SMEARING OUT OF INHOMOGENEITIES IN A WEAKLY IONIZED PLASMA IN A MAGNETIC FIELD (AMBIPOLAR DIFFUSION)

A. V. GUREVICH

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor November 3, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) 44, 1302–1306 (April, 1963)

It is shown that the process of smearing out of an inhomogeneity in a plasma, called ambipolar diffusion, is of a complex nature in a magnetic field and cannot be described by the normal diffusion equation. The velocity of the inhomogeneity across the magnetic field depends significantly on the initial size of the inhomogeneity and may be many times larger than the transverse electron diffusion velocity.

DURING the smearing out of inhomogeneities in a plasma, a difference in electron and ion concentration arises as a result of the different velocities of these particles. The changes in concentration, however, produce an electric field which inhibits further separation of the electrons and ions. As a result, the inhomogeneity is "smoothed out" in such a manner that the electron and ion concentrations are always approximately equal to each other. This process is also called ambipolar diffusion. Schottky [1] showed that in the absence of a magnetic field ambipolar diffusion resembles the normal diffusion process and proceeds at the rate of the normal ion diffusion velocity multiplied by a factor $1 + T_e/T_i$, owing to the effect of the electrons.

In the presence of a magnetic field the diffusion becomes anisotropic. The electrons, as before, diffuse more rapidly along the magnetic field lines, so that ambipolar diffusion in this direction has the velocity of the slower particles, the ions. Across the magnetic field, on the other hand, the ions diffuse more rapidly; ambipolar diffusion in a strictly transverse direction also takes place at the speed of the slowest particles, the electrons 1) multiplied by $1 + T_i/T_e$ on account of the effects of the ions[2]

How then is an inhomogeneity of arbitrary form smeared out? It would seem that knowing the diffusion velocity in the transverse and longitudinal directions, one could readily derive the rules governing the smearing out of any inhomogeneity. However this kind of naive discussion, presented in some works (see [3]), is shown below to be invalid and leads to serious misrepresentations of the nature of the effect. An analysis of this very question is discussed in the present work.

We consider an inhomogeneity in a weakly ionized plasma (i.e., when an important role is played by collisions of electrons and ions with neutral molecules) situated in a uniform magnetic field. We assume that the inhomogeneity does not vary appreciably over the length of a mean free path and does not change significantly during the mean free time of the charged particles, so that it is possible to use the macroscopic equations for describing the particle motion in the plasma 2). These equations, after linearization, take the following form:

$$\partial\delta n_e/\partial t + n_0\nabla\mathbf{v}_e = 0, \tag{1a}$$ *

$$\partial\delta n_i/\partial t + n_0\nabla\mathbf{v}_i = 0, \tag{1b}$$

$$m\nu_{em}n_0\mathbf{v}_e = -en_0\mathbf{E} - (en_0/c)\,[\mathbf{v}_e\mathbf{H}_0] - \varkappa T_e\nabla\delta n_e, \tag{1c}$$

$$M\nu_{im}n_0\mathbf{v}_i = en_0\mathbf{E} + (en_0/c)\,[\mathbf{v}_i\mathbf{H}_0] - \varkappa T_i\nabla\delta n_i. \tag{1d}$$

Here δn_e (or δn_i) is the excess concentration of electrons (or ions) in the inhomogeneities, $n_0 = n_{e0} = n_{i0}$ is the mean electron and ion concentration ($\delta n \ll n_0$), e is the charge on the electron, m and M are the electron and ion masses respectively, κ is Boltzmann's constant, T_e and T_i are the electron and ion temperatures respectively, $\mathbf{E}$ is the electric field strength, and ν_{em} and ν_{im} are

1)Simon[2] has shown that in the presence of metallic electrodes, increased diffusion of a plasma in the transverse direction is possible. In the present work we consider only an unbounded plasma.

2)On the basis of this condition inertial ($\partial \mathbf{v}/\partial t$) and viscous ($\eta\nabla^2\mathbf{v}$) terms can be omitted from the macroscopic equation of motion for the electrons and ions, since they are small in comparison to the term representing direct retardation produced by molecules.

*$[\mathbf{v}_e\mathbf{H}_0] = \mathbf{v}_e \times \mathbf{H}_0$.

the collision frequencies of the electrons and ions with neutral molecules (see, for example, [4]). Equations (1c) and (1d) apply to a system of coordinates in which the magnetic field $\mathbf{H}_0$ is constant; the gas of neutral molecules is assumed to be at rest relative to the magnetic field [3)].

The system (1) has to be completed by including the field equations. If the following condition is also satisfied,

$$(v\omega_0/c\nu)^2 \ll 1, \qquad (2)$$

where v is the thermal velocity, ν the collision frequency and ω_0 the Langmuir (plasma) frequency, then the effect of the solenoidal electric field on the motion of the charged particles can be neglected. In this case only the longitudinal electric field is of importance, and,

$$\Delta\varphi = 4\pi e\,(\delta n_i - \delta n_e). \qquad (3)$$

Condition (2) is generally valid in a weakly ionized plasma; this is the case which will be considered below [4)].

It is natural to seek the solution of the system of linear equations (1) and (3) by expanding the unknown functions in Fourier integrals with respect to the co-ordinates,

$$\delta n_{\mathbf{k}} = \int \delta n\,(\mathbf{r}) e^{-i\mathbf{k}\mathbf{r}}\, d^3r$$

etc. The equations for the Fourier components of the potential $\varphi_{\mathbf{k}}$ and the velocities $\mathbf{v}_{\mathbf{k}e}$ and $\mathbf{v}_{\mathbf{k}i}$ are algebraic. Therefore the functions $\varphi_{\mathbf{k}}$, $\mathbf{v}_{\mathbf{k}e}$ and $\mathbf{v}_{\mathbf{k}i}$ are expressible simply in terms of $\delta n_{\mathbf{k}e}$ and $\delta n_{\mathbf{k}i}$. Substituting the expressions for $\mathbf{v}_{\mathbf{k}e}$ and $\mathbf{v}_{\mathbf{k}i}$ into the equations for $\delta n_{\mathbf{k}e}$ and $\delta n_{\mathbf{k}i}$, we obtain

$$\begin{aligned} \partial\delta n_{\mathbf{k}e}/\partial t + \alpha_{He}\,[D_e k^2 \delta n_{\mathbf{k}e} + 4\pi\sigma_e\,(\delta n_{\mathbf{k}e} - \delta n_{\mathbf{k}i})] &= 0, \\ \partial\delta n_{\mathbf{k}i}/\partial t + \alpha_{Hi}\,[D_i k^2 \delta n_{\mathbf{k}i} + 4\pi\sigma_i\,(\delta n_{\mathbf{k}i} - \delta n_{\mathbf{k}e})] &= 0. \end{aligned} \qquad (4)$$

In order to simplify the form of the equations, the following symbols have been introduced

$$\alpha_{He} = \frac{1 + (\omega_H/\nu_{em})^2 \cos^2\beta}{1 + (\omega_H/\nu_{em})^2}, \qquad \alpha_{Hi} = \frac{1 + (\Omega_H/\nu_{im})^2 \cos^2\beta}{1 + (\Omega_H/\nu_{im})^2}.$$

Furthermore $D_e = \kappa T/m\nu_{em}$, $\sigma_e = e^2 n_0/m\nu_{em}$ and $\omega_H = eH_0/mc$ are respectively the coefficients of longitudinal diffusion, the conductivity, and Larmor frequency for electrons, while D_i, σ_i and Ω_H are the corresponding quantities for the ions; β is the angle between $\mathbf{k}$ and $\mathbf{H}_0$.

The solution of the linear equations (4) can clearly be written in the form,

$$\begin{aligned} \delta n_{\mathbf{k}e}(t) &= \delta n_{\mathbf{k}e}^{(1)} e^{-q_1 t} + \delta n_{\mathbf{k}e}^{(2)} e^{-q_2 t}, \\ \delta n_{\mathbf{k}i}(t) &= \delta n_{\mathbf{k}i}^{(1)} e^{-q_1 t} + \delta n_{\mathbf{k}i}^{(2)} e^{-q_2 t}, \end{aligned} \qquad (5)$$

where q_1 and q_2 are the roots of the characteristic equation. If the typical dimensions of the perturbed region are large compared with the Debye radius, R_D,

$$(kR_D)^2 = k^2 \varkappa T_e T_i / 4\pi e^2 n_0\,(T_e + T_i) \ll 1, \qquad (6)$$

then the expressions for q_1 and q_2 have the simple form:

$$q_1 = 4\pi\sigma_e\alpha_{He} + 4\pi\sigma_i\alpha_{Hi},$$

$$q_2 = k^2 \frac{(\sigma_e D_i + \sigma_i D_e)\,\alpha_{He}\alpha_{Hi}}{\sigma_e\alpha_{He} + \sigma_i\alpha_{Hi}} = k^2 \frac{\varkappa\,(T_e + T_i)}{M\nu_{im}/\alpha_{Hi} + m\nu_{em}/\alpha_{He}}. \qquad (7)$$

For $\delta n_{\mathbf{k}e}^{(1)}$, $\delta n_{\mathbf{k}e}^{(2)}$, $\delta n_{\mathbf{k}i}^{(1)}$, and $\delta n_{\mathbf{k}i}^{(2)}$, we obtain in this case

$$\begin{aligned} \delta n_{\mathbf{k}e}^{(1)} &= \frac{\sigma_e\alpha_{He}}{\sigma_e\alpha_{He} + \sigma_i\alpha_{Hi}}\,[\delta n_{\mathbf{k}e}(0) - \delta n_{\mathbf{k}i}(0)], \\ \delta n_{\mathbf{k}i}^{(1)} &= \frac{\sigma_i\alpha_{Hi}}{\sigma_e\alpha_{He} + \sigma_i\alpha_{Hi}}\,[\delta n_{\mathbf{k}i}(0) - \delta n_{\mathbf{k}e}(0)], \end{aligned} \qquad (8a)$$

$$\delta n_{\mathbf{k}e}^{(2)} = \delta n_{\mathbf{k}i}^{(2)} = [\sigma_e\alpha_{He}\delta n_{\mathbf{k}i}(0) + \sigma_i\alpha_{Hi}\delta n_{\mathbf{k}e}(0)]/(\sigma_e\alpha_{He} + \sigma_i\alpha_{Hi}). \qquad (8b)$$

Here $\delta n_{\mathbf{k}e}(0)$ and $\delta n_{\mathbf{k}i}(0)$ are the Fourier components of the initial perturbed density of the electrons and ions.

From (5), and (8) it is evident that the root q_1 corresponds to the dispersal of the initial charge in the plasma. The root q_2 corresponds to the smearing out of the inhomogeneity in such a manner that the concentrations of ions and electrons remain identical; this process can be called "ambipolar diffusion." As is known, the dispersal of excess charges procedes at a much higher rate than diffusion: $q_1/q_2 \sim 1/(kR_D)^2$. Therefore for arbitrary initial electron and ion concentrations, there is established after a time $\tau \sim 1/q_1$ an inhomogeneity in which the concentrations δn_e and δn_i are identical (8b), following which the ambipolar diffusion commences. Here

$$\delta n(\mathbf{r}, t) = (2\pi)^{-3} \int \delta n_{\mathbf{k}}(0) \exp\{i\mathbf{k}\mathbf{r} - D(\beta)\, k^2 t\}\, d^3k,$$

$$\begin{aligned} D(\beta) = \varkappa\,(T_e + T_i)\{&M\nu_{im}\,[1 + (\Omega_H/\nu_{im})^2]/[1 + (\Omega_H/\nu_{im})^2 \cos^2\beta] \\ &+ m\nu_{em}\,[1 + (\omega_H/\nu_{em})^2]/[1 + (\omega_H/\nu_{em})^2 \cos^2\beta]\}^{-1}, \end{aligned} \qquad (9)$$

[3)] We consider here isothermal diffusion. It is therefore permissible to ignore the motion of the neutral molecules in this case only when the inhomogeneities are not too large: i.e., $L^2 \ll l^2\,(n_m/n_0)$, where L is the dimension of the inhomogeneity, l is the mean free path of the particles and n_m is the number density of the molecules.

[4)] We note that in a strongly ionized plasma, where the major role is played by collisions between the charged particles themselves, condition (2) is generally not satisfied and in this case one cannot neglect the influence of the solenoidal electric fields.

where β is the angle between the wave vector $\mathbf{k}$ and the magnetic field, and $\delta n_k(0)$ is the Fourier component of the inhomogeneity in the ion or electron density after the conditions described by (8b) have been established.

We note that if the smearing out of the inhomogeneity could be described by the usual diffusion equation, then the coefficient $D(\beta)$ would be represented by

$$D(\beta) = D_{\parallel} \cos^2 \beta + D_{\perp} \sin^2 \beta, \tag{10}$$

where $D_{\parallel}$ and $D_{\perp}$ are the coefficients of longitudinal and transverse diffusion. In our case the coefficient $D(\beta)$ cannot as a rule, be represented by such an expression, so that the smearing out process in a plasma, i.e., ambipolar diffusion, does not have the same characteristics as ordinary diffusion and is of a more complex nature; it also depends appreciably on the initial structure of the inhomogeneity.

In particular, if the inhomogeneity does not extend for any great length along the magnetic field, or more accurately, if

$$R_{\parallel}^2 \ll R_{\perp}^2 (M\nu_{im}/m\nu_{em}) [1 + (\Omega_H/\nu_{im})^2], \tag{11}$$

where $R_{\parallel}$ and $R_{\perp}$ are the typical dimensions of the inhomogeneity in directions parallel and perpendicular to the magnetic field, then the second term in the denominator in the expression for $D(\beta)$ is small. In this case normal diffusion takes place [Eq. (10)] with,

$$D_{\parallel} = \varkappa (T_i + T_e)/M\nu_{im},$$

$$D_{\perp} = \varkappa (T_i + T_e)/M\nu_{im} [1 + (\Omega_H/\nu_{im})^2]. \tag{12}$$

It is readily seen that condition (11) holds during the whole development process of the inhomogeneity. Therefore if the initial inhomogeneity is such that the condition (11) is fulfilled, then ambipolar diffusion in both the longitudinal and transverse directions proceeds at the diffusion rate of the ions. The effect of the electrons is only to introduce an additional factor of $(1 + T_e/T_i)$.

If on the other hand the initial inhomogeneity extends over appreciable distances along the magnetic field, so that the reverse of condition (11) is satisfied, then diffusion proceeds in directions perpendicular to a strong magnetic field at a considerably reduced rate, that of the diffusion of the electrons:

$$D_{\perp} = \varkappa (T_i + T_e)/(M\nu_{im} + m\omega_H^2/\nu_{em}).$$

The diffusion proceeds along the magnetic field, as before, at the diffusion rate of the ions. We note also that anisotropy in diffusion for very extensive inhomogeneities begins at weaker magnetic fields [when $H_0 > (Mm\nu_{im}\nu_{em}c^2)^{1/2}/e$], than for the reverse case (11) (when $H_0 > M\nu_{im}c/e$).

Thus, the smearing out of inhomogeneities in directions at right angles to the magnetic field in a plasma depends greatly on the form of the inhomogeneity and generally speaking, proceeds much faster than the transverse diffusion velocity of electrons. The balancing of the electron and ion concentrations in the region of the inhomogeneity in this case is achieved by the formation of current loops which are produced as a result of the convective motion of electrons along the lines of force of the magnetic field. If the inhomogeneity extends over large distances along the field, then the balancing process is complicated, and the diffusion of electrons transverse to the magnetic field begins to play a dominant role.

Since the smearing out process in a magnetic field is accompanied by the production of electric currents, magnetic perturbations are in turn produced by such currents. The Fourier components for the perturbed magnetic field have the form

$$\delta \mathbf{H}_k = - \frac{4\pi e \delta n_k (\sigma_e D_i + \sigma_i D_e)}{c(\sigma_e \alpha_{He} + \sigma_i \alpha_{Hi}) [1 + (\omega_H/\nu_{em})^2] [1 + (\Omega_H/\nu_{im})^2]}$$
$$\times \left\{ \left(\frac{\omega_H}{\nu_{em}} \right)^2 \cos \beta \left[\frac{\mathbf{k}}{k} \frac{\mathbf{H}_0}{H_0} \right] + \frac{\omega_H}{\nu_{em}} \left(1 + \frac{\omega_H \Omega_H}{\nu_{em}\nu_{im}} \cos^2 \beta \right) \left(\frac{\mathbf{H}_0}{H_0} - \frac{\mathbf{k}}{k} \cos \beta \right) \right\}.$$

In the case of longitudinal diffusion ($\mathbf{k} \parallel \mathbf{H}_0$) there are no perturbations in the magnetic field, as is to be expected.

The solution for ambipolar diffusion given above is only valid in a first approximation up to $(kR_D)^2$. In the next approximation the electron and ion concentrations no longer remain equal and the inhomogeneity is polarized to an extent represented by,

$$\varphi_k = \frac{\varkappa \delta n_k}{e n_0} \frac{T_e M \nu_{im} \alpha_{He} - T_i m \nu_{em} \alpha_{Hi}}{M \nu_{im} \alpha_{He} + m \nu_{em} \alpha_{Hi}}.$$

From this result it is evident that the potential of the electric field in the inhomogeneity reverses sign, depending on the shape of the inhomogeneity.

We also note that the system (1) and (3) can be reduced (if we restrict ourselves to inhomogeneities for which $\delta n_e \approx \delta n_i = \delta n$) to a single equation, provided condition (6) is valid:

$$(\sigma_e \Delta_{He} + \sigma_i \Delta_{Hi}) \frac{\partial \delta n}{\partial t} = (\sigma_e D_i + \sigma_i D_e) \Delta_{He} \Delta_{Hi} \delta n. \tag{13}$$

Here

$$\Delta_{He} = \partial^2/\partial z^2 + [1 + (\omega_H/\nu_{em})^2]^{-1} (\partial^2/\partial x^2 + \partial^2/\partial y^2),$$

and Δ_{Hi} is an analogous operator except that ω_H/ν_{em} in it is replaced by Ω_H/ν_{im}. Equation (13) is also the equation of ambipolar diffusion in a weakly ionized plasma in a magnetic field. It goes without saying that the solution of (13) is identical to (9). In the absence of a magnetic field, Eq. (13) leads to the normal diffusion equation with a diffusion coefficient $D = (\sigma_e D_i + \sigma_i D_e)/(\sigma_e + \sigma_i)$, which agrees with Shottky's results [1].

In conclusion the author expresses his thanks to V. L. Ginzburg and V. P. Silin for valuable discussions.

[1] W. Schottky, Physik. Z. **25**, 342 (1924). N. A. Kaptsov, Elektricheskie yavleniya v gazakh i v vakuume (Electrical Phenomena in Gases and in Vacuum), Gostekhizdat, 1950.

[2] M. A. Biondi and S. C. Brown, Phys. Rev. **75**, 1700 (1949). A. Simon, Phys. Rev. **98**, 317 (1955). A. Simon, An Introduction to Thermonuclear Research, Pergamon Press, London, 1959.

[3] B. N. Gershman, Radiotekhnika i elektronika (Radio Engineering and Electronics) **1**, 720 (1956). V. P. Dokuchaev, Radiofizika (Radio Physics) **3**, 50 (1960).

[4] S. Chapman and T. Cowling, Mathematical Theory of Non-uniform Gases, Cambridge, 1952. V. L. Ginzburg, Rasprostranenie élektromagnitnykh voln v plazme (Propagation of Electromagnetic Waves in Plasma) Fizmatgiz, 1960.

Translated by F. L. Curzon
209