NONSYMMETRIC ULTRAVIOLET ASYMPTOTIC BEHAVIOR OF HIGHER-ORDER GREEN'S FUNCTIONS IN RENORMALIZED FIELD THEORY

I. F. GINZBURG

Mathematics Institute and Computation Center, Siberian Section of the Academy of Sciences, U.S.S.R.

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The nonsymmetric ultraviolet asymptotic behavior of many-particle Green's functions in the unphysical region ($p_i^2 \to \infty$) is determined by the diagrams involving an exchange of the minimum number of particles. The asymptotic expressions are given with a power-law accuracy.

DURING the past few years certain interest has been shown in the ultraviolet asymptotic behavior of higher-order Green's functions in renormalized field theories. In the weak coupling case these problems have been investigated from the standpoint of the renormalization group (cf. [1], where references to earlier work are found) up to logarithmic terms. In Weinberg's paper [2] the asymptotic behavior has been estimated up to a power-law accuracy. In what follows a similar estimate with a power-law accuracy is obtained by a method differing from the one in [2]. The methods developed here are useful for an analysis of physical phenomena at high energies.

1. FORMULATION OF THE PROBLEM

Let us consider a process involving N particles with momenta $p_1, \ldots, p_N$. This process is described by the Green's function

$$G(p_1, \ldots, p_N) = -i\left\langle 0 \left| \frac{\delta^N S}{\delta\psi(p_1)\ldots\delta\psi(p_N)} \right| 0 \right\rangle, \quad (1.1)$$

$$\sum_{i=1}^{n} p_i = l = -\sum_{k=n+1}^{N} p_k, \quad \sum_{k=1}^{N} p_k = 0. \quad (1.2)$$

Let the momenta $p_{n+1}, \ldots, p_N$ be fixed and let the momenta $p_1, \ldots, p_n$ simultaneously tend to infinity so that all their components are large and of the same order

$$|p_i^2| \sim |p_i p_j| \sim p^2 \gg m^2, \ l^2, \ 1 \leqslant i, j \leqslant n. \quad (1.3)$$

Below we shall investigate the asymptotic behavior of G(p) in this region.

One can put the function G in correspondence with the series of the usual renormalized perturbation theory:

$$G = \sum G_\sigma; \quad (1.4)$$

where G is one of the Feynman diagrams of the process under consideration [1]). (The divergences have been eliminated for the expressions of G obtained from the usual rules of perturbation theory.) The sum in Eq. (1.4) is to be understood in some generalized sense, which we shall not analyze here.

We divide all the diagrams G into non-overlapping classes and we assume that the power estimates obtained for each diagram of a certain class are also valid for the sum of all diagrams of that class. This is a strong assumption and it is not clear whether it is possible to prove its validity. Furthermore we restrict ourselves to the consideration of renormalized theories only.

We enumerate the properties of such a theory which are important for the present work (cf. e.g. [3]).

1. To each vertex μ of the diagram there corresponds a certain number

$$\omega(\mu) = \frac{1}{2}\sum_l (r_l + 2) - 4. \quad (1.5)$$

Here the sum runs over all lines originating in the vertex μ. In particular r_l equals 0, 1, or 2 for spin-0 particles (and photons), spin-1/2 particles and spin-1 particles (and also if the Lagrangian contains the coordinate derivative of a spin 0 particle), respectively, etc. The theory is renormalizable if all $\omega(\mu) \leq 0$. However interactions for which $\omega(\mu) < 0$ are unlikely, so that for simplicity we assume $\omega(\mu) = 0$ [2]).

2. Strongly (weakly) connected diagrams can

[1])In what follows the same notation will be used for a particle, its momentum and the corresponding line in the diagram, as well as for a diagram and its contribution to a Green's function, etc.

[2])It is easy to see that our result also holds for $\omega(\mu) < 0$.

(can not) be converted into unconnected diagrams by removing one line. For each strongly connected diagram F (after eliminating the singularities of lower order inside the diagram) one can find the index of the diagram or the conventional rate of growth with momentum. Under these assumptions this index depends only on the nature of the external lines l_{ext} of the diagram:

$$\omega(F) = 4 - \tfrac{1}{2}\sum_{l_{ext}} (r_l + 2). \qquad (1.6)$$

2. THE CONNECTION BETWEEN THE ASYMPTOTIC BEHAVIOR AND THE TOPOLOGY OF THE DIAGRAMS

To each pair of diagrams G_σ and G_τ we put in correspondence two other diagrams $M_{\sigma\tau}$ and $M_{\tau\sigma}$ in the following manner: we construct the diagram $\bar{G}_\sigma$ which is the Hermitian conjugate of G_σ and then connect all the lines p_i $(1 \le i \le n)$ of the diagrams $\bar{G}_\sigma$ and G_τ with each other (cf. Fig. 1)[3]:

$$M_{\sigma\tau} \sim \int \bar{G}_\sigma D^c(p_1)\, dp_1 \ldots D^c(p_n)\, dp_n G_\tau \delta\Big(\sum_{l=1}^{n} p_l - l\Big). \qquad (2.1)$$

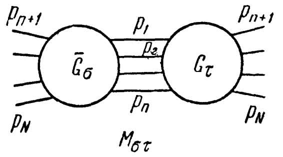

FIG. 1

If G_σ and G_τ are strongly connected diagrams, the index of the diagram $M_{\sigma\tau}$ is defined as [4]:

$$\omega(M_{\sigma\tau}) = \omega(M_{\tau\sigma}) = 4 - \sum_{l=n+1}^{N} (r_l + 2). \qquad (2.2)$$

On the other hand, assuming that

$$G_\sigma(p) \sim p^{-n_\sigma} H_\sigma(p),$$
$$\int |H_\sigma(p)|^2 \frac{dp}{p} p^\varepsilon = \begin{cases} <\infty, & \varepsilon < 0 \\ \infty, & \varepsilon = 0 \end{cases}, \qquad (2.3)$$

one can obtain from Eq. (2.1) by direct computation

$$\omega(M_{\sigma\tau}) = -n_\sigma - n_\tau + \sum_{l=1}^{n} (r_l + 2) - 4. \qquad (2.4)$$

Carrying this out for the diagram $M_{\sigma\sigma}$ one can see that

$$n_\sigma = \frac{1}{2} \sum_{l=1}^{N} (r_l + 2) - 4.$$

Let us denote the numbers of particles with r_l equal to 0 and 1 by b and f, for $1 \le l \le n$ and by b′ and f′ for $n + 1 \le l \le N$, respectively [5]. Then

$$n_\sigma = b + \tfrac{3}{2} f - 2 + \big(b' + \tfrac{3}{2} f' - 2\big). \qquad (2.5)$$

We note, however, that if the diagram G_σ is weakly connected, then Eq. (2.5) is in general, no longer valid for it. In this case an estimate of the type (2.5) is valid for each strongly connected part of G_σ. It is easy to see in this case that if the large momentum p is contained in each of these strongly connected parts G_σ, then the estimate (2.5) remains valid [6]. Let some strongly connected subdiagrams G_σ [7] not contain p and let the momentum transfer in the vertex connecting these subdiagrams with the subdiagrams containing p be small (i.e., constant as $p^2 \to \infty$). Let us denote by b_σ and f_σ the numbers of boson and fermion lines connecting such strongly connected subdiagrams of G_σ with the rest of the diagram G_σ (in which all strongly connected subdiagrams contain the momentum p). For example, in Fig. 2.a, for n = 3 we have $b_\sigma + f_\sigma = 3$ and in Figs. 2.b and c for n = 3 we have $b_\sigma + f_\sigma = 2$ (the circles denote strongly connected diagrams). Then it is easy to see, as before, that

$$\omega(M_{\sigma\tau}) = 4 - \Big[b_\sigma + b_\tau + \tfrac{3}{2}(f_\sigma + f_\tau)\Big], \qquad (2.6)$$

$$n_\sigma = b + \tfrac{3}{2} f - 2 + \big(b_\sigma + \tfrac{3}{2} f_\sigma - 2\big). \qquad (2.7)$$

The quantity n_σ takes on its minimal value when $b_\sigma + 3/2 f_\sigma$ is minimal. Depending on the

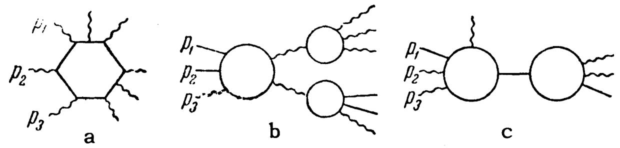

FIG. 2

[3]Here $D^c(p)$ is the causal function: for bosons it is $D^c(p)$ and for fermions it is $S^c(p)$.

[4]r_l corresponds to the particle p_l.

[5]For simplicity we assume that there are no lines with $r_l = 2$.

[6]Here we use the natural assumption that the power-law dependence of the complete Green's functions $D^c(p)$ on p is the same as for the free $\mathfrak{D}^c(p)$, i.e., for $\varepsilon > 0$

$$\lim_{p\to\infty} \mathfrak{D}^c(p)\,[D^c(p)]^{-1} p^{-\varepsilon} = 0.$$

[7]A subdiagram of G_σ is a strongly connected part of G_σ which is connected to the rest of G_σ through a line connecting vertices with fixed momentum transfer.

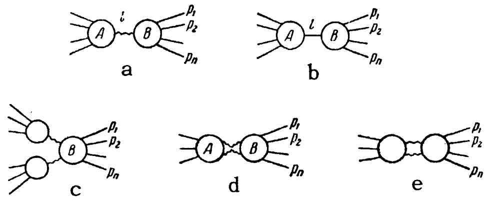

FIG. 3

nature of the particles p_i ($1 \le i \le n$), such a minimum can be attained by virtue of the conservation laws either for $b_\sigma = 1$ or for $f_\sigma = 1$, or for $b_\sigma = 2$ (corresponding to diagrams of the types represented in Figs. 3, a and b and Figs. 3, c and d, respectively.

Let us introduce the notation

$$\left(b_\sigma + \tfrac{3}{2} f_\sigma - 2\right)_{min} = g(n). \qquad (2.8)$$

Then the minimal possible n_σ is

$$(n_\sigma)_{min} = b + \tfrac{3}{2} f - 2 + g(n). \qquad (2.9)$$

This minimum is reached for those diagrams G_σ, where the "bubbles" from which the lines p_i with $1 \le i \le n$ and p_j with $n + 1 \le j \le N$ originate are connected into weakly connected diagrams with a minimal amount of lines. According to the composition of the group of fast particles these are diagrams of the forms represented in Figs. 3, a and b and Figs. 3,c and d.

The contributions from these diagrams decrease most slowly for $p^2 \to \infty$, and for large p^2 their contribution to G is the essential one. Asymptotically (for $p^2 \to \infty$) the behavior of the function G is determined only by the diagrams of the form Figs. 3, a—d. With the restrictions mentioned earlier one can consider that for $p^2 \to \infty$ these diagrams are already summed (the circles A and B on Figs. 3,a—d will then correspond to the complete sums of the perturbation theory series for the corresponding subdiagrams). By including in G_A the propagator joining A and B we obtain:

$$G = G_A(p_{n+1}, \ldots, p_N)\, G_B(p_1, \ldots, p_n; l). \qquad (2.10)$$

3. DISCUSSION

The result (2.10) means that for large p^2 the interaction can be considered peripheral. For n = 2 (2.10) goes over into the result obtained earlier in [4]. If there are particles with vanishing mass in the theory (photons), it is important to consider together with the given process also processes involving the emission of an arbitrary number of soft photons. It is easy to see that our result (2.10) remains valid also in this case, but Eq. (2.9) will be valid only to order α (cf., e.g., [3,5]).

The finer, logarithmic, details of the behavior of the function G for large p^2 can be obtained by considering only the "right halves" G_B of the diagrams in Figs. 3,a—d, i.e., diagrams with a smaller number of particles than G. It is possible that in this case the situation gets less complicated, because all arguments of G_B with the exception of one (l) are large, i.e., we are dealing with "almost symmetric" asymptotic behavior. For weak coupling a recipe for obtaining such asymptotic behavior is indicated in [1].

The result (2.9) almost completely coincides with the results of Weinberg [2], which were obtained by means of a much more involved method. The difference consists in the fact that in [2] diagrams of the type in Fig. 3,e are admitted, in addition to the diagrams in Figs. 3, c, d, i.e., Eq. (2.10) is not obtained in the case b′ = 2. The result (2.9) is also close to the result of Medvedev and Polivanov [6], obtained for n = N, by means of the axiomatic approach.

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