ON A CERTAIN TYPE OF MAGNETOHYDRODYNAMIC WAVES

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Non-linear one-dimensional magnetohydrodynamic motions involving variation of the magnitude as well as the direction of the transverse magnetic field are considered. Corresponding solutions for the simplest cases are obtained.

1. One-dimensional unsteady motion has been thoroughly investigated in magnetohydrodynamics for the case of propagation of small perturbations (linearized theory). As far as the nonlinear motion of a compressible fluid is concerned, only a few special solutions are known, in which either the plane of polarization of the magnetic field vector **H** and the velocity of the medium **v** (magnetic-sound waves) or the moduli of these vectors (Alfven waves) are invariant[1-5]. The relevant limitations are imposed on the initial and boundary conditions of problems that reduce to a consideration of such waves.[1)] In the present work some solutions of the magnetohydrodynamic equations are found which permit both the moduli and the orientations in the transverse plane of the vectors **H** and **v** to vary, and consequently one can consider initial and boundary conditions of a more general type.

The equations of plane one-dimensional isentropic motion can be written in the form,

$$\frac{\partial \mathbf{H}_\perp}{\partial t} + \frac{\partial (v_z \mathbf{H}_\perp)}{\partial z} = H_z \frac{\partial \mathbf{v}_\perp}{\partial z}; \tag{1}$$

$$\partial\rho/\partial t + \partial(v_z\rho)/\partial z = 0; \tag{2}$$

$$\frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} + \frac{1}{\rho}\frac{\partial}{\partial z}\left(p + \frac{H_\perp^2}{8\pi}\right) = 0, \qquad p = p(\rho); \tag{3}$$

$$\frac{\partial \mathbf{v}_\perp}{\partial t} + v_z \frac{\partial \mathbf{v}_\perp}{\partial z} = \frac{H_z}{4\pi\rho}\frac{\partial \mathbf{H}_\perp}{\partial z}, \qquad H_z = \text{const}. \tag{4}$$

Here ρ is the density, p the pressure, and $\mathbf{H}_\perp$ and $\mathbf{v}_\perp$ are the transverse components of **H** and **v**.

We consider here waves having a transverse magnetic field ($H_z = 0$) and make use of the condition for "freezing-in" of the magnetic field which follows from Eqs. (1) and (2),

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{H}}{\rho}\right) + v_z \frac{\partial}{\partial z}\left(\frac{\mathbf{H}}{\rho}\right) = 0. \tag{5}$$

If the vector $\mathbf{b} = \mathbf{H}/\rho$ is constant, then (2) and (3) lead to the problem of ordinary gas dynamics, in which p is replaced by $p_m = p + b^2\rho^2/8\pi$[1]. It is readily seen, however, that this latter assertion is fully justified even if only b^2 = const; this assumption has been made below.

The angular orientation of **H** in the transverse plane is φ and can vary (evidently, this possibility has not been considered previously); then, in addition to the system (2) and (3), which enables v_z, ρ, and H^2 to be evaluated, we have, according to (5), the following equation for φ:

$$\partial\varphi/\partial t + v_z\partial\varphi/\partial z = 0. \tag{6}$$

Finally, Eq. (4) is identical in form with (5) and (6), i.e., in the general case transverse motions of the medium are permissible, in which $\mathbf{v}_\perp$ is an arbitrary function of φ.

As expected[2)], φ and $\mathbf{v}_\perp$ are "entropic" quantities which are conserved in Lagrangian coordinates. A significant simplification as compared to problems of one-dimensional, non-isentropic motion arises from the fact that Eqs. (2) and (3) can be treated independently of (4) and (6) (since p in Eq. (3) varies with the entropy S).

2. We assume for example that the quantities v_z, ρ, and H^2 form a simple wave, i.e., they depend on the variable

$$\xi = z - (v_z \pm c_m)\,t, \qquad v_z = \pm\int \frac{c_m}{\rho}\,d\rho$$

($c_m^2 = dp_m/d\rho$). It is then not difficult to integrate Eq. (6), changing from the variables z and t to the variables ξ and t:

$$\varphi = F\left(t\rho c_m \pm \int_{\xi_0}^{\xi} \rho\,d\xi\right), \tag{7}$$

[1)]For example, when a conducting piston moves with a constant velocity, only waves of the type mentioned can occur, besides discontinuities[5].

[2)]Actually, for $H_z = 0$, all forces (Lorentz and elastic) are directed along the z axis and are independent of φ and $\mathbf{v}_\perp$. In particular, the assertion that when v is perpendicular to **H** the polarization of **H** remains fixed (see for example [1]), is wrong.

where F is an arbitrary function. A similar expression is also valid for $v_\perp$.

In so far as H^2 is propagated with a speed $v_z \pm c_m \neq v_z$, the φ-wave is displaced relative to the H^2-wave, and is at the same time deformed. For perturbations of finite extent, the regions over which the modulus and direction of H are variable become completely separated; following this, the φ-wave is no longer deformed and represents a special case of Alfven waves with $H_z = 0$.[3)]

If there is a discontinuity (shock wave) in the H^2-wave then no discontinuity is produced in φ itself (the so-called rotational discontinuity), but only discontinuities in the derivatives of φ. This result applies, provided the discontinuity in the H^2-wave is formed before the two waves referred to above (φ and H^2) separate from each other. An analytic solution which describes the variation of φ in the vicinity of the shock wave can be readily obtained if the wave is assumed to be stationary. In fact, by setting $\varphi(t = 0) = f(z)$ at the initial instant, we obtain,

$$\varphi = \begin{cases} f(z - v_1 t), & z < 0 \\ f[(z - v_2 t)\, v_1/v_2], & 0 < z < v_2 t \\ f(z - v_2 t), & z > v_2 t \end{cases} \qquad (8)$$

Here $v_{1,2}$ are respectively the values of v_z in front of and behind the discontinuity (which in the chosen coordinate system is located at $z = 0$).

In the case of waves of finite extent, the solutions considered describe, in particular, the interactions between transverse magnetic-sound waves and Alfven waves (and also interactions between the shock wave and the rotational discontinuity at $H_z = 0$). As is evident from the discussion, such an interaction results in rotation of the plane of polarization of the magnetic-sound wave.

In conclusion we note the analogy which exists between the propagation of magnetohydrodynamic waves on the one hand, and the propagation of electromagnetic waves in nonlinear media on the other. Without presenting any solutions in the present work, we mention only that waves similar to the ones considered above can occur in a longitudinally magnetized ferrite, with different propagation velocities of the modulus and angular orientation of the transverse magnetic field.

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[3)]In contrast to the linear problem, this separation does not always occur in an unlimited time: in the case where a discontinuity is formed, disturbances reflected from it once again reach the region over which φ is variable.

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253