OSCILLATIONS OF A WEAKLY INHOMOGENEOUS PLASMA

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The natural oscillations of a weakly inhomogeneous plasma are studied. The dispersion equations which determine the natural frequencies of such oscillations are similar to the quasi-classical Bohr quantization laws [see Eqs. (3), (55), (56)]. The high-frequency electron oscillations of a weakly inhomogeneous plasma are considered by means of such dispersion equations. The quantization rules are also used for the study of the oscillation spectrum of a weakly inhomogeneous plasma confined by a strong magnetic field. Sufficient conditions for instability of the plasma are deduced and in a number of concrete cases analyzed.

1. INTRODUCTION AND FUNDAMENTALS OF THE PROBLEM

RECENTLY the attention of many theoreticians who have been studying plasma physics has been drawn to the problem of the construction of a theory of electromagnetic waves in a weakly inhomogeneous plasma. (In particular, the necessity of such a theory is brought about by the urgent requirement for analysis of the question as to the stability of the plasma). Here attempts are made at an almost direct transfer of the methods applied in the description of a homogeneous plasma. Of course, such a procedure meets with a whole series of difficulties, among which can be noted, for example, the dependence of the natural frequencies of the vibrations on the coordinates.

On the other hand, it is well known that in the theory of propagation of electromagnetic waves in a weakly inhomogeneous medium, it has been possible to apply the method of geometric optics,[1] which corresponds to the quasiclassical approximation of quantum mechanics. In the present report we set forth the application of such a method to the theory of the natural oscillations of a weakly inhomogeneous plasma.

For simplicity, let us consider the case of a one-dimensional dependence of the properties of the plasma on the single coordinate x. Then, by taking into account the dependence of the field on time and the other coordinates in the form $\exp(-i\omega t - ik_x y + ik_z z)$, one can reduce the field equation in a many cases to the form

$$\lambda^2 y'' + q(\omega, x) y = 0, \tag{1}$$

where λ is a small quantity, for example, of the order of the Debye or Larmor radius. When one says that λ is small, it is meant that λ is small in comparison with the characteristic length of change $q(x)$.

The asymptotic solutions[2] of Eq. (1) corresponding to large λ have the form[1)]

$$y = \frac{C}{(q(\omega, x))^{1/4}} \exp\left\{\pm \frac{i}{\lambda} \int dx \sqrt{q(\omega, x)}\right\}. \tag{2}$$

The theory of the natural oscillations of a plasma is interested not only in the form of the characteristic solutions, but also in the frequency spectrum. Such a problem has been solved in principle for equations of type (1) by the construction of the quasi-classical approximation theory of quantum mechanics (see, for example, [2]). Here the eigenvalue spectrum is determined by the Bohr quantization conditions:

$$\int dx \sqrt{q(\omega, x)} = \pi n \lambda. \tag{3}$$

Here n is an integer that is large compared with unity.

A few words must be said about the limits of integration in the left-hand side of Eq. (3). If $q > 0$ throughout the entire plasma and the boundary condition of vanishing of the field on the boundary of the plasma is used, then the integral is taken over the entire plasma. If there is a single turning point at which $q > 0$, then the integral is taken from the turning point throughout the whole region $q < 0$. Finally, if there are several turning points then the

1)An exception is the vicinity of the point where $q(x) = 0$.

integral over each region q > 0 will lead to a separate equation (3). The latter means that in the quasi-classical approximation the vibrations in each such region are independent. This is brought about by the fact that, as is known from quantum mechanics, the width of the quasi-classical barrier separating the classically achievable regions must be extremely large, and therefore the corresponding transmission coefficient is extremely small.

In quantum mechanics, $q(\omega, x)$ is always a real quantity. On the contrary, in the theory of natural oscillations of a plasma, q is frequently complex. The asymptotic solution of (2) has the same form for complex q. The situation concerning the quantization conditions (3), which give the dispersion equations of the plasma oscillations, is more complicated. A simplifying circumstance is the fact that the imaginary part of $q(\omega, x)$ is frequently small. This is always the case when the absorption of the plasma is small. Under these conditions, use can be made of dispersion equations of the type (3). Therefore, equations of the type (3) are applied below for the determination both of the real and imaginary parts of the frequencies of oscillation, and in particular for the study of the stability of the plasma.[2)]

Naturally, such consideration can be applied not only to the plasma oscillations described by Eq. (1), but everywhere where the natural oscillations of a weakly inhomogeneous plasma, which is described by a set of linear differential equations dependent on a small (or large) parameter, are involved.

2. HIGH FREQUENCY ELECTROMAGNETIC OSCILLATIONS

Following these general remarks, we now turn to a consideration of concrete examples. First let us consider high-frequency electronic oscillations of the plasma. In this case, we restrict ourselves to oscillations which are propagated in the direction of the inhomogeneity. Then one can write the following relation between the electric field and the displacement

$$D(x) = \left[1 - \frac{\omega_{Le}^2(x)}{\omega^2}\right] E(x) + 3\frac{d^2}{dx^2} E r_D^2(x) \frac{\omega_{Le}^4(x)}{\omega^4}, \quad (4)$$

where

$$\omega_{Le}^2(x) = 4\pi e^2 N(x)/m, \qquad r_D^2(x) = \varkappa T(x)/4\pi e^2 N(x)$$

(this expression is valid under conditions for which the "wavelength" is large in comparison with the Debye radius). Equating the electric displacement to zero, we obtain an equation for oscillations of the longitudinal field which is obviously an equation of type (1). Therefore, the corresponding dispersion equation can be written in the form

$$\frac{1}{\sqrt{3}}\int \frac{dx}{r_D(x)} \frac{\omega^2}{\omega_{Le}^2(x)} \sqrt{1 - \frac{\omega_{Le}^2(x)}{\omega^2}} = \pi n, \quad (5)$$

Here the turning points are determined by the relation

$$\omega_{Le}^2(x) = \omega^2 \quad (6)$$

and correspond to a division of the plasma into regions of transparency and opacity.

In order to make the results obtained through the dispersion equation (5) more graphic and perceptible, let us first make a simplifying assumption on the constancy of the plasma pressure:

$$N(x)\varkappa T(x) = \text{const} = N_0 \varkappa T_0.$$

We further assume that the density of the plasma changes according to the law $N(x) = N_0[1 + (x/d)^2]$. Then the dispersion equation (5) takes the following form:

$$\frac{1}{\sqrt{3}} \frac{d}{r_D(0)} \frac{|\omega|}{\omega_{Le}(0)} \left[\frac{\omega^2}{\omega_{Le}^2(0)} - 1\right] = \pi n. \quad (7)$$

Inasmuch as the corresponding turning points are $\pm d[(\omega^2/\omega_{Le}^2(0)) - 1]^{1/2}$, then it is apparent that ω^2 must be larger than ω_{Le}^2. Keeping this in mind, and also the fact that $r_D(0)/d$ is small in comparison with unity, we get from Eq. (7):

$$\omega^2 = \omega_{Le}^2(0)\{1 + 2n\sqrt{3}\, r_D(0)/d\}. \quad (8)$$

We note that in the case of a spatially homogeneous particle distribution in the region $-L/2 \le x \le +L/2$, the dispersion equation (5) leads to the following spectrum:

$$\omega^2 = \omega_{Le}^2(0)\{1 + 3(\pi n r_D(0)/L)^2\}. \quad (9)$$

3. LOW FREQUENCY PLASMA OSCILLATIONS IN A PLANE PLASMA LAYER CONFINED BY A MAGNETIC FIELD

This section is concerned with a plasma confined by a magnetic field. We shall speak only of the dependence of the plasma parameters on the inhomogeneity x. We assume the magnetic pressure to be considerably greater than the plasma pressure. Then we can establish that the magnetic field changes over distances that are much greater than the distances characteristic of change in the

[2)]For complex q, the turning points lie in the complex plane x. Therefore, in the quantization rules (if, of course, they are possible) $\int q\,dx = 2\pi n\lambda$ the contour of integration must enclose such points in the complex plane (see [2]).

particle distribution. Therefore, with great accuracy, we can neglect the effect of spatial change of the magnetic field on the motion of the plasma particles.

The case in which the Larmor radius of the particles can be regarded as small is a comparatively simple one. Then, by solving the kinetic equation with the self-consistent field, and by expanding the solution in powers of the Larmor radius, it is easy to obtain the following expression for the ionic charge density under the assumption that $\mathbf{E} = -\operatorname{grad}\Phi$:

$$\frac{e^2N_e}{m\Omega_e^2}\left[\frac{d^2\Phi}{dx^2}+\frac{d\Phi}{dx}\frac{d\ln N}{dx}\right] - \Phi\frac{e^2}{m}\int\limits_{-\infty}^{+\infty}\frac{dv_z}{\omega-k_zv_z}\left\{k_z\frac{\partial\widetilde{f}_0}{\partial v_z}-\frac{k_y}{\Omega_e}\frac{\partial\widetilde{f}_0}{\partial x}\right\}. \qquad (10)$$

Here it is assumed that the plasma ions in the ground state are described by the distribution

$$f_0(v_z, v_\perp^2, x)+\frac{v_y}{\Omega}\frac{\partial f_0}{\partial x},$$

and $\widetilde{f}_0 = \int dv_x dv_y f_0$. The z axis is directed parallel to the direction of the constant magnetic field B. In obtaining Eq. (10), it is also taken into account that ω and k_zv_z are small in comparison with the Larmor frequency Ω. Moreover, it is assumed that $k_yR \ll 1$, where R is the Larmor radius. A similar expression holds also for electrons.

We limit ourselves to the case of values of the projection on the z axis of the phase velocity of the oscillations that are small in comparison with the thermal velocity of the electrons and large in comparison with the thermal velocity of the ions (or of sound for a non-isothermal plasma in which $T_e \gg T_i$; this is precisely the case considered below). Here the corresponding expansion can be introduced in the integrand of Eq. (10) both for electrons and ions. As a result, the following equation is obtained for the case of a Maxwellian electron distribution; this equation describes the potential of the electric field of the plasma oscillations:

$$\frac{d^2\Phi}{dx^2}-k_z^2\Phi+\frac{4\pi NMc^2}{B^2}\left[\frac{d^2\Phi}{dx^2}+\frac{d\Phi}{dx}\frac{d\ln N}{dx}\right]+\frac{Q(x,\omega)}{r_D^2(x)}\Phi=0; \qquad (11)$$

$$Q(x,\omega)=\varkappa T\frac{k_yc}{|e|B}\frac{1}{\omega}\frac{d\ln N}{dx}-1 + i\sqrt{\frac{\pi}{2}}\frac{1}{|k_z|}\sqrt{\frac{m}{\varkappa T}}\left(-\omega+\varkappa T\frac{k_yc}{|e|B}\frac{d}{dx}\ln\frac{M}{\sqrt{T}}\right). \qquad (12)$$

We note immediately that the imaginary part of the right-hand side of Eq. (12) is assumed to be small in comparison with the real one. Therefore, in the determination of the frequency, one does not have to consider the imaginary part in the first approximation.

Equation (11) contains two characteristic parameters. Corresponding to this situation, we can distinguish two limiting cases. We take $4\pi NMc^2 \gg B^2$, i.e., we shall assume that the Alfvén velocity is small in comparison with the velocity of light.[3] Neglecting the first two terms of Eq. (11), and introducing a new function $\Phi = \psi/\sqrt{N}$, we get

$$\psi''+\left\{\frac{Q(x,\omega)}{R^2(x)}-\frac{1}{2}\frac{d^2N}{dx^2}\frac{1}{N}+\frac{1}{4}\left(\frac{d\ln N}{dx}\right)^2\right\}\psi=0. \qquad (13)$$

Here $R^2(x) = \kappa T/\Omega_i^2$, where Ω_i is the Larmor frequency of the ions.

Inasmuch as the particle distribution changes little at distances of the order of the Larmor radius, the first component of the curly brackets on the left-hand side of Eq. (13) considerably exceeds the remainder. In just this sense, one can speak of the Larmor radius as a small parameter. Therefore, the asymptotic solution for the potential of the electric field can be written in the form

$$\Phi=(CR^{1/2}/N^{1/2}Q^{1/4})\exp\left\{\pm i\int dx\sqrt{Q(x,\omega)}/R(x)\right\}. \qquad (14)$$

The corresponding dispersion equation of the plasma oscillations has the form

$$\int dx\sqrt{Q(x,\omega)}/R(x)=\pi n, \qquad (15)$$

where the limits of integration are determined as in Eq. (3). Actually, let the dependence Q(x) be such that there is no turning point Q(x) = 0 in the plasma. Then the boundary condition of vanishing of the potential on the boundary of the plasma immediately reduces to Eq. (15). The change in the boundary condition leads only to the appearance on the right-hand side of (15) of an additional real component of the order of unity,[4] which is of little importance. This is connected with the fact that many "wavelengths" of oscillations are included in the characteristic distance for change in the distribution of the plasma.

To make concrete the dependence of the number of particles and the temperature on the coordinates, let us consider the application of the dispersion equation (15). Let $Q(x)/R^2(x)$ = const, which is possible only in the case T = const, $N(x) = N_0 \times$

[3]The limit $4\pi NMc^2 \ll B^2$ can be considered in a fashion quite similar to what is set forth below.

[4]One can possibly propose special dissipative (or active) boundary conditions leading to an imaginary contribution. We shall not consider such boundary conditions below.

$\exp(-x/x_0)$. We then get the following dispersion equation with the help of (15) for a plasma located in the interval $0 < x < 1$:

$$\frac{\varkappa T}{\omega}\frac{k_y c}{|e|B}\frac{d\ln N}{dx}-1 + i\sqrt{\frac{\pi}{2}}\frac{\omega}{|k_z|}\sqrt{\frac{m}{\varkappa T}}\left(\frac{\varkappa T}{\omega}\frac{k_y c}{|e|B}\frac{d\ln N}{dx}-1\right)=\left(\pi n\frac{R}{L}\right)^2. \tag{16}$$

For the case $n^2R^2 \ll L^2$ we then get

$$\omega=\varkappa T\frac{k_y c}{|e|B}\frac{d\ln N}{dx}=-\frac{\varkappa T}{x_0}\frac{k_y c}{|e|B}. \tag{17}$$

Keeping it in mind that the imaginary part of Q is small, we get from Eq. (15) in first approximation:

$$\int\frac{dx}{R(x)}\sqrt{\operatorname{Re}Q(x,\omega)}=\pi n, \tag{18}$$

where the integration is carried out along the real axis, which corresponds to neglect of the small imaginary parts of the coordinates of the turning point.

Taking into account the following term of the expansion in Eq. (15), we get

$$\int\frac{dx}{R(x)}\frac{\operatorname{Im}Q(x,\omega)}{\sqrt{\operatorname{Re}Q(x,\omega)}}=0. \tag{19}$$

Making use of Eq. (18) as an equation of first approximation, we can materially simplify the analysis of the dispersion equation (15). In order to demonstrate this, we consider the following example. Let $\operatorname{Re}Q(x) = \text{const}$. This can be accomplished only for a number of particles N and temperature T satisfying the equation

$$T\,d\ln N/dx=-T_0/x_0=\text{const}. \tag{20}$$

Then, with the help of Eq. (18), we can easily find ($\omega = \omega' + i\omega''$)

$$\omega'=\frac{k_y c}{|e|B}\varkappa T\frac{d\ln N}{dx}\left\{1+\left[\pi n\Big/\int\frac{dx}{R(x)}\right]^2\right\}^{-1}. \tag{21}$$

In the limit $n/\int R^{-1}dx$, small in comparison with unity, the spectrum of (21) is identical with (17). Keeping in mind the smallness of such a parameter and assuming ω' determined by Eq. (21), we can write

$$Q(x)=-i\left[\frac{\omega''}{\omega'}+\sqrt{\frac{\pi}{8}}\frac{\omega'}{|k_z|}\sqrt{\frac{m}{\varkappa T}}\frac{d\ln T}{d\ln N}\right]+\left[\pi n\Big/\int\frac{dx}{R}\right]^2. \tag{22}$$

If we require the satisfaction of the equation

$$\frac{1}{\sqrt{T}}\frac{d\ln T}{d\ln N}=\frac{C}{\sqrt{T_0}}, \tag{23}$$

then the right side of Eq. (22) is seen to be independent of the coordinates. Then, substituting the imaginary part of Eq. (22) in Eq. (19), we get

$$\omega''=-\sqrt{\frac{\pi}{8}}\frac{(\omega')^2}{|k_z|}\sqrt{\frac{m}{\varkappa T}}\frac{d\ln T}{d\ln N}. \tag{24}$$

Such a spectrum corresponds to instability of the plasma for $d\ln T/d\ln N < 0$. This conclusion, and also the spectrum of oscillations were obtained by Rudakov and Sagdeev,[3] who assumed that the dependence of the temperature and number of particles on the coordinate can be arbitrary. The expansion above shows that the spectra obtained by them are possible only in the case of satisfaction of Eqs. (20) and (23), or, what amounts to the same thing, for a particular specific dependence of the temperature and density of the particles on the coordinates.[5)]

Satisfaction of Eq. (20) alone is less restrictive. In this case, we again have the spectrum (21) which determines the real part of the frequency of oscillations. For the imaginary part of the frequency, we then get the following equation with the aid of Eqs. (19), (22):

$$\omega''=-\sqrt{\frac{\pi}{8}}\frac{(\omega')^2}{|k_z|}\left\{\int\frac{dx}{R(x)}\right\}^{-1}\int\frac{dx}{R(x)}\sqrt{\frac{m}{\varkappa T}}\frac{d\ln T}{d\ln N}. \tag{25}$$

Here the condition $d\ln T/d\ln N < 0$, which is satisfied at any point, is by no means sufficient. In fact, in order that the right side of Eq. (25) be greater than zero, which corresponds to the growth of the oscillations in time, the following inequality must hold:

$$\left(\int\frac{dx}{R(x)}\right)^{-1}\int\frac{dx}{R(x)}\frac{1}{\sqrt{T}}\frac{d\ln T}{d\ln N}<0. \tag{26}$$

Keeping in mind Eq. (20), and also the definition of R(x), we can write this inequality in the form

$$(x_2-x_1)\,x_0\ln[T(x_1)/T(x_2)], \tag{27}$$

where x_1 and x_2 are the boundary points of the plasma (or the turning points).

For definiteness, let $x_2 > x_1$. Then, for $x_0 > 0$, Eq. (27) takes the form $T(x_1) < T(x_2)$. In other words, the temperature increases with increase in x. At the same time, $x_0 > 0$ corresponds to a decrease in density with increase in x. Therefore, the integral jump condition of (27) corresponds not only to the local jump condition of Rudakov and Sagdeev, but also requires that their condition be sufficiently weakly violated throughout the entire plasma.[6)]

[5)] $N = N_0 \exp\{-(2/C)\sqrt{T_0/T}\}$, $\sqrt{T/T_0} = (C/2)(\delta - x/x_0)$; if $C < 0$, then $\delta x_0 < x$, while if $C > 0$, then $\delta x_0 > x$.

[6)] It is necessary to emphasize that all our considerations, which are based on the approximation of the expression for the charge density (10), and which refer to the concrete conformation of the plasma spectrum, are limited to this approximation.

Equation (19) makes it possible to obtain a general expression for the imaginary part of the oscillation frequency,[7)]

$$\omega'' = (\omega')^2 \left\{ \int dx \frac{\varkappa T(x) k_y c}{R(x)|e|B} \frac{d \ln N}{dx} \frac{1}{\sqrt{\operatorname{Re} Q(x, \omega')}} \right\}^{-1}$$
$$\times \sqrt{\frac{\pi}{2}} \frac{1}{|k_z|} \sqrt{\frac{m}{\varkappa}} \int dx \frac{1}{R(x)\sqrt{\operatorname{Re} Q(x,\omega')}} \frac{1}{\sqrt{T(x)}}$$
$$\times \left[\frac{k_y c}{|e|B} \varkappa T(x) \frac{d}{dx} \ln \frac{N}{\sqrt{T}} - \omega' \right], \qquad (28)$$

where, according to Eq. (18), the real part of the frequency is determined by the relation

$$\int \frac{dx}{R(x)} \left[\varkappa T(x) \frac{k_y c}{|e|B} \frac{1}{\omega'} \frac{d \ln N}{dx} - 1 \right]^{1/2} = \pi n. \qquad (29)$$

In the region of transparency considered by us,

$$\varkappa T(x) \frac{k_y c}{|e|B} \frac{1}{\omega'} \frac{d \ln N}{dx} \gg 1. \qquad (30)$$

Making use of this relation, we can write down the condition for the instability of the plasma, corresponding to the positive nature of the right-hand side of Eq. (28), in the following form:

$$\int \frac{dx}{T(x)\sqrt{\operatorname{Re} Q(x,\omega')}}$$
$$\times \left[\varkappa T(x) \frac{k_y c}{|e|B} \frac{1}{\omega'} \left(\frac{d \ln N}{dx} - \frac{1}{2} \frac{d \ln T}{dx} \right) - 1 \right] > 0. \qquad (31)$$

Here and below, the integration is carried out from small values to large ones.

In the special case T = const, the relation (31) takes the form

$$\frac{1}{R} \int dx \left[\frac{\varkappa T k_y c}{|e|B\omega'} \frac{d \ln N}{dx} - 1 \right]^{1/2} > 0. \qquad (32)$$

In accord with the relation (30), the inequality (32) is violated only in one isolated case when Eq. (20) is satisfied (and $n = 0$). Thus one can establish the fact that a weakly inhomogeneous plasma with a constant temperature $T_e \gg T_i$, confined by a strong magnetic field of plane geometry, can be unstable. The frequency of the increasing oscillations is determined by the equation

$$\int dx \left[\frac{\varkappa T k_y c}{|e|B\omega'} \frac{d \ln N}{dx} - 1 \right]^{1/2} = \pi n R. \qquad (33)$$

We note that as is seen from the inequality (31), the requirement of a decrease of the temperature of the plasma in the same direction in which the density decreases in the example under consideration corresponds to stabilization of the plasma but cannot be used as a sufficient condition for stability.

Finally, we write down the dispersion equation for an arbitrary distribution function for the electrons of the form $f_0 = N(x) F(\mathscr{E}, x)$, where $\mathscr{E} = mv_z^2/2$:

$$\frac{|e|B}{\sqrt{Mc}} \int dx \left[\frac{k_y c}{B|e|} \frac{1}{\omega'} \frac{d \ln N}{dx} - \left\langle \frac{1}{mv^2} \right\rangle \right]^{1/2} = \pi n, \qquad (34)$$

$$\omega'' = \pi \frac{(\omega')^2}{|k_z|} \left\{ \int dx \frac{d \ln N}{dx} \left[\frac{k_y c}{B|e|} \frac{1}{\omega'} \frac{d \ln N}{dx} - \left\langle \frac{1}{mv^2} \right\rangle \right]^{-1/2} \right\}^{-1}$$
$$\times \int dx \left[\frac{k_y c}{B|e|} \frac{1}{\omega'} \frac{d \ln N}{dx} - \left\langle \frac{1}{mv^2} \right\rangle \right]^{-1/2}$$
$$\times \left[\omega' \frac{B|e|}{k_y c} \frac{\partial F(0,x)}{\partial \mathscr{E}} + \frac{d \ln N}{dx} F + \frac{\partial F(0,x)}{\partial x} \right]. \qquad (35)$$

Here $\langle 1/mv^2 \rangle = -\int_{-\infty}^{+\infty} dv_z\, \partial F/\partial \mathscr{E}$. If this quantity is positive, then the condition for instability can be written in the following form:

$$\int dx \left(\frac{\partial F(0,x)}{\partial \mathscr{E}} + \left[\frac{d \ln N}{dx} F + \frac{\partial F(0,x)}{\partial x} \right] \frac{k_y c}{B|e|} \frac{1}{\omega'} \right)$$
$$\times \left[\frac{k_y c}{B|e|} \frac{1}{\omega'} \frac{d \ln N}{dx} - \left\langle \frac{1}{mv^2} \right\rangle \right]^{-1/2} > 0. \qquad (36)$$

The local condition for instability can be written by requiring the positiveness of the integrand

$$\frac{\partial F(0,x)}{\partial \mathscr{E}} + \left[\frac{d \ln N}{dx} F + \frac{\partial F(0,x)}{\partial x} \right] \frac{k_y c}{B|e|} \frac{1}{\omega'} > 0. \qquad (37)$$

Such a local condition is necessary but by no means sufficient. Instability will of course take place if one requires the fulfilment of the condition (37) throughout the whole region of transparency. But such a requirement is rather excessive, as is evident from the relation (36).

Comparing the results of this section with the results of the work of Rudakov and Sagdeev,[3] it should be noted that in our consideration the frequency of the oscillations is a number determined only by the integral parameters of the plasma. On the other hand, the frequency in [3] is a point function.[8)]

4. LOW FREQUENCY OSCILLATIONS IN A CYLINDRICAL PLASMA COLUMN CONFINED BY A MAGNETIC FIELD

This section will also be devoted to a plasma confined by a magnetic field (with $T_e \gg T_i$). How-

[7)]In the case $\omega'' \ll \omega'$, Eq. (19) takes the form

$$\int \frac{dx}{R} \frac{1}{\sqrt{\operatorname{Re} Q}} \left\{ \operatorname{Im} Q(x,\omega') + \omega'' \frac{\partial}{\partial \omega'} \operatorname{Re} Q(x,\omega') \right\} = 0.$$

[8)]M. L. Levin has pointed out the well-known mechanical analog which characterizes the state of affairs. If we have a set of pendulums on different suspension threads, then each of them possesses a characteristic frequency (point function). The presence of even small coupling leads, generally speaking, to new characteristic frequencies which are no longer point functions and which are determined by the parameters of the set of pendulums as a whole. I take this occasion to express my gratitude to M. L. Levin for this useful observation.

ever, in contrast to the previous section, we shall consider here a cylindrical plasma column confined by a magnetic field which depends only on the radial coordinate. It is assumed that the particle distribution in the ground state of the plasma has the form

$$f_0(v_z,\ v_\perp,\ r) - (v_\perp/\Omega)\sin(\psi - \varphi)\,\partial f_0/\partial r, \tag{38}$$

where $\mathbf{r} = (r, \psi, z)$, $\mathbf{v} = (v_\perp, \varphi, v_z)$ and the z axis is directed along the direction of the magnetic field.

We assume that the plasma pressure (more precisely, the transverse pressure)

$$P = \sum \int d\mathbf{v}\, \frac{mv_\perp^2}{2} f_0 \tag{39}$$

significantly exceeds the magnetic pressure. Then, using the relation

$$\frac{d\ln B^2}{dr} + \frac{P}{B^2/8\pi}\frac{d\ln P}{dr} = 0, \tag{40}$$

we can, as in the previous section, neglect the dependence of the magnetic field on the coordinate in comparison with the sensitive dependence of the distribution on consideration of the motion of the particles of the plasma.

It is not difficult to establish the fact that, within the framework of the assumptions used in obtaining the formula (10), the following expression results for the charge density of the ions in the case of cylindrical symmetry:

$$\frac{e^2N}{M\Omega^2}\left[\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr}\left(1 + \frac{d\ln N}{d\ln r}\right) - \frac{l^2}{r^2}\Phi\right] - \Phi\,\frac{e^2}{M}\int\limits_{-\infty}^{+\infty}\frac{dv_z}{\omega - k_z v_z}\left\{k_z\frac{\partial \widetilde{f}_0}{\partial v_z} - \frac{l}{\Omega}\frac{1}{r}\frac{\partial \widetilde{f}_0}{\partial r}\right\}. \tag{41}$$

Here Φ depends only on r. The dependence on the time and the other coordinates taken in the form $\exp(-i\omega t + ik_z z + il\psi)$, is separated out and will not be considered further.

Considering that $4\pi NMc^2 \gg B^2$, and also limiting ourselves to the case of the projection of the phase velocity on the z axis that is small in comparison with the thermal velocity of the electrons and large in comparison with the sound velocity, we get the following equation for the field potential Φ;

$$\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr}\left(1 + \frac{d\ln N}{d\ln r}\right) - \frac{l^2}{r^2}\Phi + \frac{Q(r,\omega)}{R^2(r)}\Phi = 0, \tag{42}$$

where for a Maxwellian electron distribution

$$Q(r,\omega) = \varkappa T(r)\frac{lc}{B|e|}\frac{1}{\omega}\frac{1}{r}\frac{d\ln N}{dr} - 1 + i\sqrt{\frac{\pi}{2}}\frac{\omega}{|k_z|} \times \sqrt{\frac{m}{\varkappa T(r)}}\left[\varkappa T(r)\frac{lc}{B|e|}\frac{1}{\omega}\frac{1}{r}\frac{d}{dr}\ln\frac{N}{\sqrt{T}} - 1\right]. \tag{43}$$

Making the substitution $\psi = \Phi\sqrt{rN}$, we get

$$\psi'' + \left\{\frac{Q(r,\omega)}{R^2(r)} - \frac{l^2}{r^2} + \frac{1}{4}\left(\frac{1}{r} - \frac{N'}{N}\right)^2 - \frac{1}{2}\frac{N''}{N}\right\}\psi = 0. \tag{44}$$

We shall assume lR to be small.[9] Then the asymptotic solution for the potential of the electric field has the form

$$\Phi = \frac{CR^{1/2}}{r^{1/2}N^{1/2}Q^{1/4}}\exp\left\{\pm i\int dr\,\frac{\sqrt{Q(r,\omega)}}{R(r)}\right\}, \tag{45}$$

while the corresponding dispersion equation which defines the oscillation spectrum of the plasma can be written in the form

$$\int\frac{dr}{R(r)}\sqrt{Q(r,\omega)} = \pi n. \tag{46}$$

Inasmuch as the imaginary part of $Q(r, \omega)$ is small, one can write the following equation defining the real part of the oscillation frequency:

$$\int\frac{dr}{R(r)}\left[\frac{\varkappa T}{\omega'}\frac{lc}{B|e|}\frac{1}{r}\frac{d\ln N}{dr} - 1\right]^{1/2} = \pi n. \tag{47}$$

For the imaginary part of the frequency of oscillation, we get the following equation directly:

$$\omega'' = \sqrt{\frac{\pi}{2}}\frac{(\omega')^3}{|k_z|}\sqrt{\frac{m}{\varkappa}} \times\left\{\int\frac{dr}{R(r)}\frac{1}{\sqrt{\mathrm{Re}\,Q(r,\omega)}}\varkappa T(r)\frac{lc}{B|e|}\frac{1}{r}\frac{d\ln N}{dr}\right\}^{-1} \times\int\frac{dr}{R(r)}\frac{1}{\sqrt{\mathrm{Re}\,Q(r,\omega)}}\frac{1}{\sqrt{T(r)}} \times\left[\frac{\varkappa T(r)}{\omega'}\frac{lc}{B|e|}\frac{1}{r}\frac{d}{dr}\ln\frac{N}{\sqrt{T}} - 1\right]. \tag{48}$$

It is evident that the following inequality holds in the region of transparency over which the integration here is carried out:

$$\frac{\varkappa T}{\omega'}\frac{lc}{B|e|}\frac{1}{r}\frac{d\ln N}{dr} \geqslant 1. \tag{49}$$

Making use of this relation, we obtain the following condition for positive character of the imaginary part of the frequency (the instability condition):

$$\int\frac{dr}{T(r)}\frac{1}{\sqrt{\mathrm{Re}\,Q(r,\omega)}}\left(\frac{\varkappa T(r)}{\omega'}\frac{lc}{B|e|}\frac{1}{r}\frac{d}{dr}\ln\frac{N}{\sqrt{T}} - 1\right) > 0. \tag{50}$$

In a fashion similar to the formula (3.2), the integration here and below is carried out in the direction from small r to large.

In the case of a plasma temperature independent of the coordinates the inequality (50) has the form

$$\int dr\left[\frac{\varkappa T}{\omega'}\frac{lc}{|e|B}\frac{1}{r}\frac{d\ln N}{dr} - 1\right]^{1/2} = \pi|n|R > 0, \tag{51}$$

which is always satisfied (except in the case $N \sim e^{-r^2/r_0^2}$ and the value $n = 0$ is possible here).

[9]In the case of large lR/r in the formulas written out below we must have $Q(r,\omega) - (lR/r)^2$ in place of $Q(r,\omega)$.

Consequently, a plasma column with an electron distribution function of the form (38) and which has a temperature that is homogeneous over a cross section confined by the cylindrical magnetic field with straight lines of force is seen to be unstable relative to the oscillations (51).

In conclusion, let us write the dispersion equations for a non-Maxwellian electron distribution function of the form $\tilde{f}_0 = N(r)F(\mathscr{E}, r)$:

$$\frac{|e|B}{\sqrt{Mc}}\int dr\left[\frac{lc}{B|e|}\frac{1}{\omega'}\frac{1}{r}\frac{d\ln N}{dr}-\left\langle\frac{1}{mv^2}\right\rangle\right]^{1/2}=\pi n, \tag{52}$$

$$\omega''=\pi\frac{(\omega')^2}{|k_z|}\left\{\int\frac{dr}{r}\frac{d\ln N}{dr}\left[\frac{lc}{B|e|}\frac{1}{\omega'}\frac{1}{r}\frac{d\ln N}{dr}-\left\langle\frac{1}{mv^2}\right\rangle\right]^{-1/2}\right\}^{-1}$$
$$\times\int dr\left\{\omega'\frac{B|e|}{lc}\frac{\partial F(0,r)}{\partial\mathscr{E}}+\frac{1}{r}\left[\frac{d\ln N}{dr}F+\frac{\partial F(0,r)}{\partial r}\right]\right\}$$
$$\times\left[\frac{lc}{B|e|}\frac{1}{\omega'}\frac{1}{r}\frac{d\ln N}{dr}-\left\langle\frac{1}{mv^2}\right\rangle\right]^{-1/2}. \tag{53}$$

If it is taken into account that $\langle 1/mv^2\rangle$ is positive, then one can proceed further and write down the following stability condition:

$$\int dr\left\{\frac{\partial F(0,r)}{\partial\mathscr{E}}+\frac{lc}{B|e|}\frac{1}{\omega'}\frac{1}{r}\left[\frac{d\ln N}{dr}F+\frac{\partial F(0,r)}{\partial r}\right]\right\}$$
$$\times\left[\frac{lc}{B|e|}\frac{1}{\omega'}\frac{1}{r}\frac{d\ln N}{dr}-\left\langle\frac{1}{mv^2}\right\rangle\right]^{-1/2}. \tag{54}$$

By requiring that the integrand be less than zero

$$\frac{\partial F(0,r)}{\partial\mathscr{E}}+\frac{lc}{B|e|}\frac{1}{\omega'}\frac{1}{r}\left[\frac{d\ln N}{dr}F+\frac{\partial F(0,r)}{\partial r}\right]<0, \tag{55}$$

we obtain the local stability condition. This condition is necessary in contrast with the sufficient condition (54). If we require the satisfaction of the condition (55) throughout the region of transparency (in other words, to consider it as non-local), then the stability will take place in general earlier than such a condition is seen to be satisfied.

Here the problem does not arise as to what measure it is exhaustive to consider the problem of the stability of the plasma. However, one of the reasons for writing this paper were the prospects uncovered by the possibility of applying the method of "quasi-classical quantization" to the problem of plasma stability.

5. CONCLUSION. GENERALIZATION

Summing up the situation briefly, one can repeat and generalize several premises of our research.

The first from which we set out, is that the eigenvalue problem which arises in the question of interest to us, concerning the natural oscillations of weakly inhomogeneous plasma, is solved by means of the "quantization rules" which are similar to the quasi-classical quantization rules of Bohr. It must be noted that it is not difficult to ascertain from what has been set forth above that the formula (3) can be written in the form

$$\oint dx k_x(\omega, x)=2\pi n, \tag{56}$$

where $k_x(\omega, x)$ is the projection of the wave vector on the x axis and is of course a function of the coordinates.

Keeping in mind the deep analogy between mechanics and geometric optics,[4] one can establish the fact that for finite, conditionally periodic trajectories of rays in an inhomogeneous medium, the eikonal[10)] is a non-unique quantity defined with accuracy up to sums of multiples of the values of the analogs of the mechanical action variables (the eikonal variables). Therefore, the problem of obtaining the spectrum of the characteristic oscillations in a non-one-dimensionally weakly inhomogeneous plasma reduces to finding the conditional periodic motions of the rays and, what is very important, the adiabatic invariance of the eikonal variables I_α corresponding to them. The consequent quantization rule is that the eikonal variables can take on only values which are multiples of 2π:

$$I_\alpha=2\pi n_\alpha. \tag{57}$$

In this paper we have considered only the application of a one-dimensional "quantization rule" (56). The characteristic feature of the natural oscillations considered is the smallness of their "wavelength" in comparison with the distances over which the particle distribution of an equilibrium plasma changes appreciably. On the other hand, the "wavelength" is large in comparison with such "microscopic" scales of a weakly inhomogeneous plasma as the Debye or Larmor radii. The latter fact appears in the result that the "quasi-classical quantum number" is large in comparison with unity.

In the examples considered above we have limited ourselves to the case of a medium without absorption and a medium with weak absorption (or weak buildup). For the case of strong absorption or strong buildup the projection of the wave vector in formula (56) can possess a large imaginary part, while the turning points appear to be located in the complex plane. In such a case, the difference between the regions of transparency and opacity disappear. However, even in this case, by following

[10)]In geometric optics, the field has the form $\Phi = Ae^{i\psi}$, where A is a slowly changing function and ψ is the eikonal – a very large quantity which appears in analogy to the action function divided by Planck's constant.

[2], we can make clear the conditions under which the quantization rule (56) becomes possible.

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