ANALYTIC EXPRESSIONS FOR UPPER LIMITS OF COUPLING CONSTANTS IN QUANTUM FIELD THEORY

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Analytic expressions are obtained for upper limits of the coupling constant for three fields a, b, and c as a function of the particle masses m_a, m_b, and m_c.

1. The coupling constant g^2 of three fields a, b, c satisfies, as shown by Lehmann, Symanzik, and Zimmermann,[1] the inequality $g^2\Phi(a) < 1$, where $\Phi(a)$ is a certain functional of particle a. From here follows the inequality $g^2 < 1/\Phi_{\min}$, which however is useful only if the condition $\Phi_{\min} \neq 0$ is satisfied. Geshkenbeĭn and Ioffe[2] obtained an analogous inequality with a functional satisfying this condition. The form of this functional depends on the properties of particle a and on the type of the reaction describing the transition of particle a into particles b and c. For example for a boson $\Phi(a)$ has one form and for a fermion it has a substantially different form. If a is a fermion then the functional Φ differs somewhat for the scalar and pseudoscalar cases.[2]

In a conversation with me B. L. Ioffe has posed the question of the possibility of an analytical solution of the extremum problem for the functional Φ. In the present paper the solution of this problem is given and, consequently, analytic expressions are obtained for upper limits of coupling constants.

Let particle a be a boson of mass m_a. As was shown in [2] the coupling constant of the three fields a, b, and c satisfies the inequality

$$g^2 \int\limits_{(m_b+m_c)}^{\infty} \frac{1}{2\pi} \frac{|\Gamma(\varkappa^2)|^2 q(\varkappa^2)\, d\varkappa^2}{(\varkappa^2 - m_a)^2 \varkappa} < 1, \qquad (1.1)$$

where the masses m_b and m_c are such that for all possible transitions of particle a into particles b and c the sum $(m_b + m_c)$ is closest to m_a, $m_a < m_b + m_c$. $\Gamma(\kappa^2)$ is the vertex part for the transition of the boson a into the bosons b and c, and $q(\kappa^2) = [\kappa^2 - (m_b + m_c)^2]^{1/2}[\kappa^2 - (m_b - m_c)^2]^{1/2}/2\kappa$.

In terms of the dimensionless variable $x = \kappa^2/(m_b + m_c)^2$ the inequality (1.1) becomes

$$g^2 \frac{1}{2(m_b+m_c)^2} \int\limits_1^{\infty} \frac{1}{2\pi} |\Gamma(x)|^2 \frac{\sqrt{(x-1)(x-\lambda)}}{(x-\alpha)^2 x}\, dx < 1, \qquad (1.2)$$

where

$$\alpha = m_a^2/(m_b+m_c)^2, \qquad \lambda = (m_b - m_c)^2/(m_b+m_c)^2,$$
$$\alpha < 1, \quad \alpha < \lambda. \qquad (1.3)$$

The inequality (1.1) is a consequence of the inequality

$$\int\limits_{(m_b+m_c)^2}^{\infty} \operatorname{Im} D^{-1}(\varkappa^2) \frac{d\varkappa^2}{(\varkappa^2 - m_a^2)^2} \leqslant 1, \qquad (1.4)$$

where $D(\kappa^2)$ is the Green's function of the boson a. We shall derive this inequality because the derivation presented in the paper of Geshkenbeĭn and Ioffe[2] does not seem to us to be sufficiently convincing.

From the Lehmann-Källén representation[3]

$$D(\varkappa^2) = \frac{1}{\varkappa^2 - m_a^2} - \int\limits_{(m_b+m_c)^2}^{\infty} \frac{\rho(\varkappa'^2)}{\varkappa'^2 - \varkappa^2 - i\delta} d\varkappa'^2, \qquad \rho(\varkappa'^2) \geqslant 0 \qquad (1.5)$$

it follows that $D(\kappa^2)$, hence also $D^{-1}(\kappa^2)$, is an R-function.

An arbitrary R-function $F(x)$ has the representation[4]

$$F(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{1 + xx'}{x' - x} d\sigma(x') + \operatorname{Re} F(i), \qquad (1.6)$$

where $d\sigma(x') \geq 0$ is some distribution of nonnegative masses with a finite total mass (see Appendix I)

$$\int\limits_{-\infty}^{\infty} d\sigma(x') = 2\pi \operatorname{Im} F(i) < +\infty. \qquad (1.7)$$

Let us separate out from Eq. (1.6) all the delta-function terms. Then

$$F(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{1+xx'}{x'-x} d\sigma_1(x') + \operatorname{Re} F(i)$$
$$+ \frac{\mu_\infty}{2\pi} x + \frac{1}{2\pi} \sum_k \frac{1 + xx_k}{x_k - x} \mu_k, \qquad (1.8)$$

where $\mu_k > 0$ is the coefficient of $\delta(x'-x_k)$, ($\mu_\infty = 0$ if there is no delta-function term at ∞), $d\sigma_1(x') \geq 0$ and

$$\mu_\infty + \Sigma\mu_k + \int_{-\infty}^{\infty} d\sigma_1(x') = 2\pi \operatorname{Im} F(i). \qquad (1.9)$$

It follows from Eq. (1.6) that for any interval of the real axis for an arbitrary function f(x) one has

$$\lim_{\delta\to+0}\int \frac{f(x)\, 2 \operatorname{Im} F(x'+i\delta)\, dx'}{1+x'^2} = \int f(x')\, d\sigma(x'). \qquad (1.10)$$

In particular,

$$F'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(1+x'^2)}{(x'-x)^2}\, d\sigma(x') = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\operatorname{Im} F(x'-i\delta)}{(x'-x)^2}\, dx'. \qquad (1.11)$$

Since $D^{-1}(\kappa^2)$ is an R-function which is real for $\kappa^2 \leq (m_b + m_c)^2$ and whose derivative at the point m_a^2 equals 1, it follows from Eq. (1.11) that

$$\int_{(m_b+m_c)^2}^{\infty} \operatorname{Im} D^{-1}(\kappa^2)\, \frac{d\kappa^2}{(\kappa'^2 - m_a^2)} \leqslant 1,$$

i.e., the inequality (1.4). The equality became an inequality because the integral extends not from $-\infty$ to $+\infty$ but from $(m_b + m_c)^2$ to $+\infty$. The fact that $\operatorname{Im} D^{-1}(\kappa^2) = 0$ on the interval $(-\infty, (m_b + m_c)^2)$ does not change matters since the function $D^{-1}(\kappa^2)$ may have poles in that interval. It is easy to convince oneself that each pole gives a contribution $\mu_k(1 + x_k^2)/(x_k - x)^2$ to the integral (1.11).

From the representation (1.6) it is easily established that the R-function satisfies a dispersion relation with one subtraction (see Appendix I).

2. Let us denote the integral in Eq. (1.2) by $\Phi(\Gamma)$. The problem consists in finding the minimum of this functional; at that it is necessary to define precisely the class of functions on which the minimum is being sought. In correspondence with the work of Geshkenbeĭn and Ioffe, it is supposed that $\Gamma(x)$ is a function that is regular in the plane with the cut $(1, +\infty)$ and is real for $x < 1$, consequently $\Gamma(x)$ takes on complex conjugate values on opposite sides of the cut. We shall assume that at ∞ the function $\Gamma(x)$ grows slower than any power of $|x|^{1/2}$, i.e., that for any $\epsilon > 0$ for sufficiently large $|x|$ the function $|\Gamma(x)| < \exp(\epsilon |x|^{1/2})$. The function $\Gamma(x)$ is normalized by the condition $\Gamma(\alpha) = 1$.

Let us map conformally the cut x-plane onto the unit circle $|z| \leq 1$. We let

$$z = -\frac{t-i}{t+i}, \qquad t = \sqrt{\frac{x-1}{1-\alpha}}. \qquad (2.1)$$

This mapping takes the lower edge of the cut into the lower half circumference, the upper edge into the upper half circumference, the point $x = \alpha$ into the center $z = 0$, the point $x = \infty$ into the point $z = -1$, and the functional $\Phi(\Gamma)$ takes on the form

$$\Phi(\Gamma) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,|\Gamma[x(z)]|^2\, d\theta, \qquad z = e^{i\theta} \qquad (2.2)$$

with the weight function

$$f(\theta) = \frac{1}{2\sqrt{1-\alpha}}\, \frac{t^2\sqrt{(1-\alpha)t^2 + (1-\lambda)}}{[(1-\alpha)t^2+1](t^2+1)}, \qquad t = \tan\frac{\theta}{2}. \qquad (2.3)$$

The family of functions inside the circle $|z| < 1$, on which the minimum of the functional $\Phi(\Gamma)$ is being sought, is characterized by the following properties: 1) the $\Gamma(z)$ are regular in the closed circle $|z| \leq 1$ with the point $z = -1$ excluded, $\Gamma(0) = 1$ and $\Gamma(z^*) = (\Gamma(z))^*$; 2) for any $\epsilon > 0$ in a sufficiently small neighborhood of the point $z = -1$, $|\Gamma(z)| < \exp(\epsilon/|1+z|$.[1)]

Szego[5] and Smirnov[6] have completely solved the general extremum problem for the functional $\int_{-\pi}^{\pi} p(\theta)|F(e^{i\theta})|^2 d\theta$, where $p(\theta)$ is a prescribed nonnegative function and $F(z)$ is an arbitrary function from the class H_2 (about the class H_2 see below). We will show that the result of Szego and Smirnov applies to our case as well, but will obtain the solution by a different method in order to, in contrast to the solution of Szego and Smirnov, make it directly applicable to the case of a functional of several functions, subject to several linear relations. It is precisely this kind of extremum problem that must be solved if the particle a is a fermion.

3. Let $p(\theta)$ be a nonnegative function and the integral $\int_{-\pi}^{\pi} p(\theta)\, d\theta < +\infty$. Under these conditions there exists a set of orthonormal polynomials $\varphi_n(z)$ with the weight $p(\theta)$, i.e., a set of polynomials satisfying the conditions

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} p(\theta)\,\varphi_n(e^{i\theta})\,\varphi_m^*(e^{-i\theta})\, d\theta = \delta_{nm}. \qquad (3.1)$$

This set of polynomials is unique if one imposes the additional requirement that the coefficient of the highest power z^n in $\varphi_n(z)$ be positive. If the

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1)We note that the minimum of the integral (2.2) on the class of all regular functions (z), normalized by the condition $\Gamma(0) = 1$, is equal to zero. This can be seen by setting

$$\Gamma_n(z) = \exp\left[n\,\frac{1-z}{1+z} - n\right], \qquad |\Gamma_n(e^{i\theta})| = e^{-n}$$

then as $n \to +\infty$ the function $\Phi(\Gamma_n) \to 0$.

weight function $p(\theta)$ is an even function then all the coefficients in the polynomials $\varphi_n(z)$ are real.[5] We shall consider only such weight functions $p(\theta)$ for which the integrals $\int_{-\pi}^{\pi} |\ln p(\theta)|\, d\theta$ exist, so that the harmonic function which is equal to $\ln p(\theta)$ on the circumference is given by the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln p(\theta)\left\{\frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}\right\}d\theta, \qquad z = re^{i\varphi}.$$

The Poisson kernel $\{\ldots\}$ is equal to the real part of the Schwarz kernel $(e^{i\theta}+z)/(e^{i\theta}-z)$, consequently the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln p(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta$ is a regular function inside the circle $|z|<1$, whose real part tends to $\ln p(\theta)$ as $z \to e^{i\theta}$. It then follows that the regular inside the unit circle function

$$D(z) = \exp\left(\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln p(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right) \tag{3.2}$$

possesses the following properties: 1) $D(z) \neq 0$ for $|z|<1$; 2) $|D(re^{i\theta})|^2 \to p(\theta)$ as $r \to 1$; 3) $D(0) > 0$. The function $D(z)$ plays a fundamental role in the solution of the extremum problem. In Appendix II the following identity will be proved:

$$\sum_{n=0}^{\infty} \varphi_n^*(a^*)\,\varphi_n(z) = \frac{1}{1-a^*z}\,\frac{1}{D^*(a^*)}\,\frac{1}{D(z)}\,. \tag{3.3}$$

In particular

$$\sum_{n=0}^{\infty} |\varphi_n(z)|^2 = \frac{1}{1-|z|^2}\,\frac{1}{|D(z)|^2}, \qquad \sum_{n=0}^{\infty} |\varphi_n(0)|^2 = D^{-2}(0). \tag{3.4}$$

A function $\psi(z)$ regular inside the unit circle $= \sum_{n=0}^{\infty} c_n z^n$ is said to be a function of the class H_2 if $\sum |c_n|^2 < +\infty$. It is in terms of this class of functions that one solves the problem of the conditions necessary in order that the Fourier series of a function $F(z)$ converge to that function. In Appendix II will be proved the property established by Smirnov[6]: the Fourier series in the polynomials $\varphi_n(z)$ of a function $F_n(z)$ regular inside the unit circle converges uniformly to the function $F(z)$ in an arbitrary inner circle if and only if the product $D(z)F(z)$ belongs to the class H_2. At that the condition of completeness is satisfied

$$\sum |F_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(\theta)\,|F(e^{i\theta})|^2\,d\theta,$$

$$F_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(\theta)\,F(e^{i\theta})\,\varphi_n^*(e^{-i\theta})\,d\theta. \tag{3.5}$$

From here one immediately obtains the solution to the problem of finding the minimum of the functional

$$\Phi(F;p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(\theta)\,|F(e^{i\theta})|^2\,d\theta \tag{3.6}$$

subject to the relation $F(0) = C$, on the family of functions such that the product $D(z)F(z)$ belongs to the class H_2. Indeed, it follows from Eq. (3.5) that one must find the minimum of the sum of the series

$$\sum_{n=0}^{\infty} |F_n|^2 = \sum_{n=0}^{\infty} F_n F_n^* \tag{3.7}$$

subject to the linear relation

$$\sum_{n=0} \varphi_n(a)F_n = C = C_1 + iC_2. \tag{3.8}$$

In the general case, when the quantities appearing in Eq. (3.8) are complex, one obtains two linear relations

$$\sum(\varphi_n F_n + \varphi_n^* F_n^*) = 2C_1, \qquad -i\sum(\varphi_n F_n - \varphi_n^* F_n^*) = 2C_2. \tag{3.8'}$$

Introducing the Lagrange multipliers λ_1 and λ_2 we find that for the extremum function

$$F_n = (\lambda_1 + i\lambda_2)\,\varphi_n^*(a^*). \tag{3.9}$$

On substitution in Eq. (3.8) we obtain

$$\lambda_1 + i\lambda_2 = (C_1 + iC_2)\Big/\sum_{n=0}^{\infty} |\varphi_n(a)|^2. \tag{3.10}$$

It follows from Eqs. (3.4) and (3.3) that

$$\Phi_{min} = (\lambda_1^2 + \lambda_2^2)\sum |\varphi_n(a)|^2 = |C|^2(1-|a|^2)\,|D(a)|^2,$$

$$\Gamma_{\text{extr}}(z) = (\lambda_1 + i\lambda_2)\sum \varphi_n(a)\,\varphi_n(z) = C\,\frac{1-|a|^2}{1-a^*z}\,\frac{D(a)}{D(z)}\,. \tag{3.11}$$

If the quantities F_n, a, $\varphi_n(a)$ and C are real then the relation (3.8) defines a hyperplane in the Hilbert space of the coefficients $\{F_n\}$, and Φ_{min} is nothing else but the square of the distance from the origin of the coordinate system to that hyperplane.

4. For the extremum problem of Sec. 2 the weight function $p(\theta)$ is equal to $f(\theta)$, Eq. (2.3), $a = 0$, $C = 1$. For this weight function[2)]

$$D(0) = \left(1 + \sqrt{\frac{1-\lambda}{1-\alpha}}\right)^{1/2}\Big/\,2\sqrt{2}\,(1+\sqrt{1-\alpha}), \tag{4.1}$$

$$D(z) = \frac{(1-z)(1+z)^{1/2}}{2\sqrt{2}\,(1-\alpha)^{1/4}} \times \frac{[\sqrt{1-\lambda}+\sqrt{1-\alpha}+z(\sqrt{1-\lambda}-\sqrt{1-\alpha})]^{1/2}}{1+\sqrt{1-\alpha}+z(1-\sqrt{1-\alpha})}\,. \tag{4.2}$$

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2)The integral giving D(0) can be found in any table of integrals, the integral for D(z) is evaluated by Geshkenbeĭn and Ioffe.

If for any $\epsilon > 0$ within a neighborhood sufficiently close to the point $z = -1$ the function $\Gamma(z)$ is less than $\exp(\epsilon/|1+z|)$ then the same is true of the product $D(z)\Gamma(z)$. In addition

$$\int_{-\pi}^{\pi} |D(e^{i\theta})|^2 |\Gamma(e^{i\theta})|^2 d\theta = \int_{-\pi}^{\pi} f(\theta) |\Gamma(e^{i\theta})|^2 d\theta < +\infty.$$

In Appendix III it will be shown that it follows from these two properties that the product $D(z)\Gamma(z)$ belongs to the class H_2 and therefore the general solution (3.11) is valid in this particular case as well. Since $a = 0$, $C = 1$ we have

$$\Phi_{min} = D^2(0), \qquad \Gamma_{\text{экст}} = D(0)/D(z). \tag{4.3}$$

If one substitutes into these expressions the values $D(0)$ and $D(z)$ from Eqs. (4.1) and (4.2) one obtains the following expressions for all extremal quantities:

$$\Gamma_{\text{extr}}(x) = \frac{1}{2}\frac{(\sqrt{1-\alpha}+\sqrt{1-\lambda})^{1/2}}{1+\sqrt{1-\alpha}}\frac{1}{\sqrt{1-x}}$$
$$\times \frac{(1+i\sqrt{x-1})(\sqrt{1-\alpha}-\sqrt{1-x})}{(\sqrt{1-\lambda}-\sqrt{1-x})^{1/2}}, \tag{4.4}$$

$$\Phi_{min} = \frac{1}{8\sqrt{1-\alpha}}\frac{\sqrt{1-\alpha}-\sqrt{1-\lambda}}{(1+\sqrt{1-\alpha})^2}, \tag{4.5}$$

$$g^2 < 16\frac{[(m_b+m_c)^2-m_a^2]^{1/2}[(m_b+m_c)+\sqrt{(m_b+m_c)^2-m_a^2}]^2}{2\sqrt{m_bm_c}+\sqrt{(m_b+m_c)^2-m_a^2}}. \tag{4.6}$$

The care that must be exercised when taking the limit in solving extremum problems is illustrated by the following circumstance, interesting in its own right.

Let us suppose that the weight function tends to zero on some interval. Then $D(0)$ and Φ_{min} also tend to zero. It follows that if the lower limit in the integral defining the functional Φ should lie some distance to the right of the beginning of the cut then $\Phi_{min} = 0$, no matter how small that distance might be.

5. For physical applications a somewhat more general extremum problem is of interest, namely to find the minimum of the functional $\Phi(F_1, F_2, \ldots, F_k)$ which depends on k functions in the form

$$\Phi = \sum_{\nu=1}^{k}\frac{1}{2\pi}\int_{-\pi}^{\pi} p_\nu(\theta)|F_\nu(e^{i\theta})|^2 d\theta \tag{5.1}$$

subject to several linear relations

$$\sum_{\nu=1}^{k} a_{i\nu}F_\nu(z_i) = b_i, \qquad i = 1, 2, \ldots, l, \qquad |z_i| < 1. \tag{5.2}$$

It is assumed that all $p_\nu(\theta)$ are nonnegative functions that satisfy the conditions formulated in Sec. 3. The minimum of the functional is sought on a family of functions regular inside the unit circle, with all products $D_\nu(z)F_\nu(z)$ belonging to the class H_2. By $D_\nu(z)$ we mean the function $D(z)$ corresponding to the weight function $p_\nu(z)$.

This problem is solved in precisely the same way as the extremum problem for one function with one linear relation. Each of the functions $F_\nu(z)$ is expanded in the Fourier series $\Sigma F_{\nu n}\varphi_n^{(\nu)}(z)$ in the orthogonal polynomials corresponding to the weight function $p_\nu(\theta)$. The functional Φ goes over into the sum

$$\Phi = \sum_{n=0}^{\infty}(|F_{1n}|^2 + \ldots + |F_{kn}|^2). \tag{5.3}$$

To each of the relations (5.2) there correspond two Lagrange multipliers λ_{i1} and λ_{i2}. If one introduces $\lambda_i = \lambda_{i1} + i\lambda_{i2}$ then for the extremum functions one has

$$F_{\nu n} = \sum_{i=1}^{l}\lambda_i a_{i\nu}^*\varphi_n^{(\nu)*}(z_i^*). \tag{5.4}$$

The multipliers λ_i are determined from the set of equations

$$\sum_{j=1}^{l} c_{ij}\lambda_j = b_i, \qquad i = 1, 2, \ldots, l, \tag{5.5}$$

where

$$c_{ij} = \frac{1}{1-z_iz_j^*}\sum_{\nu=1}^{k}\frac{a_{i\nu}a_{j\nu}^*}{D_\nu(z_i)D_\nu^*(z_j^*)}, \qquad c_{ji} = c_{ij}^*. \tag{5.6}$$

The solution of the extremum problem is expressed in terms of the c_{ij} as follows:

$$\Phi_{min} = \sum_{i=1}^{l} b_i^*\lambda_i = -\begin{vmatrix} 0 & b_1^* & \ldots & b_l^* \\ b_1 & c_{11} & \ldots & c_{1l} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ b_l & c_{l1} & \ldots & c_{ll} \end{vmatrix} \|c_{ij}\|^{-1}, \tag{5.7}$$

and the extremal functions as follows

$$F_\nu(z) = \frac{1}{D_\nu(z)}\sum_{i=1}^{l}\frac{a_{j\nu}^*\lambda_j}{1-z_j^*z}\frac{1}{D_\nu^*(z_j^*)}. \tag{5.8}$$

In the fermion problem $k = 2$, $l = 3$, and the weight functions are such that $D_1(z)$ and $D_2(z)$ can be evaluated explicitly. As a result a closed form expression can be given for Φ_{min} in terms of the particle masses m_a, m_b, and m_c. [2]

I am grateful to B. L. Ioffe for useful discussions.

APPENDIX I

An arbitrary positive function $v(\zeta)$ harmonic inside the circle $|\zeta| < 1$ can be represented in the form

$$v(\zeta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}d\psi(\theta), \quad \zeta = re^{i\varphi}, \quad (I.1)$$

where $d\psi(\theta)$ is a certain distribution of nonnegative masses. We omit the proof as there can be no doubt as to the truth of this assertion. It is clear that the total mass

$$\int_{-\pi}^{\pi} d\psi(\theta) = 2\pi v(0) < +\infty. \quad (I.2)$$

It follows from Eq. (I.1) that the analytic function $f(\zeta) = v(\zeta) - iu(\zeta)$, for which $v(\zeta)$ serves as its real part, can be represented in the form

$$f(\zeta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\theta}+\zeta}{e^{i\theta}-\zeta}d\psi(\theta) + i\,\mathrm{Im}\,f(0). \quad (I.3)$$

In other words, a function regular inside the unit circle and with a positive real part can be expressed in the form Eq. (I.3).

This fact has at one time (1910) been established by Caratheodory and Herglotz. If one goes over from the unit circle $|\zeta| < 1$ to the upper half-plane $\mathrm{Im}\,x > 0$ by means of the transformation $\zeta + 1 = 2i/(x+1)$, and from the function $f(\zeta)$ to the R-function $F(x) = if(\zeta)$, then one obtains the representation, Eq. (1.6). From Eq. (1.6) and the identity

$$\frac{1+x'x}{x'-x} = \frac{1+x'^2}{x'-x} - x'$$

it follows that

$$F(x) = F(x_0) + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{x'-x} - \frac{1}{x'-x_0}\right](1+x'^2)\,d\sigma(x'), \quad (I.4)$$

but $(1+x'^2)\,d\sigma(x')$ may be replaced by $2\,\mathrm{Im}\,F(x'+i\delta)dx'$ and therefore

$$F(x) = F(x_0) + \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{x'-x} - \frac{1}{x'-x_0}\right]\mathrm{Im}\,F(x'+i\delta)\,dx', \quad (I.5)$$

i.e., every R-function satisfies a dispersion relation with no more than one subtraction. Let us note that in the relation (I.5) it is not assumed that the function $F(x)$ has no real poles. One has the relation

$$\int_{-\infty}^{\infty}\frac{2\,\mathrm{Im}\,F(x'+i\delta)}{1+x'^2}dx' = \int_{-\pi}^{\pi}d\sigma(x') = 2\pi\,\mathrm{Im}\,F(i). \quad (I.6)$$

Sometimes the following property of an R-function is useful: it satisfies the double inequality

$$\frac{\sin\varphi}{C}\frac{1}{r}|F(x)| < \frac{Cr}{\sin\varphi}, \quad x = re^{i\varphi}, \quad r > 1, \quad (I.7)$$

where C is a constant that depends on the function (see [7], p. 72).

APPENDIX II

Functions of the class H_2 are characterized by the following property: if $f(z) = \sum_{\nu=0}^{\infty} c_\nu z^\nu$ then

$$\lim_{\rho\to1}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\rho e^{i\theta})|^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(e^{i\theta})|^2 d\theta = \sum_{\nu=0}^{\infty}|c_\nu|^2 < +\infty.$$

If $f(z)$ and $g(z)$ are functions of the class H_2 then

$$\lim_{\rho\to1}\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\rho e^{i\theta})(g(\rho e^{i\theta}))^* d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(e^{i\theta})(g(e^{i\theta}))^* d\theta. \quad (II.1)$$

It is easy to see that the function $D(z)$ belongs to the class H_2 (see [5], p. 285). Let us prove that if $\Sigma F_\nu\varphi_\nu(e^{i\theta})$ is the Fourier series of the function $F(e^{i\theta})$ in the polynomials $\varphi_\nu(t)$ and the product $D(z)F(z)$ belongs to the class H_2 then the completeness relation

$$\sum_{\nu=0}^{\infty}|F_\nu|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}p(\theta)|F(e^{i\theta})|^2 d\theta < +\infty \quad (II.2)$$

holds. From completeness will follow the uniform convergence of $\Sigma F_\nu\varphi_\nu(z)$ to $F(z)$ inside any inner circle.

Let us denote by $\varphi_{\rho,\nu}(z)$ a set of orthonormal polynomials with the weight $|D(\rho e^{i\theta})|$. Since as $\rho \to 1$ the quantity $|D(\rho e^{i\theta})|^2$ tends to $p(\theta)$ it follows that for any fixed ν the function $\varphi_{\rho,\nu}(z)$ tends uniformly to $\varphi_\nu(t)$ in the closed circle $|z| \le 1$. Let us denote by $F_{\rho,\nu}$ the coefficients in the Fourier expansion of the function $F(\rho e^{i\theta})$ with respect to this set of polynomials:

$$F_{\rho,\nu} = \frac{1}{2\pi}\int_{-\pi}^{\pi}|D(\rho e^{i\theta})|^2 F(\rho e^{i\theta})\varphi^*_{\rho,\nu}d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}(D(\rho e^{i\theta})F(\rho e^{i\theta}))(D(\rho e^{i\theta})\varphi_\nu(\rho e^{i\theta}))^* d\theta.$$

Since the products $D(\rho e^{i\theta})F(\rho e^{i\theta})$ and $D(\rho e^{i\theta})\varphi_\nu(\rho e^{i\theta})$ belong to the class H_2 we have, as a consequence of Eq. (II.1),

$$\lim_{\rho\to1}F_{\rho,\nu} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(D(e^{i\theta})F(e^{i\theta})(D(e^{i\theta}))\varphi_\nu(e^{i\theta}))^* d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}p(\theta)F(e^{i\theta})\varphi^*_\nu(e^{-i\theta})d\theta = F_\nu.$$

Thus the completeness relation is valid for the function $F(\rho e^{i\theta})$

$$\sum_{\nu=0}^{\infty}|F_{\rho,\nu}|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|D(\rho e^{i\theta})|^2|F(\rho e^{i\theta})|^2 d\theta.$$

As $\rho \to 1$ the integral on the right tends to

$(2\pi)^{-1} \cdot \int_{-\pi}^{\pi} p(\theta)|F(e^{i\theta})|^2 d\theta$ since $D(z)F(z) \in H_2$ and, consequently,

$$\sum_{\nu=0}^{\infty} |F_\nu|^2 \geqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} p(\theta)\,|F(e^{i\theta})|^2\,d\theta.$$

By the Bessel inequality

$$\sum_{\nu=0}^{\infty} |F_\nu|^2 \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} p(\theta)\,|F(e^{i\theta})|^2\,d\theta,$$

i.e., the equality sign holds.

Let us take for $F(z)$ the function $[(1-a^*z)\times D(a)D(z)]^{-1}$, $|a| < 1$. It is easy to verify by direct calculation that $F_\nu = \varphi_\nu^*(a^*)$.

Since $D(z)F(z) = (1-a^*z)^{-1}D(a)$ belongs to the class H_2 it follows that for $|z| < 1$

$$\sum_{\nu=0}^{\infty} \varphi_\nu^*(a^*)\,\varphi_\nu(t) = [(1-a^*z)\,D(a)\,D(z)]^{-1}.$$

This identity is given by Szego (see [5], p. 311) and by Smirnov.[6] The difference lies in the fact that we solve the extremum problem on the basis of this identity whereas Szego and Smirnov prove the identity on the basis of the solution of the extremum problem.

APPENDIX III

Let $\psi(z) = \Sigma c_n z^n$ be a function regular in the closed circle $|z| \leq 1$ with the exception of the point $z = -1$ and satisfying the following conditions: 1) for any $\epsilon > 0$ in a sufficiently small neighborhood of the point $z = -1$ one has $|\psi(z)| < \exp(\epsilon/|1+z|)$; 2) $\int_{-\pi}^{\pi} |\psi(e^{i\theta})|^2 d\theta < +\infty$. Let us show that the function $\psi(z)$ belongs to the class H_2, i.e., that $\Sigma|c_n|^2 < +\infty$.

The transformation $w+i = 2i/(1+z)$ maps the circle $|z| < 1$ into the upper half-plane $\mathrm{Im}\, w > 0$; at that the point $z = -1$ goes to ∞. We introduce the function $F(w) = i\sqrt{2}\,\psi(z)/(w+i)$. The function $F(w)$ is regular in the upper half-plane, for any $\epsilon > 0$ and sufficiently large $|w|$ the inequality $|F(w)| < \exp(\epsilon|w|)$ is satisfied and, finally, the integral $\int_{-\infty}^{\infty} |F(w)|^2 dw < +\infty$.

As was shown by Wiener and Paley (see [8], p. 9), for a function with these properties the integral over the large upper semicircle in the Cauchy formula tends to zero, i.e.,

$$F(w) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{F(w)}{w'-w}\,dw.$$

From here there follows for the function $\psi(z)$ the representation

$$\psi(z) = \frac{1}{2\pi i}\int_{|z'|=1} \frac{\psi(z')\,dz'}{z'-z}, \qquad \text{(III.1)}$$

I.e., the Cauchy formula is valid for the function $\psi(z)$ not only when the integration is over an internal circle but also when the integration is over the boundary circle.

It follows from Eq. (III.1) that

$$c_n = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \psi(e^{i\theta})\,e^{-in\theta}\,d\theta,$$

i.e., c_n coincides with the Fourier coefficient of the function $\psi(e^{i\theta})$. In accordance with the Bessel inequality

$$\sum|c_n|^2 < \frac{1}{2\pi}\int_{-\pi}^{\pi} |\psi(e^{i\theta})|^2\,d\theta < +\infty.$$

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201