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CENTRALLY-SYMMETRIC GRAVITATIONAL FIELDS

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It is shown that the well-known statement of Birkhoff to the effect that any centrally-symmetric gravitational field in vacuum should be a static field is true only under some additional conditions. These conditions are equivalent to certain assumptions regarding the wave properties of the Einstein field equation solutions. A general solution of the problem is presented.

THE importance of the particular problem of determining the field in the case of central symmetry in Einstein's theory of gravitation is well known. It is most frequently necessary to consider a centrally-symmetric field in vacuum, defined by a Schwartzschild metric. As regards the problem of centrally-symmetric fields, not necessarily static ones, in vacuum, their existence has been rejected on the basis of the following statement, first expressed by Birkhoff [1]: every centrally-symmetric field in vacuum is static, and therefore, with accuracy up to a coordinate transformation, it is defined by the Schwartzschild metric.

This statement is generally accepted and is included in all serious monographs on the general theory of relativity [2-7] (with the exception of Fock's book [8]), has previously also been accepted by the author of this paper, [9] is often used in relativistic mechanics and cosmology as a basis for crucial conclusions, and is correct under certain conditions, but only under those conditions.

In this paper a rigorous analysis is presented of the conditions under which the solution of the equations of the centrally-symmetric field in vacuum is sought:

$$R_{\alpha\beta} = 0, \tag{1}$$

a physical interpretation of these conditions is given, and the general solution of Eqs. (1) is determined which will in general be nonstatic and will contain a functional arbitrariness which cannot be eliminated.

It is easy to cite a formal example of a metric satisfying the field equations in vacuum (1) which is centrally-symmetric, and nonetheless nonstatic. For this purpose it is sufficient to take, for instance, the Schwartzschild metric in polar coordinates

$$eds^2 = \frac{r}{a-r}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{r-a}{r}dt^2 \quad (e = \pm 1), \tag{2}$$

where a is the gravitational radius, consider it in the space-time region inside the "hypersphere" ($r < a$) and carry out the substitution $r \to t$. We then obtain the metric

$$eds^2 = \frac{t-a}{t}dr^2 - t^2(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{t}{a-t}dt^2, \tag{3}$$

which (for $t < a$) will have a Minkowski type signature $(---+)$, will as before obviously satisfy the field equations in vacuum, will as before be centrally-symmetric, but will be nonstatic. Besides, it is well known [8] that for real objects studied at present the region $r < a$ does not exist, since the gravitational radius turns out to be always within the real body producing the centrally-symmetric gravitational field; in this case there appears in the field equations on the right-hand side an energy-momentum tensor and the problem is not solved in vacuum. Only in the case of the thus far hypothetical ultra-dense stars could one speak of real fields determined by the metric (3); we note that an attempt at a physical interpretation of this metric has been made in the work of Novikov. [10]

In this article we deal not with this trivial transformation of the Schwartzschild metric, but with a more rigorous investigation of the field equations in vacuum for the given class of functions within which the solution is sought. This mathematical formulation has, first, a simple physical meaning, and second, allows one to determine the general form of the metric of a centrally-symmetric field in vacuum; at the same time we explain when such a solution will be static.

1. FIELD-EQUATION SOLUTIONS THAT ADMIT SHOCK WAVES

Investigation of the solutions of (1) automatically leads to the conclusion that the components of the metric tensor $g_{\alpha\beta}(x)$ admit second derivatives, and if in addition use is made of the Bianchi identities, then one must assume that the $g_{\alpha\beta}$ have third-order derivatives. However this statement is not sufficiently specific, if the region in which the components of the metric tensor have continuous derivatives of a given order is not specified. Let us agree on the following terminology: 1) a function is called a function of the class C^r in a given four-dimensional region if in this region all its derivatives up to orders $\leq r$ exist and are continuous (in particular, we shall denote infinitely differentiable functions by C^∞ and analytical functions by C^a), 2) a function is called piecewise differentiable of the class C^r in a given four-dimensional region if a) it has derivatives of orders $\leq r$ throughout this region except perhaps on some surfaces, and b) its derivatives (of order $\leq r$) uniformly approach definite, not necessarily identical limits on approaching these surfaces from either side. In other words, these derivatives can be discontinuous when passing through such surfaces.

Let us now assume that the components of the metric tensor which are solutions of (1) are such that they admit gravitational waves propagating with the fundamental velocity; then the wave front will be an isotropic three-dimensional hypersurface in four-dimensional space-time. At the same time, as has been shown by Lichnerowicz [11], Finzi [12], Pirani [13], and by O'Brien and Synge [14], this hypersurface is characteristic (in the sense of the Hadamard characteristics for a system of partial differential equations), and it is possible to have on it discontinuities of the second and third derivatives of the components $g_{\alpha\beta}(x)$; we note that it is not necessary to identify these hypersurfaces of the wave front with the Schwartzschild singularity hypersurfaces (r is the gravitational radius) —they have nothing in common.

Let us consider the concept of a "shock wave" in the general theory of relativity. Even in Newtonian gravitation the potential and its first derivatives on the surface of the mass shell are continuous, while the second derivatives have discontinuities (inside the shell Poisson's equations hold, outside the shell—the Laplace equations). Following the hydrodynamical analogy, we define a shock wave in the general theory of relativity as a three-dimensional hypersurface on which some second derivatives of the components of the metric $g_{\alpha\beta}(x)$ have discontinuities.

The equations of the shock waves have the form

$$g^{\alpha\beta} f_{,\alpha}\ f_{,\beta} = 0.$$

It can be readily shown that in the case of the Einstein field equations in vacuum (1) (see [6], V, Sec. 7) not all second derivatives are uniquely determined by these equations, a fact which leads to the possibility of shock waves existing in vacuum.

Thus, if we wish to seek a solution with shock waves for Eq. (1), then it is essential to assume that these solutions are being sought within the following class of functions: 1) $g_{\alpha\beta}(x)$ of class C^1 (the $g_{\alpha\beta}$ admit continuous first derivatives), and 2) $\partial_\gamma g_{\alpha\beta}$ ($\equiv \partial g_{\alpha\beta}/\partial x_\gamma$) are piecewise differentiable classes C^2 (the derivatives $\partial_{\gamma\delta} g_{\alpha\beta}$ and $\partial_{\gamma\delta\lambda} g_{\alpha\beta}$ exist but are not everywhere continuous).

If instead of satisfying these conditions we seek solutions of the field equations of type (1) in the class of functions C^2, C^∞, or C^a, then it cannot be guaranteed that some wave solutions of a more general type will not result at the same time.

We shall apply these ideas to the analysis of the solutions of the equations of a centrally symmetric field in vacuum, and we shall show that the method used in the proof of Birkhoff's statement is based on the use of solutions sought at least in class C^2, and consequently wave solutions in the above sense are discarded beforehand. Subsequently we shall find solutions with the imposition of minimum conditions on the class of admissible solutions, and we shall show that they are more general than the Schwartzschild metric or metric (3). The impossibility of a rigorous proof of Birkhoff's statement without violating the Lichnerowicz conditions was first noted by Unt. [15]

2. ANALYSIS OF THE PROOF OF BIRKHOFF'S THEOREM

As is well known, the metric of a centrally symmetric field (generally a nonstatic field) can be written [5] in the form

$$e ds^2 = A dx^{1^2} + 2B dx^1 dx^4 + C\,(dx^{2^2} + \sin^2 x^2 dx^{3^2}) + D dx^{4^2}, \tag{4}$$

where A, B, C, and D are some functions of x^1 and x^4, and the metric is determined with accuracy up to the coordinate transformations:

$$x^{1'} = f(x^1, x^4), \qquad x^{2'} = x^2 + k\pi,$$
$$x^{3'} = x^3 + s\pi, \qquad x^{4'} = \theta(x^1, x^4), \tag{5}$$

where k and s are whole numbers, and f and θ are arbitrary functions of their arguments; $e = \pm 1$ is the so-called "indicator" of ds^2, chosen such that the linear element ds be real for arbitrary choice of four-direction (dx^α).

Numerous methods exist for obtaining the metric (4) starting from various assumptions. Here we also mention that this form of the metric can be obtained starting from purely group invariant concepts, as has been shown in the work of the author [9] and of Eisland.[16] For what follows it is important to note that the reduction of the metric of centrally symmetric fields to the form (4) can always be carried out only under the assumption that $g_{\alpha\beta}(x)$ belongs to class C^1 functions. (This can, for instance, be shown if it is taken into account that the necessary and sufficient condition for the existence of the three-membered group of motion is expressed by Killing's equations [9] in which only first derivatives of $g_{\alpha\beta}(x)$ enter.) In exactly the same way one can check in a trivial manner that the only form of coordinate transformations in which the structure of (4) is retained will be determined by formula (5) where f and θ will be at least of class C^1.

Any proof of Birkhoff's statement reduces to the following two stages, with accuracy up to the choice of a coordinate system: 1) by utilizing the arbitrariness in the choice of the functions f and θ of the transformations (5), the nondiagonal term B of metric (4) is caused to vanish, and the component C is reduced to the form $\pm x^{1^2}$ [or $\pm x^{4^2}$ in the case of metric (3)]; 2) by writing the field equations in vacuum (1), one obtains a Schwartzschild-type metric [or (3)] as a solution of this system.

This method is correct, but we shall be interested in the class of functions to which the solutions thus obtained belong. We shall reproduce the course of such a proof, starting from the first stage. Assuming that transformation (5) is not degenerate, i.e.,

$$\Delta \equiv f_1\theta_4 - f_4\theta_1 \neq 0 \qquad (f_i \equiv \partial f/\partial x^i), \tag{6}$$

differentiating (5), we can as a consequence of (6) define:

$$dx^1 = \Delta^{-1}(\theta_4 dx^{1'} - f_4 dx^{4'}), \qquad dx^2 = dx^{2'}, \qquad dx^3 = dx^{3'},$$
$$dx^4 = \Delta^{-1}(-\theta_1 dx^{1'} + f_1 dx^{4'}),$$

after which metric (4) will have the form

$$eds^2 = A^* dx^{1'^2} + 2B^* dx^{1'} dx^{4'} + C^*(dx^{2'^2} + \sin^2 x^{2'} dx^{3'^2}) + D^* dx^{4'^2}, \tag{7}$$

where

$$A^* = \Delta^{-2}(A\theta_4^2 - 2B\theta_1\theta_4 + D\theta_1^2),$$
$$B^* = \Delta^{-2}(-A\theta_4 f_4 + Bf_1\theta_4 + Bf_4\theta_1 - D\theta_1 f_1),$$
$$C^* = C, \qquad D^* = \Delta^{-2}(Af_4^2 - 2Bf_1f_4 + Df_1^2),\ e = \pm 1. \tag{8}$$

Now we impose the conditions

$$B^* = 0, \qquad C^* = -x^{1'^2}, \tag{9}$$

thereby completing the first stage of the proof [in the case of metric (3) we must impose the condition $C^* = -x^{4'^2}$; in all other respects the whole reasoning is literally repeated]. By virtue of (8) the first of these conditions is equivalent to the equation

$$A\theta_4 f_4 - B(f_1\theta_4 + f_4\theta_1) + D\theta_1 f_1 = 0 \tag{10}$$

and the second will as a consequence of (5) have the form

$$C = -x^{1'^2} = -f^2, \qquad f = e^*\sqrt{-C}, \qquad e^* = \pm 1, \tag{11}$$

i.e.,

$$f_1 = -e^* C_1/2\sqrt{-C}, \qquad f_4 = -e^* C_4/2\sqrt{-C}. \tag{12}$$

Therefore, substituting for f_1 and f_4 in (10), we arrive at the unique equation

$$(BC_1 - AC_4)\theta_4 + (BC_4 - DC_1)\theta_1 = 0. \tag{13}$$

It follows from (12) that if $C_1 = C_4 = 0$ in some region, then condition (6) is not fulfilled and the transformation will be degenerate. However, it is easy to verify that this case contradicts conditions (1) and therefore does not merit investigation.[10]

The linear equation (13) with first-order partial derivatives is consistent when the known conditions of the existence theorem and the uniqueness of the ordinary equation

$$dx^1/(BC_4 - DC_1) = dx^4/(BC_1 - AC_4)$$

are satisfied.

However, in view of the fact that we are not interested in the uniqueness of the solution, and the only important aspect is the existence of any solution, the requirements presented in (13) can even be relaxed and it is sufficient to require simply that the coefficients of θ_1 and θ_4 in (13) be continuous.[17] This means that the function C belongs to the class C^1 (which still does not exclude the possibility of wave solutions). Choosing θ as the integral of (13), we bring the metric (7) to the form

$$eds^2 = A^* dx^{1'^2} - x^{1'^2}(dx^{2'^2} + \sin^2 x^{2'} dx^{3'^2}) + D^* dx^{4'^2}. \tag{14}$$

After that, writing Eqs. (1), we obtain without difficulty the Schwarzschild metric, as usual. Let us estimate the solution and show that it is certainly not a wave solution. In fact, as has been shown in Sec. 1, in order for the metric (14) not to exclude wave solutions upon integration of Eqs. (1), it is necessary that in any case the coefficients A^* and D^* be functions of the class C^1, i.e., that they admit continuous first partial derivatives. However, when we substitute for f_1, f_4, and θ_4 with the aid of (12) and (13) it follows from Eqs. (8) that, for example,

$$D^* = (BC_1 - AC_4)^2/C_1^2(DC_1^2 - 2BC_1C_4 + AC_4^2)\,\theta_1^2.$$

In order for the function D^* to be of class C^1 it is necessary that C_{11}, C_{14}, and C_{44} exist and be continuous, i.e., returning to the metric (4) one can state that some of the functions A, B, C, and D are assumed to belong to class C^2.

Thus the method of proving Birkhoff's statement is essentially based on the assumption that some components of the metric (4) belong to the class C^2, i.e., generally speaking, they may not admit wave solutions. Under this assumption the method is entirely correct, but only with this stipulation.

In order to find a metric which does not exclude wave solutions (in the sense of Sec. 1) it is necessary to impose instead of conditions (9) other conditions which would be satisfied for the functions A, B, C, and D of class C^1 and A_i, B_i, C_i, and D_i —piecewise differentiable functions of class C^2.

3. CENTRALLY SYMMETRIC FIELD IN VACUUM, ADMITTING SHOCK WAVES

We consider the metric (4), and carrying out no coordinate transformations for the time being, we take C outside the brackets; after this, the metric can be written in the form

$$eds^2 = C(x^1, x^4)\,[\mathrm{I}(x^1, x^4) + \mathrm{II}(x^2, x^3)], \qquad (15)$$

where each of the quadratic binary forms

$$\mathrm{I}(x^1, x^4) = \frac{1}{C}[A dx^{1^2} + 2B dx^1 dx^4 + D dx^{4^2}],$$

$$\mathrm{II}(x^2, x^3) = -dx^{2^2} - \sin^2 x^2 dx^{3^2} \qquad (16)$$

depends only on two variables, different for each form, and has respectively signatures of the form $(-+)$ and $(--)$.

Such metrics can be called conformally-reducible and, using transformation (4), the first of these forms can be written in the form

$$\mathrm{I}(x^{1'}, x^{4'}) = \sigma(\gamma^2 dx^{1'^2} - dx^{4'^2}), \qquad \sigma = \pm 1. \qquad (17)$$

It is extremely important to note that unlike in the case of transformations (9), the writing of I in the form (17) does not lead to the requirement of increasing the class of differentiability of the components A, B, C, and D.

In fact, since the only admissible transformation will as before be (4), we can repeat the entire reasoning of the preceding section and reduce the form $\mathrm{I}(x^1, x^4)$ to

$$\mathrm{I}(x^{1'}, x^{4'}) = \sigma(A^* dx^{1'^2} + 2B^* dx^{1'} dx^{4'} + D^* dx^{4'^2}), \qquad (18)$$

where

$$A^* = (A\theta_4^2 - 2B\theta_1\theta_4 + D\theta_1^2)/C\Delta^2,$$
$$B^* = -(A\theta_4 f_4 - B\theta_4 f_1 - B\theta_1 f_4 + D\theta_1 f_1)/C\Delta^2,$$
$$D^* = (Af_4^2 - 2Bf_4 f_1 + Df_1^2)/C\Delta^2.$$

Consequently, to reduce form (18) to (17) it is necessary and sufficient to satisfy the conditions

$$A\theta_4 f_4 - B(\theta_4 f_1 + \theta_1 f_4) + D\theta_1 f_1 = 0,$$
$$Af_4^2 - 2Bf_4 f_1 + Df_1^2 = -C\sigma\Delta^2, \qquad \Delta = f_1\theta_4 - f_4\theta_1. \qquad (19)$$

The first of these equations, homogeneous with respect to the derivatives of θ and f, allows one to determine explicitly the value of $t = f_4/f_1$ (or $1/t$). The second equation is a quadratic polynomial in t. Consequently, excluding t, we arrive at one first-order partial differential equation in θ_1 and θ_4, (which reduces to a linear equation). The conditions for the existence of a solution of this equation lead to the continuity condition for A, B, C, and D. Choosing the sign of σ, we obtain a real solution. Consequently, such a transformation is possible and can be carried out if the metric (4) belong to the class C^1: the possibility of the appearance of wave solutions is not excluded beforehand. The radical difference between the reduction of metric (4) to (14) and to the metric (17) consists in the fact that in the first instance a functional and not a differential requirement $(-x^{1'^2} = C)$ enters into the coordinate transformation; this requirement introduces after differentiation an assumption about the second derivatives of C, whereas in the second instance this is not required.

Thus, in the class of functions C^1 one can reduce the metric (4) of a centrally symmetric field to the form

$$eds^2 = \alpha[\sigma(\gamma^2 dx^{1^2} - dx^{4^2}) - dx^{2^2} - \sin^2 x^2 dx^{3^2}], \qquad (20)$$

where

$$\alpha = \alpha(x^1, x^4), \qquad \sigma = \pm 1, \qquad \gamma = (x^1, x^4)$$

and the signature of the metric for arbitrary value of σ will be $(---+)$.

Since any centrally symmetric field is conformally reducible, one can make use of the results obtaining for such fields in the case of empty space. Such a problem has been solved for arbitrary Einstein spaces (for n = 4 and a Minkowski-type signature). The formulation of the problem, the method of solution, and investigations of some of the possible cases are given by the author in [9] Sec. 46 and [18]. A generalization for field equations $R_{\alpha\beta} = \kappa g_{\alpha\beta}$ $(\kappa \neq 0)$, and an investigation of cases that have not been considered is given in [19].

For an arbitrary conformally reducible gravitational field in vacuum the following statements obtain.

1) If the metric of a conformally reducible field

$$eds^2 = \alpha\,[\pm(\gamma^2 dx^{1^2} - dx^{4^2}) - (dx^{2^2} + \beta^2 dx^{3^2})],$$

$$\alpha = \alpha(x^1, x^2, x^3, x^4), \qquad \gamma = \gamma(x^1, x^4), \qquad \beta = \beta(x^2, x^3) \quad (21)$$

satisfies the Einstein levels in vacuum (1), then

$$\alpha^{-1} = \psi(x^1, x^4) + \varphi(x^2, x^3). \quad (22)$$

2) Depending on the form of the functions φ, ψ, β, and γ, several different possible cases are distinguished. In order to determine which of these cases is possible for a centrally symmetric field in vacuum we shall note the specific features that distinguish metric (20) from (21).

From (20) and (22) we see that the metric (20) has the following singularities:

$$\varphi = \text{const}, \qquad \beta = \sin x^2.$$

These two conditions determine the type of possible solution uniquely (see [9], p. 373). We thus have the following result: if the components of a metric tensor of a centrally symmetric field in vacuum are functions of class C^1, and their first derivatives—piecewise differentiable functions of class C^2, then the general form of the metric of such a field, in the special coordinate system, can be written in the form

$$eds^2 = \psi^{-2}\,[\sigma(\gamma^2 dx^{1^2} - dx^{4^2}) - dx^{2^2} - \sin^2 x^2 dx^{3^2}],$$

$$\sigma = \pm 1, \quad (23)$$

where the functions ψ and γ depend only on the variables x^1 and x^4 and are defined by the equations:

$$\psi_1 = \nu\gamma, \qquad \psi_4^2 = C_1\psi^3 + \sigma\psi^2 + \nu^2, \qquad \gamma_4\psi_4 - \gamma\psi_{44} = \nu', \quad (24)$$

where $\nu(x^1)$ is an arbitrary function of x^1, C_1 = const, and $\sigma = \pm 1$. Equations (24) cannot, in general, be integrated in terms of elementary functions, and lead to known solutions only for certain ψ and γ (cf. Sec. 4).

We note that the process of integration of the field equations (1) for the metric (20) is based solely on the assumption that the $g_{\alpha\beta}(x)$ have second derivatives, but does not require their continuity in the whole investigated region, since the order of the class of functions $g_{\alpha\beta}(x)$ is not increased.

From the definition of the wave solutions given in Sec. 1 it follows that the metric (23) can admit wave solutions of the shock-wave type.

4. INVESTIGATION OF THE SOLUTION WITH SHOCK WAVES

First we note that, depending on the sign of σ ($\sigma = \pm 1$), the role of the coordinate time in (23) will be taken on by the variable x^1 or x^4 [in analogy with the Schwarzschild metric and the nonstatic metric (3) in the case of solutions of class C^2]. One can also readily verify by direct calculation that the metric (23) turns the field equations in vacuum (1) into an identity if conditions (24) and their differential consequences are used:

$$\psi_{44} = \tfrac{3}{2}C_1\psi^2 + \sigma\psi, \qquad \gamma_4\psi_4 = \nu' + \gamma\psi(\tfrac{3}{2}C_1\psi + \sigma),$$
$$\psi_{11} = \nu'\gamma + \nu\gamma_1, \qquad \gamma_{44} = \gamma(3C_1\psi + \sigma). \quad (25)$$

One can readily check that the metric (23) can for ψ = const be reduced to the Minkowski metric. If the constant C_1 equals zero, then as a consequence of (24) and (25) all the components of the curvature tensor vanish: there is no gravitational field. Consequently, the constant $C_1 \neq 0$ indeed characterizes the mass producing the field.

If $\nu = 0$, then it is readily seen that the field equations can be integrated in terms of elementary functions*

$$\psi = \frac{1}{C_1}\left[\text{th}^2\left(\lambda - \frac{x^4}{2}\right) - 1\right], \quad \text{if} \quad \sigma = 1,$$
$$\psi = \frac{1}{C_1}\left[\text{tg}^2\left(\lambda + \frac{x^4}{2}\right) + 1\right], \quad \text{if} \quad \sigma = -1,$$

where $\lambda = \lambda(x^1)$.

In addition, it is easy to check that in this case $g_{\alpha\beta}(x)$ is reduced for certain transformations of the class C^2 to a form in which continuous second derivatives of $g_{\alpha\beta}(x)$ are admitted; but this means that the metric can be reduced either to the Schwarzschild metric ($\sigma = 1$) or to the quasistatic metric (3).

Thus, solutions with shock waves are possible only in the case when C_1 and ν are different from zero. But in this case the third of Eqs. (24) becomes an identity as a consequence of the first two equations of this system and of their differential consequences (25) which assume the existence but not the continuity of the second derivatives. Then, if it is assumed that

$$\psi = \frac{1}{C_1}\left(4\wp - \frac{\sigma}{3}\right), \qquad \sigma = \pm 1, \quad (26)$$

the second of Eqs. (24) is reduced to the form

$$\wp_4^2 = 4\wp^3 - g_2\wp - g_3, \quad (27)$$

where

*th = tanh; tg = tan.

$$g_2 = -\frac{1}{12}, \qquad g_3 = \frac{C_1^2}{16}\left(\nu + \frac{2\sigma}{27C_1^2}\right). \qquad (28)$$

Equation (27) means that (cf. [20]) $\wp$ is an elliptical Weierstrass function: $\wp(x^4 + \lambda)$, where $\lambda = \lambda(x^1)$, and g_2 and g_3 are the so-called invariants of the function $\wp$. The first equation of (24) serves as the definition of the function γ.

Summarizing, one can state that the general form of the metric of a centrally-symmetric field in vacuum admitting shock waves in a special system of coordinates can be written down in the form (23), where ψ and γ depend only on x^1 and x^4 and are expressed in terms of the Weierstrass elliptical function $\wp(x^4 + \lambda)$ (27) with the invariants (28).

It is obvious how one can express $\wp$ in terms of other elliptical functions. We note that all these results are obtained in a particularly simple fashion if one uses the method of the Lie group theory of motion in Riemann spaces but in the class C^1 of functions.

The equation of the hypersurface on which discontinuities of the second derivatives of the metric tensor, i.e., of the components of the curvature tensor determining the gravitational field, are possible, will at the same time be the equation of the shock wave and will for the metric (23) be written in the form

$$\frac{\sigma}{\gamma^2}(\partial_1\omega)^2 - (\partial_2\omega)^2 - \frac{1}{\sin^2 x^2}(\partial_3\omega)^2 - \sigma(\partial_4\omega)^2 = 0.$$

It is thus not excluded that centrally-symmetric pulsations of the gravitating masses can produce gravitational shock waves—a conclusion which is interesting not only for cosmological problems.

Let us note that the function ψ is in a certain sense an analog of the "radius vector" r for the Schwartzschild metric. When $\psi \to \infty$, the curvature of space-time differs by an arbitrarily small amount from zero (the curvature tensor tends to zero).

It is also easy to verify that any solution (23) of the field equations in vacuum belongs to gravitational fields of the type I, following the classification proposed by the author.

It follows directly from (23) that such a field admits in vacuum a four-membered group of motions with the operators

$$X_s f \equiv \underset{s}{\xi}^\alpha \partial_\alpha f, \qquad (\alpha, s = 1, \ldots, 4),$$

where*

$$\underset{1}{\xi}^\alpha = \delta_2^\alpha \cos x^3 - \delta_3^\alpha \sin x^3 \operatorname{ctg} x^2,$$

$$\underset{2}{\xi}^\alpha = \delta_2^\alpha \sin x^3 + \delta_3^\alpha \cos x^3 \operatorname{ctg} x^2,$$

$$\underset{3}{\xi}^\alpha = \delta_3^\alpha, \qquad \underset{4}{\xi}^\alpha = -\frac{\psi_4}{\gamma}\delta_1^\alpha + \nu\delta_4^\alpha$$

and it is not permissible to reduce in the class of functions C^2 the fourth operator to the form δ_1^α (or δ_4^4) which is characteristic of static and quasi-static metrics.

*ctg = cot.

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