difficulty. This hypothesis consists of the assumption that for the given values of j and t the partial wave amplitude is not a number but a matrix, corresponding to the various transformations of the t-channel system (for example, the transformation of a pion pair into a pion pair, or into nucleon pairs with different spins s and orbital angular momenta l). Among the possible states of this system one may find states for which the value $j = 0$ is not physical (for example the triplet state of a nucleon pair for $j = l + 1$). If the pole is present in only those elements of the matrix $f(j,t)$ which correspond to transitions into such states, then the passage of the pole trajectory through the point $j = 0$ does not give rise to the appearance of a "ghost."

However, Gell-Mann arrived at the conclusion that one of the nucleon-nucleon scattering amplitudes nevertheless remained infinite and to eliminate this difficulty introduced another pole trajectory, which he called the Q-trajectory. We shall show below that this latter difficulty actually does not arise and that the original Gell-Mann hypothesis is sufficient to eliminate the "ghost." Moreover, if one accepts the usual ideas about the analytic properties of partial wave amplitudes as functions of j then it turns out that the matrix f(j, t) cannot simultaneously have the pole $\alpha(t_0) = 0$ in its "sense" and "nonsense" (for j = 0) elements. Therefore the hypothetical part of Gell-Mann's assumption consists only in the alternative choice of the version desirable for the theory.

2. PARTIAL WAVE AMPLITUDES IN THE t CHANNEL

If we restrict ourselves to the three processes: pion-pion, pion-nucleon, and nucleon-nucleon scattering, whose amplitudes are denoted respectively by U_{ba} ($U_{\pi\pi}$, $U_{\pi N}$, U_{NN}), then we must consider eight invariant amplitudes—functions of the kinematic invariants: U, A, B, H_i, which enter into the following well known expressions:[5,6]

$$U_{\pi\pi} = U(s, t), \tag{1}$$

$$U_{\pi N} = A(s, t)\,\bar{u}(-p_1')\,u(p_1) + B(s, t)(p_{2\mu} - p_{2\mu}')\,\bar{u}(-p_1')\,\gamma_\mu u(p_1), \tag{2}$$

$$U_{NN} = \sum_{i=1}^{3} H_i(s, t)\,[\bar{u}(p_2)\,O^i u(-p_2')]\,[\bar{u}(-p_1')\,O^i u(p_1)], \tag{3}$$

where $t = (p_1 + p_1')^2$, $s = (p_1 - p_2')^2$, O^i are the matrices forming the five Fermi covariants

$$O^1 = 1, \qquad O^2 = \gamma_\mu,$$

$$O^3 = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \qquad O^4 = i\gamma_5\gamma_\mu, \qquad O^5 = \gamma_5;$$

p_1 and p_2 are the 4-momenta of the particle, and p_1' and p_2' of the antiparticle in the t channel before and after the collision respectively (for discussion of scattering of the particles, i.e., for the s channel, one must replace p_1' by $-p_1'$ and p_2' by $-p_2'$). The normalization of the spinor amplitudes u(p) is such that

$$\bar{u}(p)\,u(p) = p_0/|p_0|,$$

and the amplitudes U_{ba} are normalized such that the effective differential cross section is expressed in the form:

$$d\sigma_{ba} = \frac{k_2}{k_1}\frac{1}{W^2}|U_{ba}|^2\,do,$$

where k_1 and k_2 stand for the magnitudes of the momenta before and after collision, W is the total energy, and do is the solid angle element in the c.m.s.

In Eqs. (1)—(3) a definite isospin T in the t channel is assumed. We shall be interested in the value T = 0.

Next we must find the expansion of the amplitudes in partial waves in the c.m.s., in the physical region of the t channel. Although the form of these expansions is known[6,7] we present a derivation in order to be able to follow their characteristic features. To this end we express the quantities bilinear in the spinor amplitudes u that enter Eqs. (2) and (3) in terms of two-component spinors that define nucleon polarization. Making use of the explicit expressions for u(p)

$$u(p_1) = \frac{1}{\sqrt{2m}}\begin{pmatrix}\sqrt{\varepsilon + m}\,v_1\\ \sqrt{\varepsilon - m}\,(\boldsymbol{\sigma}\mathbf{n}_1)\,v_1\end{pmatrix},$$

$$u(-p_1') = \frac{1}{\sqrt{2m}}\begin{pmatrix}-\sqrt{\varepsilon - m}\,(\boldsymbol{\sigma}\mathbf{n}_1)w_1\\ \sqrt{\varepsilon + m}\,w_1\end{pmatrix},$$

where σ_i are the Pauli matrices, m is the mass, ϵ is the energy, and $\mathbf{n}_1$ is a unit vector in the direction of motion of the incident nucleon in the c.m.s. of the nucleon and antinucleon, and of analogous expressions for $u(p_2)$ and $u(-p_2')$, we arrive at expressions in which the spin states of the nucleon-antinucleon pair appear in terms of the quantities

$$\chi_1 = \overset{*}{w}_1 w_1, \qquad \chi_2 = \overset{*}{v}_2 w_2,$$
$$\boldsymbol{\chi}_1 = \overset{*}{w}_1 \boldsymbol{\sigma} v_1, \qquad \boldsymbol{\chi}_2 = \overset{*}{v}_2 \boldsymbol{\sigma} w_2;$$

χ_1 and χ_2 represent singlet states, and $\boldsymbol{\chi}_1$ and $\boldsymbol{\chi}_2$ represent triplet states. (Let us note that the spin states of the nucleon are determined by the spinor v, and those of the antinucleon by the spinor $w^*\sigma_2$.) The amplitudes $U_{\pi N}$ and U_{NN} may then be expressed in the form:

$$U_{\pi N} = F_1(\boldsymbol{\chi}_1\mathbf{n}_1) + F_2([\mathbf{n}_1\boldsymbol{\chi}_1]\,[\mathbf{n}_1\mathbf{n}_2]), \tag{4*}$$

$$U_{NN} = f_1\chi_1\chi_2 + f_2(\boldsymbol{\chi}_1\mathbf{n}_1)(\boldsymbol{\chi}_2\mathbf{n}_2) + f_3([\mathbf{n}_1\boldsymbol{\chi}_1]\,[\mathbf{n}_2\boldsymbol{\chi}_2]) + f_4([\mathbf{n}_1\,[\mathbf{n}_1\boldsymbol{\chi}_1]]\,[\mathbf{n}_2\,[\mathbf{n}_2\boldsymbol{\chi}_2]]) + f_5\{(\mathbf{n}_1\boldsymbol{\chi}_1)([\mathbf{n}_2\boldsymbol{\chi}_2]\,[\mathbf{n}_2\mathbf{n}_1]) + (\mathbf{n}_2\boldsymbol{\chi}_2)([\mathbf{n}_1\boldsymbol{\chi}_1]\,[\mathbf{n}_1\mathbf{n}_2])\}, \tag{5}$$

where F_i and f_i are functions of the invariants t and $z = \mathbf{n}_1 \cdot \mathbf{n}_2$.

It is seen from Eq. (4) that the pion pair can go over only into triplet states of the nucleon-antinucleon pair, with F_1 being the amplitude for the annihilation of the longitudinal component of the triplet state vector $\boldsymbol{\chi}_1$ (spin projection along the direction of the momentum $m_s = 0$), and F_2 being the amplitude for the annihilation of the transverse component ($|m_s| = 1$). Analogously it can be seen from Eq. (5) that f_1 is the amplitude for transition from the singlet to the singlet state. The remaining amplitudes correspond to transitions between triplet states: f_2—between longitudinal components ($m_{s1} = m_{s2} = 0$), f_3 and f_4—between transverse components ($|m_{s1}| = |m_{s2}| = 1$) and f_5—between longitudinal and transverse and vice versa ($m_{s1} = 0$, $|m_{s2}| = 1$ or $|m_{s1}| = 1$, $m_{s2} = 0$).

*$[\mathbf{n}_1\boldsymbol{\chi}_1] = \mathbf{n}_1 \times \boldsymbol{\chi}_1$.

By comparing Eqs. (2) with (4) and (3) with (5) it is not hard to express the invariant amplitudes in terms of the F_i and f_i:

$$A = -\frac{2m}{\sqrt{t-4m^2}} F_1 + \frac{4m^2}{\sqrt{t(t-4m^2)}} z' F_2,$$

$$B = -\frac{2m}{\sqrt{t(t-4\mu^2)}} F_2, \tag{6}$$

where μ is the pion mass,

$$z' = (\mathbf{n}_1\mathbf{n}_2) = -(2s + t - 2m^2 - 2\mu^2)/\sqrt{(t-4m^2)(t-4\mu^2)},$$

$$H_1 = \frac{4m^2}{t-4m^2}\left\{f_2 + zf_4 - \frac{t+4m^2}{2m\sqrt{t}} zf_5\right\}, \tag{7}$$

$$H_2 = \frac{4m^2}{t-4m^2}\left\{-f_4 + \frac{2m}{\sqrt{t}} f_5\right\}, \quad H_3 = \frac{4m^2}{t-4m^2}\left\{f_4 - \frac{\sqrt{t}}{2m} f_5\right\},$$

$$H_4 = \frac{4m^2}{t-4m^2} f_3,$$

$$H_5 = \frac{4m^2}{t}\left\{-f_1 - \frac{4m^2}{t-4m^2} zf_3 + zf_4 - \frac{\sqrt{t}}{2m} zf_5\right\}, \tag{8}$$

where

$$z = (\mathbf{n}_1\mathbf{n}_2) = -1 - \frac{2s}{t-4m^2}. \tag{9}$$

Equations (4) and (5) are in a convenient form for expansion in partial wave amplitudes, i.e., in eigenstates of the total angular momentum of the system. Since the spin wave functions χ or $\boldsymbol{\chi}$ enter linearly into $U_{\pi N}$ and bilinearly into U_{NN} we may write

$$U_{\pi N} = \boldsymbol{\chi}_1 \mathbf{F}(\mathbf{n}_1, \mathbf{n}_2, t),$$

$$U_{NN} = \chi_1\chi_2 f_1(z, t) + \chi_{1i}\chi_{2k} F_{ik}(\mathbf{n}_1, \mathbf{n}_2, t). \tag{10}$$

The expansions of $\mathbf{F}$ and F_{ik} have the following form (see, for example, [8]):

$$\mathbf{F}(\mathbf{n}_1, \mathbf{n}_2, t) = 4\pi \sum_{jM\lambda} f^j_{\pi\lambda}(t) Y^*_{jM}(\mathbf{n}_2) \mathbf{Y}^\lambda_{jM}(\mathbf{n}_1),$$

$$F_{ik}(\mathbf{n}_1, \mathbf{n}_2, t) = 4\pi \sum_{jM\lambda_1\lambda_2} f^j_{\lambda_1\lambda_2}(t) [\mathbf{Y}^{\lambda_2 *}_{jM}(\mathbf{n}_2)]_k [\mathbf{Y}^{\lambda_1}_{jM}(\mathbf{n}_1)]_i, \tag{11}$$

where the Y_{jM} are spherical harmonics and $\mathbf{Y}^\lambda_{jM}$ are vector spherical harmonics; λ, λ_1, λ_2 take on three values corresponding to the three types of vector spherical harmonics: longitudinal

$$\mathbf{Y}^1_{jM}(\mathbf{n}) = \mathbf{n} Y_{jM}(\mathbf{n})$$

and two transverse of different parities

$$\mathbf{Y}^2_{jM}(\mathbf{n}) = \frac{1}{\sqrt{j(j+1)}} \nabla Y_{jM}(\mathbf{n}),$$

$$\mathbf{Y}^3_{jM}(\mathbf{n}) = \frac{1}{\sqrt{j(j+1)}} [\mathbf{n}\nabla] Y_{jM}(\mathbf{n})$$

[by $\nabla Y(\mathbf{n})$ is understood $\rho(\partial/\partial\rho) Y(\boldsymbol{\rho}/\rho)$].

Introducing these expressions into Eq. (10) and using the identity

$$\sum_M Y^*_{jM}(\mathbf{n}_2) Y_{jM}(\mathbf{n}_1) = \frac{2j+1}{4\pi} P_j(z)$$

(where P_j is the Legendre function) we obtain for $U_{\pi N}$ the following expansion:

$$U_{\pi N} = \sum_j (2j+1)\left\{f^j_{\pi 1}(\boldsymbol{\chi}_1\mathbf{n}_1) + \frac{1}{\sqrt{j(j+1)}} f^j_{\pi 2}(\boldsymbol{\chi}_1\nabla_1)\right\} P_j(z'), \tag{12}$$

from which, by comparison with Eq. (4), we find

$$F_1 = \sum_{j=0}^{\infty} (2j+1) f^j_{\pi 1}(t) P_j(z'),$$

$$F_2 = \sum_{j=1}^{\infty} \frac{(2j+1)}{\sqrt{j(j+1)}} f^j_{\pi 2}(t) P'_j(z') \tag{13}$$

(the prime denotes differentiation with respect to the argument).

From Eq. (11) we obtain for U_{NN}

$$U_{NN} = \sum_j (2j+1)\Big\{f^j_{00}\chi_1\chi_2 + f^j_{11}(\boldsymbol{\chi}_1\mathbf{n}_1)(\boldsymbol{\chi}_2\mathbf{n}_2) + \frac{1}{j(j+1)} f^j_{22}(\boldsymbol{\chi}_1\nabla_1)(\boldsymbol{\chi}_2\nabla_2)$$
$$+ \frac{1}{j(j+1)} f^j_{33}([\mathbf{n}_1\boldsymbol{\chi}_1]\nabla_1)([\mathbf{n}_2\boldsymbol{\chi}_2]\nabla_2) + \frac{1}{\sqrt{j(j+1)}} f^j_{12}[(\boldsymbol{\chi}_1\mathbf{n}_1)(\boldsymbol{\chi}_2\nabla_2) + (\boldsymbol{\chi}_2\mathbf{n}_2)(\boldsymbol{\chi}_1\nabla_1)]\Big\} P_j(z). \tag{14}$$

Comparing Eqs. (14) and (5) we find

$$f_1 = \sum_{j=0}^{\infty} (2j+1) f^j_{00}(t) P_j(z), \quad f_2 = \sum_{j=0}^{\infty} (2j+1) f^j_{11}(t) P_j(z),$$

$$f_3 = \sum_{j=1}^{\infty} \frac{2j+1}{j(j+1)} \{f^j_{33}(t)[zP'_j(z)]' - f^j_{22}(t) P''_j(z)\},$$

$$f_4 = \sum_{j=1}^{\infty} \frac{2j+1}{j(j+1)} \{f^j_{22}(t)[zP'_j(z)]' - f^j_{33}(t) P''_j(z)\},$$

$$f_5 = \sum_{j=1}^{\infty} \frac{2j+1}{j(j+1)} f^j_{12}(t) P'_j(z). \tag{15}$$

After adding to Eqs. (12) and (14) the expansions of the amplitude U

$$U = \sum_{j=0}^{\infty} (2j+1) f^j_{\pi\pi}(t) P_j(z), \tag{16}$$

we have all the necessary expansions.

The table shows a scheme for the matrix f^j_{ba} and serves to clarify the notation introduced for the partial wave amplitudes. The states a and b are denoted by the symbols π, 1, 2, 3, 0.

3. PROPERTIES OF THE PARTIAL WAVE AMPLITUDES NEAR $\alpha = 0$

From Eq. (11) it is not hard to obtain the inverse relations that give the partial wave amplitudes in terms of the amplitudes f_i and F_i:

	2 pions	Nucleon-antinucleon			
		triplet			singlet
		even $((-1)^j)$		odd $((-1)^{j+1})$	
		longitudinal	transverse		
	π	1	2	3	0
π	$f^j_{\pi\pi}$	$f^j_{\pi 1}$	$f^j_{\pi 2}$	0	0
1		f^j_{11}	f^j_{12}	0	0
2			f^j_{22}	0	0
3				f^j_{33}	0
0					f^j_{00}

$$f^j_{\pi\lambda} = \frac{1}{4\pi}\int \mathbf{F}(\mathbf{n}_1, \mathbf{n}_2, t)\, \mathbf{Y}^{\lambda *}_{jM}(\mathbf{n}_1)\, Y_{jM}(\mathbf{n}_2)\, do_1 do_2,$$

$$f^j_{\lambda_1\lambda_2} = \frac{1}{4\pi}\int F_{ik}(\mathbf{n}_1, \mathbf{n}_2, t)\, [\mathbf{Y}^{\lambda_1^*}_{jM}(\mathbf{n}_1)]_i\, [\mathbf{Y}^{\lambda_2}_{jM}(\mathbf{n}_2)]_k\, do_1 do_2.$$

These expressions, when $\mathbf{Y}^{\lambda}_{jM}$ is replaced by its explicit form, reduce easily to integrals involving only F_i, f_i, and the Legendre functions. Using for F_i and f_i dispersion relations in the variable z, we obtain expressions for the partial-wave amplitudes which can be easily continued into the region of complex j.[1] Without giving the explicit form of the f^j_{ba} we shall remark on their general structure which distinguishes the amplitudes f^j_{12} and $f^j_{\pi 2}$ from the remaining ones. All amplitudes contain integrals of the form[6,7]

$$A^j = k(j)\int_{z_0}^{\infty} A(t, z)\, Q_l(z)\, dz,$$

where A is the absorptive part of the corresponding amplitude, Q_l is the Legendre function of the second kind and $l = j, j \pm 1$, and the coefficient k(j) is a rational function of j. Therefore, in accordance with our general ideas, we take it that in the region of interest the partial amplitudes have as a function of j no singularities other than poles. However the functions f^j_{12} and $f^j_{\pi 2}$ contain in k(j) the factor $\sqrt{j(j+1)}$, i.e., they have "kinematic" branch points at j = 0 and j = −1. Extracting these we write

$$f_{\pi 2}(j, t) = \sqrt{j(j+1)}\, \varphi_{\pi 2}(j, t),$$

$$f_{12}(j, t) = \sqrt{j(j+1)}\, \varphi_{12}(j, t), \qquad (17)$$

where $\varphi_{\pi 2}$ and φ_{12} will be taken to have, like the remaining partial wave amplitudes, no singularities other than poles.

Let us consider now the structure of the matrix $f_{ba}(j, t)$ near the vacuum pole $j_P = \alpha(t)$. As can be seen from the table, f_{ba} separates into the matrix $f_{b'a'}$ ($a', b' = \pi, 1, 2$) and the diagonal elements f_{33}, f_{00}. In the general case the poles appear in the elements of one of the three submatrices; the vacuum pole can appear only in the elements of $f_{b'a'}$.

Let

$$f_{ba} = \frac{g_{ba}(j, t)}{j - \alpha(t)} \qquad (a, b = \pi, 1, 2). \qquad (18)$$

As a result of the unitarity condition the residues r_{ba} of the functions f_{ba} at the pole $j = \alpha(t)$ [$r_{ba} = g_{ba}(\alpha, t)$] are related as follows:[9]

$$r_{aa} r_{bb} = (r_{ba})^2. \qquad (19)$$

Let us take into account the fact that f_{2a} is of the form, Eq. (17). Let

$$\varphi_{a2} = \frac{h_{a2}(j, t)}{j - \alpha(t)}, \qquad h_{a2}(\alpha, t) = \rho_{a2}.$$

It then follows from Eq. (19) that ($a = \pi, 1$)

$$r_{aa} r_{22} = \alpha(\alpha + 1)\, \rho^2_{a2}. \qquad (20)$$

In order that Eq. (20) be satisfied as $\alpha \to 0$ it is necessary that either r_{22}, or $r_{\pi\pi}$ and r_{11} tend to zero. However the state π (2 pions) and the state 1 (longitudinal triplet) are both physical states for j = 0 and the pole at $\alpha = 0$ would represent a "ghost." The state 2, on the other hand, (transverse triplet, $|m_s| = 1$) makes physical sense only for $j \geq 1$. It is therefore natural to take the alternative

$$r_{11} = \alpha\rho_{11}, \qquad r_{\pi\pi} = \alpha\rho_{\pi\pi}, \qquad (21)$$

where ρ_{11} and $\rho_{\pi\pi}$ remain finite at $\alpha = 0$.

Moreover, it follows from Eqs. (21) and (19) that

$$r_{\pi 1}^2 = \alpha^2 \rho_{11}\rho_{\pi\pi}, \qquad r_{\pi 1} = \alpha\rho_{\pi 1} \tag{22}$$

where

$$\rho_{\pi 1}^2 = \rho_{11}\rho_{\pi\pi}, \tag{23}$$

i.e., $r_{\pi 1}$ also vanishes at $\alpha = 0$.

4. ASYMPTOTIC EXPRESSIONS FOR SCATTERING AMPLITUDES

It now remains to obtain, on the basis of the expansions (13), (15), and (16), which are valid in the physical region of the t channel, expressions for the amplitudes U, F_i, and f_i in the physical region of the s channel at large energies $(t < 0, |z| \to \infty)$ determined by the P pole. We note that at that the amplitudes f_{33}^j and f_{00}^j may be ignored. Correspondingly, according to Eq. (15) one must ignore the amplitude f_1.

The amplitudes F_1 and f_2 have expansions identical in structure and coinciding with the expansion of U —the amplitude for spinless particles. According to the rules found in [1] for the continuation of the amplitude U one must:

1) Expand Eq. (16) into parts even and odd in z, replacing $P_j(z)$ respectively by $\frac{1}{2}[P_j(z) \pm P(-z)]$. At that $U = U^+ + U^-$.

2) For each of these parts separately one must carry out the Sommerfeld-Watson transformation, taking the partial amplitudes $f^j(t)$ as having been continued into the region of complex j. The result is

$$U^+ = \frac{1}{4i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{2j+1}{\sin \pi j} f_{\pi\pi}(j, t)\,[P_j(-z) + P_j(z)]\,dj - \frac{\pi}{2}\sum_{\alpha > -1/2} \frac{2\alpha+1}{\sin \pi\alpha} r_{\pi\pi}^{(\alpha)}\,[P_\alpha(-z) + P_\alpha(z)] \tag{24}$$

and an analogous expression for U^-.

3) Extract the contribution of the P pole. It is contained in U^+. Taking $j_P = \alpha(t) > -\frac{1}{2}$ and replacing P_α by its asymptotic expression for large $|z|$ we obtain

$$U = -\frac{\pi C(\alpha)}{\sin \pi\alpha} r_{\pi\pi} z^\alpha, \tag{25}$$

where

$$C(\alpha) = \frac{(2\alpha+1)\,\Gamma(2\alpha+1)\,(1+e^{i\pi\alpha})}{2^{\alpha+1}\,[\Gamma(\alpha+1)]^2} \qquad (C(0) = 1). \tag{26}$$

We see that if $r_{\pi\pi}$ remains finite for $\alpha = 0$ ("ghost") then U becomes infinite. If, instead, Eq. (21) holds then U remains finite.

The expression for U, Eq. (25), obviously goes over into the expression for F_1 and f_2 respectively as $r_{\pi\pi}$ is replaced ty $r_{\pi 1}$ and r_{11}, and so F_1 and f_2 also will be finite at $\alpha = 0$ if Eqs. (21) and (22) hold.

Guided by the same rules in obtaining the remaining amplitudes we encounter the following difference. The corresponding sums do not contain the value $j = 0$ (since the states 2 and 3 start with the value $j = 1$). Therefore the integrals into which these sums are transformed must be taken along a contour that passes to the right of the point $j = 0$.

For the amplitude f_3 this integral will have the form

$$f_3 = -\frac{i}{4}\int_{a-i\infty}^{a+i\infty} \frac{2j+1}{j(j+1)\sin \pi j} f_{22}(j, t)\,[P_j''(z) + P_j''(-z)]\,dj, \tag{27}$$

where $0 < a < 2$ (taking into account the fact that we are interested in the even part of the amplitude only and that to the right of the line Re $j = a$ the function $f(j, t)$ has no singularities; for $t < 0$ one may choose $a < 1$). It is relevant that the integrand in Eq. (27) has a fixed pole at $j = 0$. Therefore, if the P pole $\alpha(t)$ is close to zero one must take into account the contributions from both poles with the result

$$f_3 = -\pi\left\{\frac{C(\alpha)(1-\alpha)}{(1+\alpha)\sin \pi\alpha} r_{22} z^{\alpha-2} + \frac{1}{\pi} f_{22}(0, t)\, z^{-2}\right\}. \tag{28}$$

As $\alpha \to 0$ we have $f_{22}(0, t) \approx -r_{22}/\alpha$ and the dominant term for large $|z|$ will be

$$f_3 = -r_{22} z^{-2} \ln z. \tag{29}$$

Gell-Mann [4] did not take into account the second term in Eq. (28) and therefore arrived at an infinite expression for f_3 at $\alpha = 0$. Let us note that for $\alpha < 0$ the dominant contribution in Eq. (28) comes from the second term, i.e., from the fixed pole.

For the amplitude f_4 we obtain according to Eq. (15):

$$f_4 = \frac{i}{4}\int_{a-i\infty}^{a+i\infty} dj\, \frac{2j+1}{j(j+1)\sin \pi j} f_{22}(j, t) \times \{[zP_j'(z)]' + [(-z)\,P_j'(-z)]'\}. \tag{30}$$

Here the integrand has no fixed pole at $j = 0$. Therefore

$$f_4 = -\frac{\pi\alpha C(\alpha)}{(1+\alpha)\sin \pi\alpha} r_{22} z^{\alpha-1}. \tag{31}$$

This expression, too, is finite at $\alpha = 0$. By comparing Eq. (31) with Eqs. (28) and (29) we see that for large $|z|$ the amplitude f_3 is small compared to f_4. It then follows from Eq. (8) that one may ignore the amplitude H_4 of the axial vector interaction, as was asserted by Gribov and Pomeranchuk.[10]

Proceeding in the same way we obtain from Eq. (15) for the amplitude f_5

$$f_5 = \frac{i}{4} \int_{a-i\infty}^{a+i\infty} dj \frac{2j+1}{\sqrt{j(j+1)} \sin \pi j} f_{12}(j, t) [P_j'(z) - P_j'(-z)]. \quad (32)$$

Because of Eq. (17) the integrand in Eq. (32) has no branch point at $j = 0$ (nor a fixed pole at $j = 0$). Therefore

$$f_5 = -\frac{\pi\alpha C(\alpha)}{\sin \pi\alpha} \rho_{12} z^{\alpha-1} \quad (33)$$

and is finite at $\alpha = 0$.

The amplitude F_2 is obtained from Eq. (33) by replacing ρ_{12} by $\rho_{\pi 2}$.

Introducing the above obtained expressions for F_i and f_i into Eqs. (6) and (8) we obtain the general expressions for the spin structure of the amplitudes $U_{\pi N}$ and U_{NN} found in [10,11], which, consequently, remain valid also in the region where the P pole passes through the value $\alpha = 0$.

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256