SPECTRAL REPRESENTATIONS FOR THE FIVE-POINT FUNCTION IN PERTURBATION THEORY

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The analytical properties are studied of the inelastic scattering amplitude corresponding to a fifth order loop diagram with arbitrary masses on the lines. A double spectral representation of the Mandelstam type is obtained. Several restrictions on the masses and the fixed invariants, necessary for this representation to hold, are found. In the physical region even the weakest of these restrictions are violated and the representation does not hold owing to the presence of singularities in the complex plane of the spectral invariant.

1. INTRODUCTION

LET us consider the matrix element for a process in which i initial particles go into f final particles (for simplicity we deal throughout with scalar particles):

$$\langle f|S|i\rangle = \delta_{fi} - 2\pi i\delta(\Sigma p_i - \Sigma p_f)\prod_{i,f}(4E_iE_f)^{-1/2}F_{fi}(p_i, p_f).$$

The amplitude F_{fi} for this process is a relativistic invariant; it can be a function of only relativistic invariants constructed out of the four-momenta of the initial and final particles. Altogether there are $2^{i+f-1} - 1$ such invariants; the requirement that the momenta be on the mass shell and the conservation laws leave independent $3(i+f) - 10$ of these invariants.

The unitarity of the S matrix allows one to show where the Im F_{fi} is certainly different from zero: as a function of the total energy (or its square) in the barycentric system F_{fi} has poles corresponding to single-particle intermediate states, and a ("physical") cut corresponding to multiparticle intermediate states in the unitarity relation.

As is well known, in the dispersion-relations method F_{fi} is studied as a function of the invariants as the latter vary in their complex planes. It is assumed that the amplitudes for all crossed processes that can be obtained from the original by all possible permutations of initial and final particles are different, generally speaking, boundary values of one common analytic function F. Since each invariant will turn out to be the square of the total energy for some one crossed process, it follows from the unitarity relation that F will have singularities corresponding to poles and physical cuts of each invariant. It seems very probable and desirable that the analytic properties of F should consist of just these singularities so that the analytic structure of F could be represented on the appropriate sheet by a multiple, generally speaking, Cauchy integral.

The usual methods of quantum field theory make it possible to establish spectral representations of F_{fi} in only one of the invariants and in a very limited region of physical values for the remaining invariants, and only for processes where the number of particles does not exceed four (see, however, [1]). For this reason a number of authors [2-4] have studied the analytic properties of contributions to the amplitude from perturbation theory diagrams. Methods have been developed that make it possible to obtain a spectral representation for contribution from loop diagrams of third and fourth order of perturbation theory. (We call loop diagrams those diagrams for which the internal lines form one closed loop.)

We address ourselves here to the study of loop diagrams of fifth order. The question of the existence, in the physical region of a process with five or more particles, of spectral representations (even if only double) is of particular significance. Double spectral representations are needed even for the convergence of the partial wave expansion of the inelastic amplitude, as a consequence of the more complicated kinematics of the inelastic process. A method of investigation analogous to that used by Vladimirov [4] in application to loop diagrams of fourth order results in a single (13) and double (23) dispersion relation. By analytic continuation in the invariants a number of necessary conditions are found for the existence of these relations. The weakest of these restrictions—conditions of stability—are violated in the physical region of the process: 2 particles → 3 particles. As a result even the single dispersion relations have complex cuts and poles.

Part of our results has been obtained by Kostyrko.[5] After the present work in its first version was submitted for publication the authors became acquainted with the work of Cook and Tarski,[6] in which the same problem is solved by a different method.

2. LOOP DIAGRAM OF FIFTH ORDER

It is convenient to start the investigation from the α parametrized representation of the contribution to the amplitude of the loop diagram (Fig. 1a):

$$F^{(5)}(x_{ij}) = \int_0^\infty \Big(\prod_{k\leqslant 5} d\alpha_k\Big)\,\delta\Big(1-\sum_{k\leqslant 5}\alpha_k\Big) D^{-3}(\alpha_k, x_{ij}),$$

$$D(\alpha_k, x_{ij}) = 1 + 2\sum_{i<j\leqslant 5}\alpha_i\alpha_j x_{ij}. \qquad (1)$$

The advantage of this representation[1] lies in the fact that $F^{(5)}$ explicitly depends only on invariant combinations of external 4-momenta:

$$x_{ij} = (m_i - m_j)^2 - p_{ij}^2/2m_i m_j.$$

All of the invariants encountered in Eq. (1) are listed in Fig. 1b. Encircled are those that, as a consequence of the mass shell condition

$$p_{ij}^2 = m_{ij}^2, \qquad (i, j) = (1, 2);\ (2,3);\ (3,4);\ (4, 5);\ (5,1),$$

are constant ("mass" invariants). The remaining five invariants defined by the equations

$$p_{35}^2 = (p_{34}+p_{54})^2,\ \ p_{25}^2 = (p_{12}+p_{51})^2,\ \ p_{24}^2 = (p_{23}+p_{34})^2,$$
$$p_{14}^2 = (p_{15}+p_{45})^2,\ \ p_{13}^2 = (p_{12}+p_{23})^2$$

are variable ("dynamic" invariants).

In place of using the form, Eq. (1), we express $F^{(5)}$ as

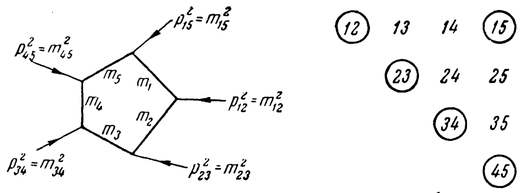
$$F^{(5)}(x_{ij}) = 2^{-1}\frac{\partial^2}{\partial\lambda^2}F_\lambda(x_{ij})\big|_{\lambda=1};\quad F_\lambda = \int_{T_4} d\alpha_1\, d\alpha_2\, d\alpha_3\, d\alpha_4\, D_\lambda^{-1};$$

$$D_\lambda = \lambda + 2\sum_{i<j\leqslant 5}\alpha_i\alpha_j x_{ij}, \qquad \alpha_5 = 1-\alpha_1-\alpha_2-\alpha_3-\alpha_4. \qquad (2)$$

It is seen that after the integration over α_5 is carried out with the help of the delta function the region T_4 of integration over the remaining parameters consists of the single simplex

$$T_4 : \alpha_k \geqslant 0, \quad \alpha_1+\alpha_2+\alpha_3+\alpha_4 \leqslant 1.$$

Let us consider first the analytic properties of F_λ with respect to x_{35} aiming at the establishing



a b

FIG. 1

of a spectral representation in x_{35}. It is easy to see that for

$$\alpha_3\alpha_5 \geqslant \varepsilon > 0, \qquad \alpha_j \in T_4, \qquad (3)$$

$$x_{ij} > 0 \quad (i, j) \neq (3,5)\ ^{1)}. \qquad (4)$$

D_λ^{-1} is analytic in the complex x_{35} plane cut from $-\infty$ to -2λ and falls off at ∞ like $|x_{35}|^{-1}$. Consequently,

$$D_\lambda^{-1}(x_{35}) = \pi^{-1}\int_{-\infty}^{-2\lambda}\delta(D')\,dx_{35}'/(x_{35}'-x_{35}). \qquad (5)$$

Let us integrate both sides of Eq. (5) over $T_{4\epsilon}$ (distinguished from T_4 by the condition (3)), interchange the order of integration, and take the limit $\epsilon \to 0$. We then obtain

$$F_\lambda = \pi^{-1}\int_{-\infty}^{-2\lambda}\Delta F_\lambda\, dx_{35}'/(x_{35}'-x_{35}), \qquad (6a)$$

$$\Delta F_\lambda = \int_{T_4} d\alpha_1\, d\alpha_2\, d\alpha_3\, d\alpha_4\,\delta(D'). \qquad (6b)$$

(The prime on D' means that its spectral variable is $x_{35} = x_{35}' < -2\lambda$; in cases where no confusion is likely the prime will be omitted.)

In order to evaluate ΔF_λ we express D_λ in the form $D_\lambda = c_4\alpha_4^2 + 2b_4\alpha_4 + a_4$. Here, obviously,

$$c_4 = -2x_{45}, \qquad b_4 = x_{45} + \sum_{k\leqslant 3}\alpha_k H_{k4,5},$$

$$a_4 = \lambda + 2\sum_{k\leqslant 3}x_{k5}(\alpha_k - \alpha_k^2) + 2\sum_{i<j\leqslant 3}\alpha_i\alpha_j H_{ij5},$$

where for brevity we have set

$$H_{ijk} = x_{ij} - x_{ik} - x_{jk}.$$

Having in mind integration over α_4 with the help of the δ-function in Eq. (6b) we note that $D_\lambda = 0$ at the points $\alpha_4^{\pm}$, where

1) These conditions have no physical content whatsoever (in contrast to, for example, the conditions that are imposed on the mass invariants corresponding to stability of the masses on the diagram lines $x_{ij} > -2$). Since our immediate goal is the finding of at least the form of the representation on the physical sheet we use these conditions to define a region of the physical sheet where it is easiest to achieve this goal. All these conditions simplify enormously proofs of useful relations of positive- or negative-definiteness type.

$$\alpha_4^{\pm} = b_4 \mp \sqrt{\Delta}/2x_{45},$$

$$\Delta \equiv b_4^2 - a_4 c_4 = 2\lambda x_{45} + x_{45}^2 - 2x_{45}\sum_{k\leqslant 3}\alpha_k H_{45k} + \sum_{k\leqslant 3}\alpha_k^2 C_{k45} + 2\sum_{i<j\leqslant 3}\alpha_i\alpha_j M_{ij45}, \qquad (7)$$

where we have introduced the notation

$$C_{ijk} = x_{ij}^2 + x_{jk}^2 + x_{ki}^2 - 2\,(x_{ij}x_{jk} + x_{ij}x_{ki} + x_{jk}x_{ki}),$$

$$M_{ijkl} = 2x_{kl}H_{ijl} + H_{ikl}H_{jkl}.$$

It turns out that for

$$x_{14} > x_{15}, \qquad x_{24} > x_{25} \qquad (8)$$

(see first footnote) all $\Delta > 0$ and there exist two different roots α_4^+, α_4^-. At that α_4^- always lies outside T_4, and α_4^+ lies inside T_4 if and only if $a_4 \leq 0$. Therefore

$$\Delta F_\lambda = \int d\alpha_1 d\alpha_2\, d\alpha_3 \frac{d\alpha_4 \delta\,(\alpha_4 - \alpha_4^+)}{|\partial D_\lambda / \partial\alpha_4|} = \int d\alpha_1\, d\alpha_2\, d\alpha_3 \Delta^{-1/2}. \quad (9)$$

The region of integration in Eq. (9) is given by the intersection of T_4 with the region $a_4 \leq 0$. To find it let us consider $a_4 = a_4(\alpha_1, \alpha_2, \alpha_3)$. It can be shown that the conditions

$$x_{13} > x_{15}, \qquad x_{23} > x_{25}, \qquad \sqrt{x_{12}} > |\sqrt{x_{15}} \pm \sqrt{x_{25}}| \quad (10)$$

(see first footnote) are sufficient to make the limits of integration over α_3 in Eq. (9) equal to the roots $\alpha_3^{\mp}$ of the trinomial $a_4(\alpha_3)$. These roots lie inside T_3 if they are real. The condition for their reality is given by the inequality

$$d = d\,(\alpha_1, \alpha_2, \lambda) \geqslant 0 \qquad (11)$$

[where $d(\alpha_1, \alpha_2, \lambda)$ is the discriminant of the trinomial $a_4(\alpha_3)$], which thus defines the limits of integration over α_1 and α_2.

Thus the region of integration in Eq. (9) is defined by

$$\alpha_3^- < \alpha_3 < \alpha_3^+, \quad 0 < \alpha_2 < \alpha_2^*, \qquad 0 < \alpha_1 < \alpha_1^*. \qquad (12)$$

Here $\alpha_2^* = \alpha_2^*(\alpha_1, \lambda)$ and $\alpha_1^* = \alpha_1^*(\lambda)$ are the smallest roots of respectively the equations $d(\alpha_1, \alpha_2) = 0$ and $d(\alpha_1, 0) = 0$.

By direct examination it is easy to convince oneself that the quantities $\alpha_3^{\mp}$, α_2^*, α_1^* satisfy the following essential conditions:

$$\alpha_1^*(1)|_{x_{35}=-2} = 0, \quad \alpha_2^*(\alpha_1^*, \lambda) = 0,$$

$$\alpha_3^+(\alpha_1, \alpha_2^*, \lambda) = \alpha_3^-(\alpha_1^*, \alpha_2^*, \lambda).$$

These conditions are sufficient to justify the assertion that if F_λ is defined by Eqs. (6a), (9), and (12) then the differentiation with respect to λ in Eq. (2) need be performed only in the integral over α_3:

$$\frac{\partial^2}{\partial\lambda^2}F_\lambda|_{\lambda=1} = \frac{\partial^2}{\partial\lambda^2}\int_{-\infty}^{-2\lambda}\frac{dx'_{35}}{x'_{35}-x_{35}}\int_0^{\alpha_1^*(\lambda)}d\alpha_1\int_0^{\alpha_2^*(\alpha_1,\lambda)}d\alpha_2\int_{\alpha_3^-}^{\alpha_3^+}d\alpha_3\Delta^{-1/2}|_{\lambda=1}$$

$$= \int_{-\infty}^{-2}\frac{dx'_{35}}{x'_{35}-x_{35}}\int_0^{\alpha_1^*(1)}d\alpha_1\int_0^{\alpha_2^*(\alpha_1,1)}d\alpha_2\left[\frac{\partial^2}{\partial\lambda^2}\int_{\alpha_3^-(\alpha_1,\alpha_2,\lambda)}^{\alpha_3^+(\alpha_1,\alpha_2,\lambda)}d\alpha_3\Delta^{-1/2}\right]_{\lambda=1}.$$

For the integration over α_3 let us write

$$b_4^2 - c_4 a_4 \equiv \Delta = c_3\alpha_3^2 + 2b_3\alpha_3 + a_3, \quad c_3 = C_{345},$$

$$b_3 = -x_{45}H_{453} + \sum_{k\leqslant 2}\alpha_k M_{k345}$$

and, remembering that everywhere in T_3 $\Delta > 0$, obtain

$$\int_{\alpha_3^-}^{\alpha_3^+} d\alpha_3\Delta^{-1/2} = c_3^{-1/2}\ln\{\sqrt{c_3}\alpha_3 + (b_3/\sqrt{c_3}) + \sqrt{\Delta}\}\Big|_{\alpha_3^-}^{\alpha_3^+} = c_3^{-1/2}\ln(X^+/X^-)$$

[where we took into account that $a_4(\alpha_3^{\pm}) = 0$ and, consequently, $\Delta(\alpha_3^{\mp}) = b_4^2$]. After transforming the fraction in the argument of the logarithm we find

$$X^{\pm} = f\,(\alpha_1, \alpha_2) \pm \sqrt{c_3 d\,(\alpha_1, \alpha_2, \lambda)},$$

$$f\,(\alpha_1, \alpha_2) = x_{35}H_{354} - \sum_{k\leqslant 2}\alpha_k M_{k435},$$

and $d(\alpha_1, \alpha_2, \lambda)$ turns out to be the same as in Eq. (11).

3. ANALYTIC CONTINUATION

And so we have obtained the dispersion relation

$$F^{(5)}(x_{ij}) = \frac{1}{2\pi}\int_{-\infty}^{-2}\frac{dx'_{35}}{x'_{35}-x_{35}}\int_0^{\alpha_1^*(1)}d\alpha_1 \times \int_0^{\alpha_2^*(\alpha_1,1)} d\alpha_2 C_{345}^{-1/2}\frac{\partial^2}{\partial\lambda^2}\ln[X^+(\lambda)/X^-(\lambda)]|_{\lambda=1}, \qquad (13)$$

valid under the restrictions (4), (8), and (10). These restrictions can be weakened by analytically continuing the spectral function in the dispersion relation. Let us note that not only the integrand but also the limits of integration over the parameters α_1 and α_2 depend on the invariants x_{ij}, $(i, j) \neq (3, 5)$, that are being continued. The form of the representation (13) will remain unchanged until upon varying x_{ij} one encounters singularities of the integrand. It can be shown that the singularities of the logarithm are encountered before the singularities of the root. We shall therefore take the condition $C_{345} > 0$ as satisfied.

The branch points of the logarithm appear for $f \pm \sqrt{c_3 d} = 0$. To discover the zeros we express $f^2 - c_3 d \equiv \varphi(\alpha_1, \alpha_2)$ in the form $\varphi(\alpha_1, \alpha_2) = c_2\alpha_2^2 + 2b_2\alpha_2 + a_2$. Let us consider the free term $a_2 = a_2(\alpha_1)$. We have

$$a_2(0) = 4x_{35}^2 x_{34} x_{45} - 2\lambda C_{345} \equiv 2x_{35} L_{345}.$$

Let us note that

$$x_{34} > -2, \quad x_{45} > -2 \tag{14}$$

is the condition for the existence of a dispersion relation (with an anomalous threshold, generally speaking) in the invariant x_{35} for the diagram of Fig. 2a, obtained from ours by collapsing the lines 1 and 2 to a point, and the smaller root $x_{35}^-(\lambda)$ of the equation $L_{345} = 0$ is for $\lambda = 1$ the threshold in this dispersion relation. In this connection the condition, Eq. (14), may be given a direct interpretation.

When condition (14) is satisfied $a_2(0) > 0$, this being true for all x'_{35} satisfying $x'_{35} < x_{35}^-(\lambda)$. Moreover, $a_2(\alpha_1^*) = f^2(\alpha_1^*, 0) > 0$. Consequently, on the segment $[0, \alpha_1^*]$ the function $a_2(\alpha_1)$ has either two or no roots, and that number can change only after the discriminant $R(a_2)$ of the trinomial $a_2(\alpha_1)$ changes sign. At the same time for $x'_{35} < -2\lambda$ and $x_{14} \to +\infty$ one has $R(a_2) \to 4x_{35}^2 x_{14}^2 C_{345} d(0, 0) > 0$, and the roots $\alpha_1^\pm$ have opposite signs and, consequently, lie outside $[0, \alpha_1^*]$. This situation will remain unchanged until, with decreasing x_{14}, $R(a_2)$ changes sign for the second time. The equation $R(a_2) = 0$ defines for $\lambda = 1$, as one can show, the curve of singularities for the Mandelstam representation in x_{35}, x_{14} for the diagram of Fig. 2b, which is obtained from our five-point diagram by collapsing the line 2 to a point. If one requires

$$x_{ij} > -2, \quad (i, j) = (1,3);\ (3,4);\ (4,5);\ (5,1) \tag{15}$$

i.e., the stability conditions for the diagram, Fig. 2b, and also

$$\sum_{(i,j)} \arccos(x_{ij} + 1) < 2\pi, \quad (i, j) = (1,3);\ (3,4);\ (4,5);\ (5,1) \tag{16}$$

(the condition for the existence of the Mandelstam representation for the diagram, Fig. 2b), then for

$$x_{14} > x_{14}^{(-)}(\lambda), \tag{17}$$

where $x_{14}^{(-)}(\lambda)$ is the smaller root of the equation $R(a_2) = 0$, one has $a_2(\alpha_1) > 0$ preserved everywhere in $[0, \alpha_1^*]$, i.e., $\varphi(\alpha_1, 0) > 0$.

Further, $\varphi(\alpha_1, \alpha_2^*) = \varphi(0, \alpha_2^*) = f^2(0, \alpha_2^*) > 0$. In a manner analogous to the preceding we conclude that $\varphi(\alpha_2)$ has no roots in $[0, \alpha_2^*]$ till, as

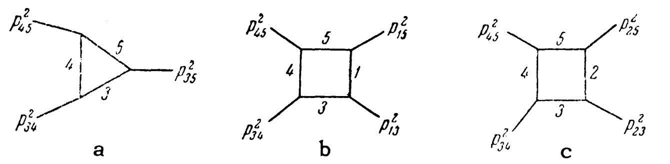


FIG. 2

x_{24} decreases from $+\infty$, the discriminant $R(\varphi)$ changes sign for the second time. It can be shown that for $\alpha_1 = 0$ and $\lambda = 1$ the equation $R(\varphi) = 0$ defines the curve of singularities for the Mandelstam representation in x_{35} and x_{24} for the diagram of Fig. 2c, obtained from ours by collapsing line 1 to a point. If, in analogy to Eqs. (15) and (16), one requires

$$x_{ij} > -2, \quad (i, j) = (2,5);\ (2,3);\ (3,4);\ (4,5) \tag{18}$$

(the conditions of stability for the diagram of Fig. 2c) and

$$\sum_{(i,j)} \arccos(x_{ij} + 1) < 2\pi, \quad (i, j) = (2,5);\ (2,3);\ (3,4);\ (4,5) \tag{19}$$

(the condition for the existence of the Mandelstam representation for the diagram of Fig. 2c), then for

$$x_{24} > x_{24}^{(-)}(\lambda, \alpha_1), \tag{20}$$

where $x_{24}^{(-)}(\lambda, \alpha_1)$ is the smaller root of the equation $R(\varphi) = 0$, one has $\varphi(\alpha_1, \alpha_2) > 0$ everywhere in $[0, \alpha_2^*]$.

Consequently the logarithm in Eq. (13) will have no singularities at least as long as the restrictions (15)—(20), which have simple physical meaning, are satisfied. The representation (13) may be analytically continued to values of x_{ij} satisfying these restrictions. Let us note that the threshold of the representation will, generally speaking, change, becoming dependent on all the other x_{ij}, $(i, j) \neq (3, 5)$. The changed value $x_{35}^{(\mathrm{thr})} > -2$ (anomalous threshold!) and turns out to be equal to

$$x_{35}^{(\mathrm{thr})} = \max_{(\alpha_1)} x_{35}^{(-)}(1, \alpha_1),$$

where $x_{35}^{(-)}(\lambda, \alpha_1)$ is the smaller root of the equation $x_{35}^{-2} R(\varphi) = 0$.

4. DOUBLE DISPERSION RELATIONS

Certain of our restrictions, namely (15), (18), and (20), are not only sufficient but also necessary for the existence of the spectral representation (13). In particular for $x_{24} < \max_{(\alpha_1)} x_{24}^{(-)}(\lambda, \alpha_1)$ and with the remaining restrictions (15)—(19) satisfied, there will be for an arbitrary α_1 in $[0, \alpha_1^*]$ two values $\alpha_2^\pm$ in $[0, \alpha_2^*]$ such that $f^2 - c_3 d = 0$. Since in the

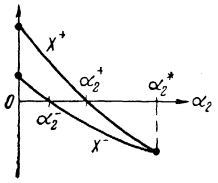


FIG. 3

limit as $x_{24} \to -\infty$ we have $f(\alpha_1^*, 0) > 0$, $f(0, \alpha_2^*) < 0$, and at the end points of the interval $[0, \alpha_2^*]$ $f^2 - c_3 d > 0$, it follows that if only the condition (20) is violated the quantities $X^{\pm} = f \pm \sqrt{c_3 d}$ will have one root each in $[0, \alpha_2^*]$, namely $\alpha_2^{\mp}$ (see Fig. 3). Consequently, the analytic properties of the logarithm in Eq. (13) in the complex variable x_{24} are described by the formula (for $\alpha_2 \geq \epsilon > 0$)

$$\ln (X^+/X^-) = \pi^{-1} \times \int_{-\infty}^{x_{24}^{(-)}(\lambda, \alpha_1)} dx'_{24} \{\theta(-X^+) - \theta(-X^-)\}/(x'_{24} - x_{24}). \qquad (21)$$

Integrating Eq. (21) over α_1 and α_2 in a region differing from (12) by the condition $\alpha_2 \geq \epsilon > 0$, interchanging the order of integration, and taking the limit $\epsilon \to 0$, we find

$$\Delta F_\lambda = (2\pi)^{-1} \int_0^{\alpha_1^*} d\alpha_1 \int_{-\infty}^{x_{24}^{(-)}} dx'_{24} (x'_{24} - x_{24})^{-1} C_{345}^{-1/2} \times \int_0^{\alpha_2^*} \{\theta(-X^+) - \theta(-X^-)\} d\alpha_2$$

$$= \pi^{-2} C_{345}^{-1/2} \int_0^{\alpha_1^*} d\alpha_1 \int_{-\infty}^{x_{24}^{(-)}} dx'_{24} (x'_{24} - x_{24})^{-1} c_2^{-1} \sqrt{R(\varphi)}, \qquad (22)$$

where $R(\varphi)$ may be expressed in the form $R(\varphi) = C_{345}(c_1\alpha_1^2 + 2b_1\alpha_1 + a_1)$. $R(\varphi)$ is positive in the region of integration over α_1 and x'_{24} in Eq. (22), and vanishes on the curved part of that region's boundary. The region of integration is shown in Fig. 4, $x_{24}^{(4)}$ and $x_{24}^{(5)}$ are the thresholds of the spectral representations in x_{24} for respectively the diagram of Fig. 2c and Fig. 1a; $x_{24}^{(4)}$ is the root of the equation $a_1 = 0$ and $x_{24}^{(5)}$ of the equation $b_1 - a_1 c_1 = 0$.

To evaluate the spectral function of the double dispersion relation it remains to change the order of integration in Eq. (22), to differentiate once with respect to λ, to integrate over α_1, and, having differentiated once more with respect to λ, to set $\lambda = 1$. As a result we obtain

$$F^{(5)}(x_{ij}) = \pi^{-2} \int dx'_{35} \int dx'_{24} (x'_{35} - x_{35})^{-1} (x'_{24} - x_{24})^{-1} K_5^{-1}(x_{ij}). \qquad (23)$$

Here the integration is over the connected part of the region $K_5 \geq 0$, containing the values x'_{35}, x'_{24}

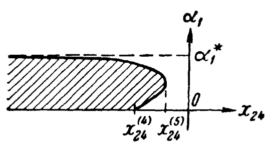


FIG. 4

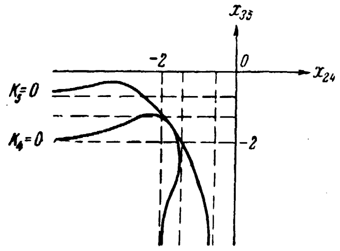


FIG. 5

$\to -\infty$. The possible appearance of this part is shown in Fig. 5. K_5 is defined by the equation $4x_{35}^2 c_2 K_5 = b_1^2 - c_1 a_1$, $\lambda = 1$, and coincides with the determinant constructed out of the x_{ij}:[3]

$$K_5(x_{ij}) = \det \| y_{ij} \| \qquad y_{ij} = x_{ij} + 1. \qquad (24)$$

Since the equation $K_5 = 0$ is quadratic with respect to any one of the x_{ij} it is clear that the singularities of the representation (23) with respect to x_{35} or x_{24} are in the form of simple poles.

5. DISCUSSION

As a result of our study of the amplitude $F^{(5)}(x_{ij})$ we have obtained the spectral representation of the type (13) under the restrictions (15)—(20), or the double spectral representation (23) under the restrictions (15)—(19). It has already been remarked that conditions (15), (17), and (18) are in any event necessary for the existence of both (13) and (23). It can be shown that if any one of them is violated the logarithm in Eq. (13) can have singularities both for a certain continuum of complex values and for isolated complex values of x'_{35}. In other words the cut in Eq. (13) must be extended from the anomalous threshold into the region of complex x'_{35}. In addition the amplitude develops complex poles. The position of the cut, or, more precisely, of the complex branch point at its end, as well as the position of the poles, depends on the values of all the other invariants.

A detailed investigation of the kinematics[7] of the inelastic process considered shows that, for example, in the channel where p_{35}^2 equals the square of the total energy, in the physical region the stability conditions for the dynamical invariants x_{13} and x_{25} [conditions (15) and (18)] are violated. Thus perturbation theory indicates that in the physical region even the one-dimensional spectral representation of the amplitude for multiple production has an extremely complicated form. The program of evaluating the contributions to the absorptive part of the amplitude from the complex cuts and poles is in many respects unclear. Therefore the possibility of finding a region (in part physical) in which the contribution from the complex singularities can be neglected seems rather attractive. Such a possibility has been realized in the work of the authors.[8]

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