ON THE THEORY OF THE FERMI LIQUID

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The ground state energy and other characteristics of a Fermi system are calculated using the method of Green's functions and Landau's theory of Fermi liquids. The calculations are carried out for a potential with a long-range and a short-range part such that $n^{1/3}r_s \ll 1$ and $n^{1/3}r_l \gg 1$, where r_s and r_l are the ranges of the short- and long-range parts and n is the density of particles.

INTRODUCTION

LANDAU'S theory of Fermi liquids permits the calculation of the energy of the system, the effective mass of the excitations, the velocity of zero and ordinary sound, and other characteristics of a Fermi liquid. However, the theoretical formulas contain a number of unknown numerical parameters which can not be determined within the framework of a phenomenological theory. These parameters can be calculated only on the basis of a microscopic theory. Up to now, such calculations have only been carried out in the gas approximation, i.e., for the case where the range of the forces is appreciably smaller than the average distance between the particles $n^{-1/3}$, where n is the density of particles.

In the present paper we calculate the vertex part and the basic characteristics of a Fermi system for the case where the interaction potential consists of two terms V_s and V_l, such that V_s can be treated in the gas approximation, i.e., the range r_s of V_s is much smaller than $n^{-1/3}$, whereas V_l satisfies the opposite condition: $r_l \gg n^{-1/3}$, where r_l is the range of V_l. The magnitudes of V_s and V_l can be arbitrarily large. The method of summing over graphs allows one to find the vertex part $\Gamma_{\alpha\beta,\gamma\delta}(p_1, p_2, p_1+k, p_2-k)$ for $|k|p_0 > \omega_k$, where p_0 is the Fermi momentum. However, the ground state energy and other characteristics of the system are expressed in terms of the vertex part $\Gamma_{\alpha\beta,\gamma\delta}$ in the limit $p_0|k|\omega_k^{-1} \to 0$, $\omega_k \to 0$, i.e., $\Gamma^{\omega}_{\alpha\beta,\gamma\delta}$. To calculate $\Gamma^{\omega}_{\alpha\beta,\gamma\delta}$, we use the basic relations of the theory of Fermi liquids.

Fermi liquids with potentials consisting of two terms with $r_s \ll n^{-1/3}$ and $r_l \gg n^{-1/3}$ have already been studied earlier in their application to the nucleus.[1,2] Vagradov and Kirzhnits[1] have taken account of the long-range force V_l with the help of the perspicuous but inaccurate method of the self-consistent field in configuration space. Amus'ya[2] calculated the energy of the system by the graph technique, but he assumed that the expression for the vertex part of the long-range forces found by summation of graphs is valid for all **k** and ω. Actually, this expression is correct only when $p_0|\mathbf{k}| > \omega$ (see below). Moreover, in [2] no account was taken of the graph Γ_c, which is of the same order of magnitude as the graphs Γ_a and Γ_b [see formula (7)]. Both in [1] and [2] V_l is an attractive potential, so that the normal state of the system is unstable (see below).

THE VERTEX PART FOR $p_0|\mathbf{k}| > \omega$

In order to calculate the energy spectrum of the system we must know the vertex part (box diagram) $\Gamma_{\alpha\beta,\gamma\delta}(p_1p_2, p_3p_4)$. Let us assume first that $V = V_s$, $V_l = 0$. Let us compare the ladder graph of second order (ladder with two rungs, Fig. 1a) with the other nonvanishing graph of second order, viz., the chain with two dotted lines (Fig. 1b). In the first case the matrix element is equal to

$$M_a \sim \int G_0(p_3-p)\, G_0(p_4+p)\, V(k-p)\, V(p)\, d^4p$$

$$\sim V^2(0) r_s^{-1},$$

$$k = p_2 - p_4 \equiv p_3 - p_1, \qquad p = \{\mathbf{p}, \omega\},$$

$$G_0(p) = [\omega - \varepsilon_0(\mathbf{p}) + \mu + i\delta \operatorname{sgn}(|\mathbf{p}| - p_0)]^{-1};$$

μ is the chemical potential, $\varepsilon_0(\mathbf{p}) = \mathbf{p}^2/2$; $\hbar = m = 1$; $p_0 = (3\pi^2 n)^{1/3}$.

In the second case

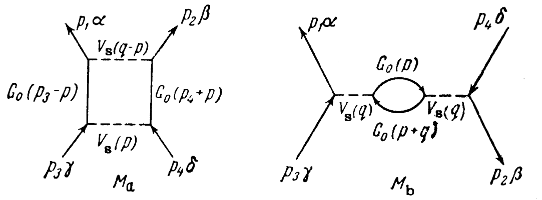

FIG. 1

$$M_b \sim V^2(k)\,\Pi_0(k),$$

$$\Pi_0(k) = \frac{2}{i(2\pi)^4}\int G_0(p+k)\,G_0(p)\,d^4p$$

$$= \frac{2}{(2\pi)^3}\int \frac{n_0(\mathbf{p}) - n_0(\mathbf{p}+\mathbf{k})}{\omega - \varepsilon_0(\mathbf{p}+\mathbf{k}) + \varepsilon_0(\mathbf{p})}\,d\mathbf{p}; \qquad (1)$$

if $|\mathbf{k}| \ll p_0$, then [3]

$$\Pi_0(k) \approx \frac{2}{(2\pi)^3}\int \frac{\partial n_0}{\partial \varepsilon}\,\frac{p_0(\mathbf{nk})}{p_0(\mathbf{nk}) - \omega_k}\,d\mathbf{p}$$

$$= -\frac{p_0}{\pi^2}\left[1 - \frac{\omega_k}{2|\mathbf{k}|p_0}\ln\left|\frac{\omega_k + p_0|\mathbf{k}|}{\omega_k - p_0|\mathbf{k}|}\right|\right.$$

$$\left. + \frac{i\pi|\omega_k|}{2|\mathbf{k}|p_0}\,\theta(p_0|\mathbf{k}| - |\omega_k|)\right]; \qquad (2)$$

here $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$, $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. $\Pi_0(k) \approx -p_0/\pi^2$ for $p_0|\mathbf{k}| \gg \omega_k$ and $\Pi_0(k) \approx p_0^3|\mathbf{k}|^2/3\pi^2\omega_k^2$ for $p_0|\mathbf{k}| \ll \omega_k$.

Thus $M_b/M_a \gtrsim n^{1/3}r_s$. Hence we can restrict ourselves to graphs of the ladder type for $nr_s^3 \ll 1$.

Let us now assume that $V_s = 0$ and $nr_l^3 \gg 1$. Obviously, we can then restrict ourselves to graphs of the chain type if $p_0|\mathbf{k}| > \omega$. If, on the other hand, $p_0|\mathbf{k}| < \omega$, we must also take into account more complicated graphs, as is seen from (2). If both $V_s \neq 0$ and $V_l \neq 0$ the complete box diagram $\Gamma_{\alpha\beta,\gamma\delta}$ for $p_0|\mathbf{k}| > \omega$ can be calculated using only the graphs of the type shown in Fig. 2 and similar graphs in which the external lines for the terminal ladder or chain are both ingoing or both outgoing, and not one ingoing, the other outgoing, as in Fig. 2.

Let us now consider the graph of Fig. 2 and all graphs of the same form which differ from one another only by the number of rungs in any one ladder or the number of links in any one chain. Summing all graphs of this type, we obtain the graph of Fig. 2 in which only one ladder is replaced by the box diagram of the gas approximation, i.e., by the sum of all graphs of the ladder type $\Gamma_{g\alpha\beta,\gamma\delta}(p_1p_2, p_3p_4)$, or one chain replaced by the box diagram $\Gamma_{l\,\alpha\beta,\gamma\delta}(p_1p_2, p_3p_4)$, i.e., the sum of all graphs of the chain type. The expression for Γ_g for a Fermi system has been found by Galitskiĭ:[4]

$$\Gamma_{g\alpha\beta,\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta}\Gamma_g,$$

$$\Gamma_g = f(\mathbf{p}',\mathbf{p}) + \int d\mathbf{q}\,f(\mathbf{p}',\mathbf{q})\,f^*(\mathbf{p},\mathbf{q})$$

$$\times\left[\frac{1}{\mathbf{q}^2 - \mathbf{p}^2 + i\delta} + \frac{N(\mathbf{q})}{\omega_P - \mathbf{q}^2 - \mathbf{P}^2/4 + i\delta N(\mathbf{q})}\right],$$

$$2p' = p_1 - p_2;\quad 2p = p_3 - p_4;\quad P = p_1 + p_2;$$

$$N(\mathbf{q}) = 1 - n(\mathbf{P}/2 + \mathbf{q}) - n(\mathbf{P}/2 - \mathbf{q}). \qquad (3)$$

Here $f(\mathbf{p}', \mathbf{p})$ is the two-body scattering amplitude multiplied by 4π. For Γ_l we can easily find the equation (see Fig. 3)

$$\Gamma_{l\,\alpha\beta,\gamma\delta} = \delta_{\alpha\gamma}\left[\delta_{\beta\delta}V_l(\mathbf{k}) + V_l(\mathbf{k})\,\Pi(k)\,\Gamma_{l\alpha'\beta,\alpha'\delta}/2\right],$$

from where we obtain

$$\Gamma_{l\,\alpha\beta,\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta}\Gamma_l,\qquad \Gamma_l = V_l(\mathbf{k})/[1 - V_l(\mathbf{k})\,\Pi(k)]. \quad (4)$$

The zero order Green's functions $G_0(p)$ in the graphs of Figs. 2 and 3 must, of course, be replaced by the exact Green's functions $G(p)$. Therefore, the expression (3) for G, which has been found using G_0, is approximate.

The polarization operator $\Pi(k)$, expressed through the exact G, can be calculated in the following way. According to [5,6], G has close to the pole $\epsilon = 0$, $|\mathbf{p}| = p_0$ the form

$$G(\mathbf{p},\varepsilon) = \frac{a}{\varepsilon - v(|\mathbf{p}| - p_0) + i\delta\,\mathrm{sgn}(|\mathbf{p}| - p_0)},$$

$$0 < a \leqslant 1,\qquad v = \frac{p_0}{m^*}.$$

The product $G(p)\,G(p+k)$ under the integral over p can, according to Landau,[6] be written in the form

$$G(p)\,G(p+k) \approx 2\pi i a^2\frac{(\mathbf{nk})}{\omega - v(\mathbf{nk})}\,\delta(\varepsilon)\,\delta(|\mathbf{p}| - p_0) + \varphi(p), \qquad (5)$$

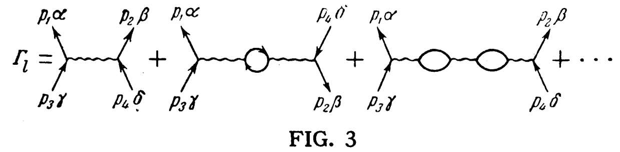

FIG. 3

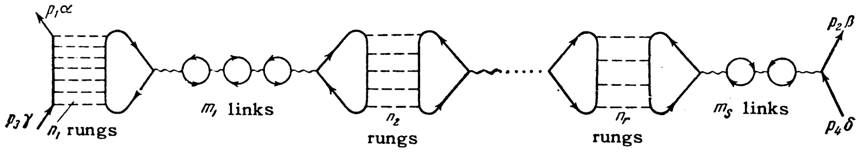

FIG. 2. Graph with r ladders and s chains. Dotted line: $V_s(p)$, wavy line: $V_l(p)$.

where $\varphi(p)$ is the regular part of $G(p)G(p+k)$. We obtain from (5)

$$\Pi(k) = c - b\left[1 - \frac{\omega}{2v|\mathbf{k}|}\ln\left|\frac{\omega + v|\mathbf{k}|}{\omega - v|\mathbf{k}|}\right| + \frac{i\pi|\omega|}{2v|\mathbf{k}|}\theta(v|\mathbf{k}| - |\omega|)\right],$$

$$c = \frac{2}{i(2\pi)^4}\int\varphi(p)\,d^4p, \qquad b = \frac{p_0^2 a^2}{v\pi^2}, \qquad \lim_{\omega/p_0|\mathbf{k}|\to 0}\Pi = c - b. \quad (6)$$

Extending the summation over graphs of the type of Fig. 2 to the other ladders and chains, we obtain the same graph in which now all ladders are replaced by the box diagram Γ_g and all chains by the box diagram Γ_l. Adding graphs analogous to that of Fig. 2, but with two ingoing or two outgoing lines at all terminal ladders or chains, we obtain for the total box diagram Γ the expression (for $\omega < p_0|\mathbf{k}|$):

$$\Gamma = \Gamma_g + \Gamma_l + \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4,$$

where Γ_1, Γ_2, Γ_3, and Γ_4 are shown in Fig. 4.

Figure 4 leads to a system of four integral equations for Γ_1, Γ_2, Γ_3, and Γ_4, which are given graphically in Fig. 5. This system is easily solved, but in our approximation we can manage without solving it. Indeed, let us estimate the graph Γ_a in Fig. 4 by replacing Γ_g by the constant amplitude f_0:

$$\Gamma_{a\alpha\beta,\gamma\delta}(p_1, p_2; p_3, p_4) = \frac{2i\delta_{\alpha\gamma}\delta_{\beta\delta}}{(2\pi)^4}\int\Gamma_g(p_1, p+k; p_3, p) \times \Gamma_l(p, p_2; p+k, p_4)\,G(p)\,G(p+k)\,d^4p$$
$$\approx 2\delta_{\alpha\gamma}\delta_{\beta\delta}f_0\Gamma_l(k)\,\Pi(k);$$
$$\text{for } \omega \ll p_0|\mathbf{k}| \quad \Gamma_l = -2f_0(b-c)\Gamma_l; \quad f_0 \sim r_s.$$

Thus $\Gamma_a/\Gamma_l \sim n^{1/3}r_s$, i.e., the inclusion of the additional box diagram Γ_g gives the small factor $n^{1/3}r_s$. In calculating Γ_1 and Γ_2 we can therefore neglect all graphs containing two or more box diagrams Γ_g, restricting ourselves to the graphs Γ_a, Γ_b, and Γ_c. Calculating these in the same way as Γ_a, but without the replacement of Γ_g by the constant f_0, we find [the terms containing φ are not written down in (7)]:

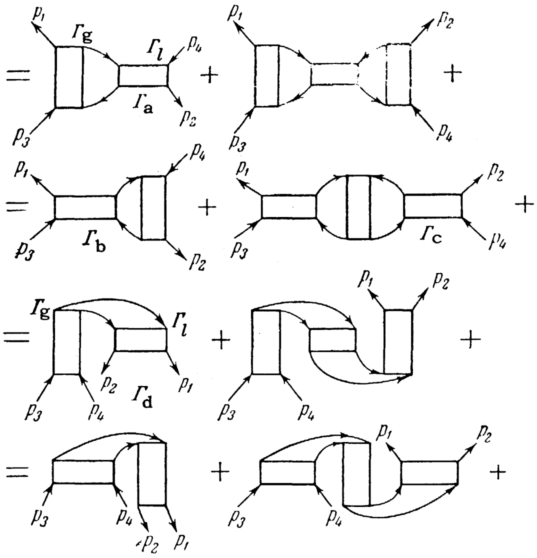

FIG. 4

$$\Gamma_a \approx \frac{(c-b)}{4\pi}\Gamma_l(k)\int\Gamma_g(\mathbf{p}_1, \omega_1; p_0\mathbf{n} + \mathbf{k}, \omega; \mathbf{p}_1 + \mathbf{k}, \omega_1 + \omega; p_0\mathbf{n}, 0)\,\frac{v(\mathbf{kn})}{\omega - v(\mathbf{kn})}\,d\Omega,$$

$$\Gamma_b \approx \frac{(c-b)}{4\pi}\Gamma_l(k)\int\Gamma_g(p_0\mathbf{n}, 0; \mathbf{p}_2, \omega_2; p_0\mathbf{n} + \mathbf{k}, \omega; \mathbf{p}_2 - \mathbf{k}, \omega_2 - \omega)\,\frac{v(\mathbf{kn})}{\omega - v(\mathbf{kn})}\,d\Omega,$$

$$\Gamma_c \approx \frac{(c-b)\,\Gamma_l(k)}{4\pi}\int\Gamma_b(\mathbf{p}_1, \omega_1; p_0\mathbf{n} + \mathbf{k}, \omega; \mathbf{p}_1 + \mathbf{k}, \omega_1 + \omega; p_0\mathbf{n}, 0)\,\frac{v(\mathbf{kn})}{\omega - v(\mathbf{kn})}\,d\Omega, \quad (7)$$

if $\Gamma_g = f_0$, then $\Gamma_b = \Gamma_a$, $\Gamma_c = 4f_0\Gamma_l^2(k)\,\Pi^2(k)$.

Let us now consider the simplest graph entering in Γ_3, namely Γ_α:

$$\Gamma_{d\alpha\beta,\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta}\Gamma_d;$$

$$\Gamma_d = \frac{i}{(2\pi)^4}\int\Gamma_g(p, P-p; p_3, p_4)\,G(P-p) \times G(p)\,\Gamma_l(p_1, p_2; p, P-p)$$
$$\approx -\frac{a^2}{(2\pi)^3}\int\Gamma_g(p, P-p; p_3, p_4)\,\Gamma_l(p_1, p_2; p, P-p) \times \frac{d\mathbf{p}}{\omega_P - \varepsilon(\mathbf{p}) - \varepsilon(\mathbf{P}-\mathbf{p}) + i\delta\,\mathrm{sgn}(|\mathbf{p}| - p_0)};$$

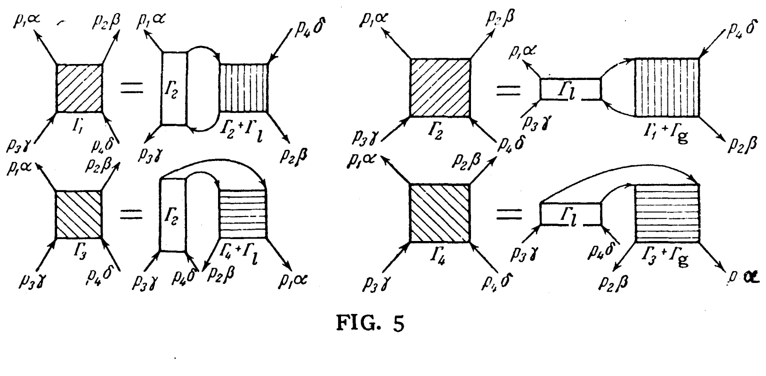

FIG. 5

the integral over $\mathbf{p}$ will converge for such values of $|\mathbf{p}|$ for which the product $\Gamma_g\Gamma_l$ begins to decrease appreciably, i.e., when $|\mathbf{p}| \sim r_s^{-1}$ ($\Gamma_l \to 0$ for $|\mathbf{p}| \sim r_l^{-1}$ and $\Gamma_g \to 0$ for $|\mathbf{p}| \to r_s^{-1}$). Therefore

$$\Gamma_d \approx f_0\Gamma_l/r_l, \qquad \Gamma_d/\Gamma_l \approx r_s/r_l \ll n^{1/3}r_s \tag{8}$$

and hence all graphs entering in Γ_3 and Γ_4 can be neglected.

As far as the long-range component of the potential, V_l, is concerned, there are two possibilities: 1) $U_l r_l^3 n^{1/3} \gtrsim 1$ [U_l is the maximal value of $|V_l(\mathbf{r})|$] and 2) $U_l r_l^3 n^{1/3} \ll 1$. In the second case we can treat the potential V_l in the Born approximation. Here we have to include only the graph of first order in V_l, so that we have for all values of $\mathbf{k}$ and ω

$$\Gamma = \Gamma_g + V_l(\mathbf{k}). \tag{9}$$

In the first case, which is of much greater interest, we have for $p_0|\mathbf{k}| > \omega$ (see above)

$$\Gamma = \Gamma_g + \Gamma_l + \Gamma_a + \Gamma_b + \Gamma_c, \tag{10}$$

$$\Gamma_g + \Gamma_a + \Gamma_b + \Gamma_c \sim r_s;$$

$$\text{for } U_l r_l^3 n^{1/3} \gg 1, \qquad \Gamma_l \approx -\Pi^{-1}(k) - \Pi^{-2}(k)V_l^{-1}(\mathbf{k}),$$

$$-\Pi^{-1} \approx (b-c)^{-1} \sim n^{-1/3}, \quad \Pi^{-2}V_l^{-1} \sim n^{-1/3}(U_l r_l^3 n^{1/3})^{-1}.$$

$\Gamma > 0$ for both signs of $V_l(\mathbf{k})$, $\Gamma_g + \Gamma_a + \Gamma_b + \Gamma_c$ depends on p_1 and p_2, while Γ_l does not.

VERTEX PART FOR ARBITRARY k, ω, AND Γ^ω

Let us first consider the case $U_l r_l^3 n^{1/3} \gtrsim 1$. In Landau's theory of Fermi liquids the limiting values of Γ: $\Gamma^k = \Gamma$ for $|\mathbf{k}| \to 0$, $\omega/p_0|\mathbf{k}| \to 0$ and $\Gamma^\omega = \Gamma$ for $\omega \to 0$, $p_0|\mathbf{k}|/\omega \to 0$, play an essential role. From the value of Γ obtained by us [formula (10)] we can evidently determine only Γ^k (it is important that in all parts of the graphs entering in Γ_l according to Fig. 3 there are no insertions of the ladder type with $p_0|\mathbf{k}'| < \omega'$). Let us first write down the symmetrized box diagram $\tilde{\Gamma}_{\alpha\beta,\gamma\delta}$ which is obtained from $\Gamma_{\alpha\beta,\gamma\delta}(p_1p_2, p_3p_4)$ by subtracting $\Gamma_{\alpha\beta,\gamma\delta}$ with permuted arguments, $p_3\gamma \rightleftarrows p_4\delta$. If Γ in (10) is regarded as a function of p_1, p_2, and k, then

$$\tilde{\Gamma}_{\alpha\beta,\gamma\delta} = \Gamma(p_1, p_2, k)\,\delta_{\alpha\gamma}\delta_{\beta\delta} - \Gamma(p_1, p_2, p_2 - p_1 - k)\,\delta_{\alpha\delta}\delta_{\beta\gamma}.$$

From this we find $\tilde{\Gamma}^k_{\alpha\beta,\gamma\delta}$:

$$\tilde{\Gamma}^k_{\alpha\beta,\gamma\delta}(p_1, p_2) = \Gamma^k(p_1, p_2, 0)\,\delta_{\alpha\gamma}\delta_{\beta\delta} - \Gamma(p_1, p_2, p_2 - p_1)\,\delta_{\alpha\delta}\delta_{\beta\gamma}. \tag{11}$$

In order to obtain $\tilde{\Gamma}^\omega_{\alpha\beta,\gamma\delta}$, we use the equation of Landau's theory of Fermi liquids which connects $\tilde{\Gamma}^\omega_{\alpha\beta,\gamma\delta}$ and $\tilde{\Gamma}^k_{\alpha\beta,\gamma\delta}$:[6]

$$\tilde{\Gamma}^\omega_{\alpha\beta,\gamma\delta}(p_1, p_2) = \tilde{\Gamma}^{(k)}_{\alpha\beta,\gamma\delta}(p_1, p_2) + \frac{p_0^2 a^2}{v(2\pi)^3}\int \tilde{\Gamma}^k_{\eta\beta,\xi\delta}(q, p_2) \times \tilde{\Gamma}^\omega_{\alpha\xi,\gamma\eta}(p_1, q)\,d\Omega_q. \tag{12}$$

The solution of (12) can be found by expanding into a series of Legendre polynomials.[6] For the zeroth harmonic we must replace $\tilde{\Gamma}^k_{\eta\beta,\xi\delta}$ under the integral in (12) by the average value of $\tilde{\Gamma}^k_{\eta\beta,\xi\delta}$ on the Fermi sphere, $\overline{\tilde{\Gamma}^k_{\eta\beta,\xi\delta}}$:

$$\overline{\tilde{\Gamma}^k_{\eta\beta,\xi\delta}} = \frac{1}{4\pi}\int \tilde{\Gamma}^k_{\eta\beta,\xi\delta}(q, p_2)\,d\Omega_q.$$

The main term in $\overline{\tilde{\Gamma}^k_{\alpha\beta,\gamma\delta}}$ will be $\Gamma^k_l(p_1, p_2, 0)$:

$$\Gamma^k_l(p_1, p_2, 0) = (b - c + V_l^{-1}(0))^{-1} \sim n^{-1/3},$$

$$\overline{\Gamma^k_l}(p_1, p_2, 0),\ \overline{\Gamma^k_{a,b,c}}(p_1, p_2, 0),\ \overline{\Gamma}_g(p_1, p_2, p_2 - p_1) \sim r_s; \tag{13}$$

the average values $\overline{\Gamma}_{a,b,c}(p_1p_2, p_2 - p_1)$ can be neglected:

$$\overline{\Gamma}_{a,b,c}(p_1, p_2, p_2 - p_1) \sim r_s/n^{2/3}r_l^2;$$

the average value $\overline{\Gamma}_l(p_1p_2, p_2 - p_1)$ will also be small:

$$\overline{\Gamma}_l(p_1, p_2, p_2 - p_1) \sim n^{-1/3}/n^{2/3}r_l^2. \tag{14}$$

We thus obtain from (12) for the l-th harmonic of $\tilde{\Gamma}^\omega$:

$$\tilde{\Gamma}^{\omega l}_{\alpha\beta,\gamma\delta} = [\Gamma^{\omega l}\delta_{\alpha\gamma}\delta_{\beta\delta} - \Gamma'^{\omega l}\delta_{\alpha\delta}\delta_{\beta\gamma}][1 + b\,\Gamma'^{\omega l}/2(2l+1)^{-1}];$$

$$\Gamma^{\omega 0} = \Gamma^{kl}[1 - b(\Gamma^{kl} - \Gamma'^{\omega l}/2)/2(2l+1)]^{-1}$$

$$\Gamma^{\omega 0} \approx \frac{\Gamma^k_l}{1 - b(\Gamma^k_l + \Gamma^k_1)};\quad \Gamma^{\omega l} \approx \Gamma^{kl} \ll \Gamma^{\omega 0}\ (l \neq 0);$$

$$\Gamma'^{\omega} = \Gamma'_l + \Gamma'_g + \Gamma'_{abc}, \tag{15}$$

where

$$\Gamma^k_1 \approx \overline{\Gamma}^k_g/2 + \Gamma^k_{abc},\quad \Gamma_{abc} = \Gamma_a + \Gamma_b + \Gamma_c,$$

$$\Gamma^k_i \equiv \Gamma^k_i(p_1, p_2, 0),$$

$$\Gamma'_i \equiv \Gamma_i(p_1, p_2, p_2 - p_1) \qquad (i = l, g, a, b, c).$$

To estimate the magnitude of c in (6) and (15), we use the relation for a found in [3,7]:

$$a^{-1} = 1 - \frac{i}{2}\int \tilde{\Gamma}^\omega_{\alpha\beta,\alpha\beta}(p, q)\,\varphi(q)\,d^4q/(2\pi)^4. \tag{16}$$

We find from (16)

$$a^{-1} \approx 1 + c\Gamma^{\omega 0},$$

and hence

$$-cb^{-2}/(\Gamma_1^k - V_l^{-1}(0)/b^2) \approx 1 - a < 1,$$

$$\Gamma_l^k \approx [b + aV_l^{-1}(0)]^{-1}, \qquad \Gamma^{\omega 0} \cong -1/a\,[b^2\Gamma_1^k - V_l^{-1}(0)]. \tag{17}$$

It follows from (17) that

$$\Gamma^{\omega 0}/\Gamma^k \sim U_l r_l^3 n^{1/3} \quad \text{for} \quad U_l r_l^3 r_s n^{2/3} \ll 1,$$

$$\Gamma^{\omega 0}/\Gamma^k \sim n^{-1/3} r_s^{-1} \gg 1 \quad \text{for} \quad U_l r_l^3 r_s n^{2/3} \gg 1.$$

Using the basic equation of the theory of Fermi liquids,[6]

$$\tilde{\Gamma}_{\alpha\beta,\gamma\delta}(p_1, p_2, k) = \tilde{\Gamma}^\omega_{\alpha\beta,\gamma\delta}(p_1, p_2) + \frac{a^2 p_0^2}{(2\pi)^3 v}\int \tilde{\Gamma}^\omega_{\alpha\xi,\gamma\eta}(p_1, q)\tilde{\Gamma}_{\eta\beta,\xi\delta}(q, p_2, k)\frac{v(\mathbf{nk})}{\omega - v(\mathbf{nk})}d\Omega_q;$$

$$\mathbf{n} = \frac{\mathbf{q}}{|\mathbf{q}|} \tag{18}$$

we can find, by the method of successive approximations, the symmetrized box diagram $\tilde{\Gamma}_{\alpha\beta,\gamma\delta}$ for arbitrary k and ω:

$$\tilde{\Gamma}_{\alpha\beta,\gamma\delta}(p_1, p_2, k) \approx \frac{\Gamma_l^k\,\delta_{\alpha\gamma}\delta_{\beta\delta}}{1 - b(\Gamma_l^k + \Gamma_1^k)f(k)} - (\Gamma_l' + \Gamma_g' + \Gamma_{abc}')\,\delta_{\alpha\delta}\delta_{\beta\gamma},$$

$$f(k) = \frac{\omega}{2v|\mathbf{k}|}\ln\left|\frac{\omega + v|\mathbf{k}|}{\omega - v|\mathbf{k}|}\right| - \frac{i\pi|\omega|}{2v|\mathbf{k}|}\theta(v|\mathbf{k}| - |\omega|). \tag{19}$$

If we introduce the spin operators $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$, we have

$$\tilde{\Gamma}_{\alpha\beta,\gamma\delta} \approx \frac{\Gamma_l^k}{1 - b(\Gamma_l^k + \Gamma_1^k)f} - \frac{1}{2}(\Gamma_l' + \Gamma_g' + \Gamma_{abc}') - \frac{(\boldsymbol{\sigma\sigma}')}{2}(\Gamma_l' + \Gamma_g' + \Gamma_{abc}'). \tag{20}$$

EFFECTIVE MASS OF THE EXCITATIONS AND ENERGY OF THE SYSTEM

The quantity $a^2\tilde{\Gamma}^\omega_{\alpha\beta,\gamma\delta}$ appears in the expressions for the effective mass of the excitations m*, the velocity of zero sound (with momentum l) $\tilde{u}_l = s_l v$, the velocity of ordinary sound u, and the ground state energy of the system E_0:[6]

$$\frac{1}{m^*} - 1 = 1 - \frac{p_0 a^2}{2(2\pi)^3}\int \tilde{\Gamma}^\omega_{\alpha\beta,\alpha\beta}(\chi)\cos\chi\,d\Omega, \tag{21}$$

$$\chi = \sphericalangle(\mathbf{p}_1, \mathbf{p}_2),$$

$$\frac{s_0}{2}\ln\left(\frac{s_0 + 1}{s_0 - 1}\right) \approx 1 + 1/\Gamma^{\omega 0}b, \tag{22}$$

$$u^2 = \frac{p_0^2}{3} + \frac{p_0^3 a^2}{6(2\pi)^3}\int \tilde{\Gamma}^\omega_{\alpha\beta,\alpha\beta}(\chi)(1 - \cos\chi)\,d\Omega, \tag{23}$$

$$E_0 = \int \mu(N)\,dN, \qquad \mu(N) = \int u^2(N)\frac{dN}{N}. \tag{24}$$

The difference $1/m^* - 1$ is, according to (21), determined by the first and not the zeroth harmonic of $\tilde{\Gamma}^\omega_{\alpha\beta,\alpha\beta}$, i.e., not by $\Gamma^{\omega 0}$, but by the small quantities $\overline{\Gamma}_l$, $\Gamma_g - \overline{\Gamma}_g$:

$$\frac{1}{m^*} = 1 - \frac{p_0 a^2}{(2\pi)^3}\int\Big[2(\Gamma_g^k + \Gamma_{abc}^k - \overline{\Gamma}_g^k - \overline{\Gamma}_{abc}^k) - (\Gamma_l' + \Gamma_g' + \Gamma_{abc}')\Big]\cos\chi\,d\Omega,$$

$$\frac{1}{m^*} - 1 = \beta_1 + \beta_2, \qquad \beta_1 \approx n^{-2/3} r_l^{-2} \qquad \beta_2 \sim n^{2/3} r_s^2. \tag{25}$$

In (22) $\Gamma^{\omega 0}b \gg 1$ for $U_l r_l^3 n^{1/3} \gg 1$; therefore [6]

$$s_0 \approx [b\Gamma^{\omega 0}/3]^{1/2}. \tag{26}$$

From (23) we find

$$u^2 \approx \frac{p_0^2}{3} + \frac{p_0^3 a^2}{3\pi^2}\Gamma^{\omega 0} = \frac{p_0^2}{3} - \frac{p_0^3 a}{3\pi^2}\frac{1}{[b^2\Gamma_1^k - V_l^{-1}(0)]}. \tag{27}$$

Substituting (27) in (24), we obtain for $\partial a/\partial n \approx 0$

$$\Delta E_0 = E_0 - E_{\text{ideal gas}} \approx \frac{3aNV_l(0)}{\alpha^2}\left[\frac{3}{4}n^{1/3} - \frac{1}{\alpha}\tan^{-1}\alpha n^{1/3} - \frac{1}{2\alpha^4}\ln(1 + \alpha^2 n^{2/3})\right],$$

$$\alpha = a^2 3^{1/3}\pi^{-4/3}\sqrt{|\Gamma_1^k| V_l(0)},$$

$$\Delta E_0 \approx -(3\pi^2)^{4/3} N n^{1/3} a^{-3}/4\Gamma_1^k \sim E_{\text{ideal gas}}/n^{1/3} r_s \text{ for } U_l r_l^3 r_s n^{2/3} \gg 1;$$

$$\Delta E_0 \approx \tfrac{1}{2}aV_l(0)Nn \sim U_l r_l^3 n^{1/3} E_{\text{ideal gas}} \text{ for } U_l r_l^3 r_s n^{2/3} \ll 1;\; E_{\text{ideal gas}} = \frac{3}{10}(3\pi^2)^{2/3} N n^{2/3}. \tag{28}$$

To have a system which is thermodynamically stable, i.e., $u^2 = \partial p/\partial\rho > 0$, it is necessary that $\Gamma^{\omega 0} > 0$, i.e.,

$$-\Gamma_1^k b^2 + V_l^{-1}(0) > 0.$$

If there are attractive forces, pairing has to be taking into account.

We see therefore that the correction to the effective mass is small in the presence of long-range forces with $U_l r_l^3 n^{1/3} \gtrsim 1$, just as in the gas model.[4,8,9] The ground state energy and the velocity of sound, on the other hand, are appreciably larger than for the ideal gas (by the factor $U_l r_l^3 n^{1/3}$ or $n^{-1/3} r_s^{-1/3}$). This increase is connected with the presence of the long-range forces. It is clear that the transition to the limit $r_l = \infty$, i.e., to Coulomb forces, is not possible in our formulas;

for Coulomb forces the system must be electrically neutral.

In the case where $U_l r_l^3 n^{1/3} \ll 1$, when Γ is given by (9), $\Gamma^{\omega} = \Gamma^{k}$ and the corrections to m^*, $\tilde{u}$, u, and E_0 are small as compared to the ideal gas values: $\sim n^{1/3} r_s$ or $\sim U_l r_l^3 n^{1/3}$.

In the case of the nucleus the inequalities $n^{1/3} r_s \ll 1$ and $n^{1/3} r_l \gg 1$ do not hold very strongly. Therefore, our model can only be regarded as a limiting case if applied to the nucleus.

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