RELAXATION OF PHOTONS AND PLASMA ELECTRONS IN A STRONG MAGNETIC FIELD

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Relaxation of plasma electrons and of the photons emitted by them as a result of radiative processes in a strong magnetic field have been studied. The time variation of the photon distribution function and the mean transverse electron energy $\overline{\epsilon}_\perp(t)$ have been derived in the relativistic case. During the photon relaxation time $\tau \sim \omega_H \Omega^{-2}\sqrt{T/mc^2}$ (Ω is the plasma frequency and ω_H the Larmor frequency) the mean transverse electron energy practically reaches its stationary value $\overline{\epsilon}_\perp(\tau) \approx \overline{\epsilon}_\perp(\infty)$, whereas the photon distribution function approaches a Rayleigh-Jeans distribution with a temperature $T = \overline{\epsilon}_\perp(\infty)$ in the frequency range $\Delta = \omega_H\sqrt{T/mc^2}$ about the frequency ω_H. For periods $t > \tau$, the electron relaxation proceeds (with a constant mean transverse energy) toward a Maxwellian distribution.

1. It has been shown previously[1,2] that the process of radiation and absorption of electromagnetic waves by plasma electrons in a strong magnetic field can exhibit a significant effect on the relaxation of electrons and transport phenomena in a plasma. However, the problem of the relaxation of the photon distribution function has scarcely been considered in these researches. It has only been established that the photon distribution function tends toward the electron distribution in a time which is much shorter than the relaxation time of the electrons.

The present research is devoted to a detailed study of the relaxation of photons and electrons in the nonrelativistic case under the assumption that radiative prccesses play a much more important role than Coulomb collisions.

2. As was shown in [3], the kinetic equations for the homogeneous case can be written for the electron distribution function f and the photon density matrix $N_{\lambda\lambda'}$ in the form

$$\frac{\partial f_1}{\partial t} = \mathrm{Sp}\sum_{2\mathbf{k}}\{(1-f_1)f_2(1+\hat{N}) - f_1(1-f_2)\hat{N}\}\hat{R}(1\mathbf{k};2)$$
$$\times\,\delta(\varepsilon_1+\hbar\omega-\varepsilon_2) - \mathrm{Sp}\sum_{2\mathbf{k}}\{(1-f_2)f_1(1+\hat{N})$$
$$-f_2(1-f_1)\hat{N}\}\hat{R}(2\mathbf{k};1)\,\delta(\varepsilon_2+\hbar\omega-\varepsilon_1),$$

$$\frac{\partial\hat{N}}{\partial t} = \frac{1}{2}\sum_{12}\{(1-f_1)f_2(1+\hat{N}) - f_1(1-f_2)\hat{N}\}\hat{R}(1\mathbf{k};2)$$
$$\times\,\delta_+(\varepsilon_1+\hbar\omega-\varepsilon_2) + \frac{1}{2}\sum_{12}\hat{R}(1\mathbf{k};2)\{(1-f_1)f_2(1+\hat{N})$$
$$-f_1(1-f_2)\hat{N}\}\,\delta_-(\varepsilon_1+\hbar\omega-\varepsilon_2), \qquad (1)$$

where $\hat{R}$ is a matrix in the space of polarization indices λ with matrix elements

$$R_{\lambda\lambda'}(1\mathbf{k};2) = (2\pi/\hbar)\,\Psi^*_\lambda(1\mathbf{k};2)\,\Psi_{\lambda'}(1\mathbf{k};2) \qquad (2)$$

and $\Psi_\lambda(1\mathbf{k};2)$ is the probability amplitude for transition of the electron from state 2 (with quantum numbers n_2, p_{2z}, which define the transverse electron energy $\epsilon_\perp = \hbar\omega_H(n+1/2)$ and the longitudinal momentum of the electron relative to the magnetic field H) to the state 1 (with quantum numbers n_1, p_{1z}) with emission of a photon with wave vector $\mathbf{k}$ and polarization λ. In Eq. (1), the trace is taken in the space of the polarization indices λ.

In the classical approximation, the quantity $\hat{R}$ has the form

$$R_{\lambda\lambda'}(1\mathbf{k};2) = (4\pi^2/\omega V)\,(\mathbf{e}_{\mathbf{k}\lambda}\mathbf{B})^*(\mathbf{e}_{\mathbf{k}\lambda'}\mathbf{B})$$
$$\times\,\Delta(p_{1z}+\hbar k_z-p_{2z})\,\Delta(p_{1y}+\hbar k_y-p_{2y}), \qquad (3)$$

$$B_x = A_1e^{-i\varphi}+A_2e^{i\varphi},\quad B_y = i(A_1e^{-i\varphi}-A_2e^{i\varphi}),$$
$$A_1-A_2 = -\frac{e}{m}p_\perp J'_{|s|}\left(\frac{\omega p_\perp}{eH}\sin\vartheta\right)\frac{s}{|s|},$$
$$A_1+A_2 = \frac{e}{m}\frac{|s|eH}{\omega\sin\vartheta}J_{|s|}\left(\frac{\omega p_\perp}{eH}\sin\vartheta\right),$$
$$B_z = \frac{e}{m}p_zJ_{|s|}\left(\frac{\omega p_\perp}{eH}\sin\vartheta\right). \qquad (3')$$

In these formulas V is the normalized volume, φ, ϑ the azimuthal and polar angles of the wave vector $\mathbf{k}$ relative to the magnetic field, and $s = n_2 - n_1$.

It is easy to prove that in the nonrelativistic case, for $\omega \sim \omega_H$,

$$\mathbf{e}_{k\lambda}^{0}\mathbf{B}=\frac{ep_{\perp}}{2m}\begin{cases}-\cos\vartheta, & \lambda=1\\ i, & \lambda=2,\end{cases}\quad s=1,$$

where $\mathbf{e}_{\mathbf{k}1}^{0}$ is the unit polarization vector lying in the plane of the vectors $\mathbf{k}$ and $\mathbf{H}$, while $\mathbf{e}_{\mathbf{k}2}^{0}$ is the unit polarization vector orthogonal to this plane.

In what follows it will be convenient to use the polarization vectors $\mathbf{e}_{\mathbf{k}\lambda}$, which are connected with the linear polarization vectors $\mathbf{e}_{\mathbf{k}\lambda}^{0}$ by the relations

$$\mathbf{e}_{\mathbf{k}1}=(-i\cos\vartheta\,\mathbf{e}_{\mathbf{k}1}^{0}+\mathbf{e}_{\mathbf{k}2}^{0})(1+\cos^2\vartheta)^{-1/2},\qquad(4)$$

$$\mathbf{e}_{\mathbf{k}2}=(\mathbf{e}_{\mathbf{k}1}^{0}-i\cos\vartheta\,\mathbf{e}_{\mathbf{k}2}^{0})(1+\cos^2\vartheta)^{-1/2}.$$

These polarization vectors correspond to an elliptically polarized wave and possess the property that for them

$$\mathbf{e}_{\mathbf{k}\lambda}\mathbf{B}=i(ep_{\perp}/2m)(1+\cos^2\vartheta)^{1/2}\delta_{\lambda 1},\qquad(5)$$

and consequently, in accord with (3), the operator $\hat{R}$ is reduced to diagonal form:

$$R_{\lambda\lambda'}=\frac{4\pi^2}{\omega V}\left(\frac{ep_{\perp}}{2m}\right)^2(1+\cos^2\vartheta)\delta_{\lambda_1}\delta_{\lambda'1}\Delta$$

$$\times(p_{1z}+\hbar k_z-p_{2z})\,\Delta(p_{1y}+\hbar k_y-p_{2y}).\qquad(6)$$

This formula refers to the case $s=n_2-n_1=1$. In the nonrelativistic case, $R_{\lambda\lambda'}$ is of the same scale, for $s=-1$, as in (6), we do not need the corresponding expression in what follows.

We represent the kinetic equation for the photon distribution function $N_{\lambda\lambda'}$ in the following form:

$$\partial\hat{N}/\partial t=-\hat{N}\hat{\Gamma}-\hat{\Gamma}^{+}\hat{N}+\hat{\nu};\qquad(7)$$

$$\hat{\Gamma}=\frac{1}{2}\sum_{12}(f_1-f_2)\,\hat{R}\,(1\mathbf{k};\,2)\,\delta_{+}(\varepsilon_1+\hbar\omega-\varepsilon_2),$$

$$\hat{\nu}=\sum_{12}f_2(1-f_1)\,\hat{R}\,(1\mathbf{k};\,2)\,\delta(\varepsilon_1+\hbar\omega-\varepsilon_2),\qquad(7')$$

where $\Gamma^{+}_{\lambda\lambda'}\equiv\Gamma^{*}_{\lambda'\lambda}$ is the Hermitian conjugate to Γ.

3. In the classical approximation, the electron radiation collision integral can, in accord with [1,3], be written in the form

$$\dot{f}^{(r)}=\frac{1}{p_{\perp}}\frac{\partial}{\partial p_{\perp}}p_{\perp}j_{\perp}^{(r)}+\frac{\partial}{\partial p_z}j_z^{(r)};\qquad(8)$$

$$j_{\perp}^{(r)}=\frac{V}{(2\pi)^3}\mathrm{Sp}\int d\mathbf{k}\,\frac{\omega}{cp_{\perp}}\left(\frac{\varepsilon}{c}-p_z\cos\vartheta\right)\left\{f+\frac{\hbar\omega}{c}\hat{N}\right.$$

$$\left.\times\left[\frac{\partial f}{\partial p_z}\cos\vartheta+\frac{1}{p_{\perp}}\left(\frac{\varepsilon}{c}-p_z\cos\vartheta\right)\frac{\partial f}{\partial p_{\perp}}\right]\right\}$$

$$\times\sum_{n'p_z'p_y'}\hat{R}\,(\mathbf{p}',\mathbf{k};\,\mathbf{p})\,\delta(\varepsilon'+\hbar\omega-\varepsilon),$$

$$j_z^{(r)}=\frac{V}{(2\pi)^3}\mathrm{Sp}\int d\mathbf{k}\,k_z$$

$$\times\left\{f+\frac{\hbar\omega}{c}\hat{N}\left[\frac{\partial f}{\partial p_z}\cos\vartheta+\frac{1}{p_{\perp}}\left(\frac{\varepsilon}{c}-p_z\cos\vartheta\right)\frac{\partial f}{\partial p_{\perp}}\right]\right\}$$

$$\times\sum_{n'p_z'p_y'}\hat{R}\,(\mathbf{p}',\mathbf{k};\,\mathbf{p})\,\delta(\varepsilon'+\hbar\omega-\varepsilon).\qquad(9)$$

In the nonrelativistic case, as is seen from these formulas, $j_z^{(r)}\ll j_{\perp}^{(r)}$, and consequently,

$$\dot{f}^{(r)}=\frac{1}{p_{\perp}}\frac{\partial}{\partial p_{\perp}}p_{\perp}j_{\perp}^{(r)}.\qquad(8')$$

The current $j_{\perp}^{(r)}$, in the nonrelativistic case, in accord with Eqs. (6), (9), can be written in the form

$$j_{\perp}^{(r)}=\frac{p_{\perp}}{2\tau_e^{(r)}}\left\{f+\zeta(t)\frac{\partial f}{\partial\varepsilon_{\perp}}\right\};\qquad(10)$$

$$1/\tau_e^{(r)}=4r_0\omega_H^2/3c,\qquad r_0=e^2/mc^2,$$

$$\zeta(t)=\frac{3}{16\pi}\hbar\omega_H\int do\,(1+\cos^2\vartheta)\,N_{11}(p_z;\,\vartheta,\varphi),\qquad(10')$$

where $N_{11}(p_z;\,\vartheta,\varphi)$ N_{11} is the value of the function $N_{11}(\mathbf{k})$ for

$$\omega=\bar{\omega}_H=\omega_H/(1-(p_z/mc)\cos\vartheta)).$$

4. We now proceed to find the values of ν and Γ in the classical nonrelativistic approximation.

In carrying out summation over n_2 in Eqs. (7′), we can restrict ourselves in the nonrelativistic case, in accord with (3) and (3′), just to the terms with $n_2=n_1\pm1$. However, since we are considering photon frequencies close to ω_H, the principal role in the nonrelativistic case ($T\ll mc^2$) is played by the component with $n_2=n_1+1$ ($s=1$), owing to the presence in Eqs. (7′) of the functions $\delta_{+}(\epsilon_1+\hbar\omega-\epsilon_2)$, $\delta(\epsilon_1+\hbar\omega-\epsilon_2)$. Moreover, keeping in mind that, in the classical approximation,

$$f_2(1-f_1)\approx f_1,\qquad f_1-f_2\approx-\hbar\omega_H\frac{m}{p_{\perp}}\frac{\partial f}{\partial p_{\perp}},$$

we get, in accord with (7′) and (6),

$$\nu_{\lambda\lambda'}|_{\omega=\bar{\omega}_H}=\frac{mr_0c^2}{2\hbar\omega_H^2}\frac{1+\cos^2\vartheta}{|\cos\vartheta|}\frac{1}{2\pi\hbar^2}\int d\mathbf{p}_1\varepsilon_{1\perp}f_1\delta(p_z-p_{1z})\delta_{\lambda1}\delta_{\lambda'1},$$

$$\Gamma_{\lambda\lambda'}|_{\omega=\bar{\omega}_H}$$

$$=\frac{mr_0c^3}{2\omega_H}(1+\cos^2\vartheta)\frac{1}{2\pi\hbar^3}\int d\mathbf{p}_1\delta_{+}(\cos\vartheta\,(p_z-p_{1z}))\,f_1\delta_{\lambda1}\delta_{\lambda'1}.$$

Introducing the notation

$$\bar{\varepsilon}_{\perp}=\int d\mathbf{p}_1\varepsilon_{1\perp}f_1\,\delta(p_z-p_{1z})\Big/\int d\mathbf{p}_1f_1\,\delta(p_z-p_{1z}),$$

$$\frac{1}{\tau}=\frac{e^2c}{4\pi\hbar^3\omega_H}\int d\mathbf{p}_1f_1\delta(p_z-p_{1z}),\qquad(11)$$

we transform these formulas to the form

$$\nu_{\lambda\lambda'} = \nu\delta_{\lambda 1}\delta_{\lambda' 1}, \quad \Gamma_{\lambda\lambda'} = \Gamma\delta_{\lambda 1}\delta_{\lambda' 1};$$

$$\nu = \frac{\bar{\varepsilon}_\perp}{\hbar\omega_H}\operatorname{Re}\Gamma, \quad \Gamma = \frac{1+\cos^2\vartheta}{2\tau\,|\cos\vartheta\,|}(1+i\xi),$$

$$\xi = \pm\pi P\int\frac{f_1 d\mathbf{p}_1}{p_z - p_{1z}}\Big/\int f_1\delta(p_z - p_{1z})\,d\mathbf{p}_1. \quad (12)$$

In the latter formula the plus sign corresponds to the value of $\cos\vartheta > 0$, while the minus sign corresponds to $\cos\vartheta < 0$.

Using Eqs. (12), we transform Eq. (7) for the photon distribution function to the form

$$\frac{\partial N_{11}}{\partial t} = -\frac{1+\cos^2\vartheta}{\tau\,|\cos\vartheta\,|}\left\{N_{11} - \frac{\bar{\varepsilon}_\perp}{\hbar\omega_H}\right\}, \quad (13)$$

$$\frac{\partial N_{12}}{\partial t} = -\Gamma^* N_{12}, \qquad \frac{\partial N_{21}}{\partial t} = -\Gamma N_{21}, \qquad \frac{\partial N_{22}}{\partial t} = 0. \quad (14)$$

From the latter formula (14) it is evident that photons with the polarization $\mathbf{e}_{k2}$ do not generally take part in radiative processes.

5. We consider the relaxation of the skew components N_{12} and N_{21} of the density matrix $N_{\lambda\lambda'}$. For this purpose, we note that, in accordance with (1) and (8′),

$$\frac{\partial}{\partial t}\int f_1 d\mathbf{p}_\perp = \int\frac{1}{p_\perp}\frac{\partial}{\partial p_\perp}p_\perp j_\perp^{(r)} d\mathbf{p}_\perp = 0$$

and, consequently, the quantity Γ does not depend on the time.

Integration of Eq. (14) yields

$$N_{21}(t) = N_{21}(0)\,e^{-\Gamma t}, \qquad N_{12}(t) = N_{21}^*(t). \quad (15)$$

If we assume that there is initially a Maxwellian distribution with temperature $T_{||}$ in the longitudinal momentum of the electrons, then we get the following for the quantities $1/\tau$, ξ, in accord with (11), (12),

$$\frac{1}{\tau} = \sqrt{\frac{\pi}{8}}\frac{\Omega^2}{\omega_H}\sqrt{\frac{mc^2}{T_{||}}}\exp\left(-\frac{p_z^2}{2mT_{||}}\right),$$

$$\Omega^2 = \frac{4\pi ne^2}{m}, \qquad \xi = 2\sqrt{\pi}\int_0^{p_z/\sqrt{2mT_{||}}} e^{u^2}du. \quad (16)$$

Thus the values of N_{12}, N_{21}, which oscillate with the frequency $\sim\xi/\tau$, are damped out in a time of the order of τ.

6. We turn to a consideration of the relaxation of N_{11}. The unknown quantity ζ, associated with the electron distribution function, appears in Eq. (13), which determines the change of N_{11} with time.

The electron distribution function f, in accordance with (8′), (10), satisfies the equation

$$\frac{\partial f}{\partial t} = \frac{1}{\tau_e^{(r)}}\frac{\partial}{\partial\varepsilon_\perp}\varepsilon_\perp\left\{f + \zeta\frac{\partial f}{\partial\varepsilon_\perp}\right\}, \quad (17)$$

where the quantity ζ, by (10′), does not depend on $\epsilon_\perp$. It is easy to get

$$\frac{d}{dt}\int d\mathbf{p}_\perp\varepsilon_\perp f = -\frac{1}{\tau_e^{(r)}}\int d\mathbf{p}_\perp(\varepsilon_\perp - \zeta)\,f,$$

$$\frac{d}{dt}\int d\mathbf{p}_\perp f = 0.$$

from Eq. (17). Making use of the definition (11) of the quantity $\bar{\epsilon}_\perp$, we then find

$$d\bar{\varepsilon}_\perp/dt = -(\bar{\varepsilon}_\perp - \zeta)/\tau_e^{(r)}. \quad (18)$$

Equations (13) and (18), together with the definition (10′) of the quantity ζ, make up the complete set of equations for the analysis of the relaxation of N_{11}:

$$\dot{\bar{\varepsilon}}_\perp = -\frac{1}{\tau_e^{(r)}}\bar{\varepsilon}_\perp + \frac{3}{8}\frac{\hbar\omega_H}{\tau_e^{(r)}}\int_{-1}^{1}dx\,(1+x^2)\,N_{11}(x),$$

$$\dot{N}_{11}(x) = -\frac{1+x^2}{\tau\,|x|}\left\{N_{11}(x) - \frac{\bar{\varepsilon}_\perp}{\hbar\omega_H}\right\}, \quad (19)$$

where $x = \cos\vartheta$ and

$$N_{11}(x) = \frac{1}{2\pi}\int_0^{2\pi}d\varphi N_{11}(p_z;\,\vartheta,\varphi).$$

Solving the second equation of (19) with the initial condition $N_{11}(x)\,|_{t=0} = N_{11}^{(0)}(x)$, we get

$$N_{11}(x,t) = \exp\left(-\frac{1+x^2}{\tau\,|x|}t\right)$$

$$\times\left\{N_{11}^0(x) + \frac{1+x^2}{\tau\,|x|}\frac{1}{\hbar\omega_H}\int_0^t\bar{\varepsilon}_\perp(t')\exp\left(\frac{1+x^2}{\tau\,|x|}t'\right)dt'\right\}. \quad (20)$$

Substituting (20) in the first equation of (19), we finally get the following equation for $\bar{\epsilon}_\perp$:

$$\dot{\bar{\varepsilon}}_\perp + \frac{1}{\tau_e^{(r)}}\bar{\varepsilon}_\perp = g_0(t) + \int_0^t g(t-t')\,\bar{\varepsilon}_\perp(t')\,dt'; \quad (21)$$

$$g_0(t) = \frac{3}{8}\frac{\hbar\omega_H}{\tau_e^{(r)}}\int_{-1}^{1}(1+x^2)\exp\left(-\frac{1+x^2}{\tau\,|x|}t\right)N_{11}^0(x)\,dx,$$

$$g(t) = \frac{3}{4\tau\tau_e^{(r)}}\int_0^1\frac{(1+x^2)^2}{x}\exp\left(-\frac{1+x^2}{\tau x}t\right)dx. \quad (21')$$

By finding in (21) the mean transverse energy of the electron $\bar{\epsilon}_\perp$ in the same way as with (20), we solve the problem of the relaxation of N_{11}.

We use the Laplace transform method in Eq. (21). Introducing

$$f(p) = \int_0^\infty e^{-pt}\bar{\varepsilon}_\perp(t)\,dt, \quad (22)$$

we find

$$pf(p) - \bar{\varepsilon}_\perp(0) + f(p)/\tau_e^{(r)} = g_0(p) + g(p)\,f(p), \quad (23)$$

where

$$g_0(p) = \int_0^\infty e^{-pt}g_0(t)\,dt, \qquad g(p) = \int_0^\infty e^{-pt}g(t)\,dt \quad (22')$$

and $\bar{\epsilon}_\perp(0)$ is the initial mean transverse energy of the electron.

From Eq. (23), we find

$$f(p) = [\bar{\varepsilon}_\perp(0) + g_0(p)]/[p + 1/\tau_e^{(r)} - g(p)]. \qquad (23')$$

From (22′), (21′), the quantities $g_0(p)$ and $g(p)$ are equal to

$$g_0(p) = \frac{3}{8}\hbar\omega_H \frac{\tau}{\tau_e^{(r)}} \int_{-1}^{1} \frac{|x|(1+x^2)}{1+x^2+\tau p|x|} N_{11}^0(x)\,dx,$$

$$g(p) = \frac{1}{\tau_e^{(r)}} - \frac{3}{4}\frac{\tau}{\tau_e^{(r)}} p \int_0^1 \frac{x(1+x^2)}{1+x^2+\tau p x}\,dx. \qquad (24)$$

Therefore, Eq. (23′) can be rewritten in the form

$$f(p) = \frac{\tau_e^{(r)}}{\tau}\frac{1}{p}\frac{\bar{\varepsilon}_\perp(0) + g_0(p)}{F(\tau p)}, \qquad (25)$$

$$F(z) = \frac{\tau_e^{(r)}}{\tau} + \frac{3}{4}\int_0^1 \frac{x(1+x^2)}{1+x^2+zx}\,dx. \qquad (26)$$

It follows from Eq. (26) that $F(z)$ is an analytic function in the complex plane z, the only singularity of which is the line cut $z \le -2$. We emphasize that the function $F(z)$ has no zeroes in the entire complex plane of the variable z. The function $g_0(p)$, in accord with (24), is also an analytic function of $z = p\tau$ with the cut $z \le -2$, and, except for this, has no singularities in the z plane. Therefore, using the inversion formula

$$\bar{\varepsilon}_\perp(t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{pt} f(p)\,dp, \qquad \sigma > 0, \qquad (27)$$

and (25), one can transform the contour of integration to the form shown in the drawing.

Since

$$\int_{C_1} e^{pt} f(p)\,dp \to 0 \text{ for } t \to \infty,$$

then the limiting value of the mean transverse electron energy is equal to

$$\bar{\varepsilon}_\perp(\infty) = \frac{1}{2\pi i}\oint_{C_0} e^{pt} f(p)\,dp = \frac{\tau_e^{(r)}}{\tau}\frac{\bar{\varepsilon}_\perp(0) + g_0(p)|_{p=0}}{F(0)}. \qquad (28)$$

From Eqs. (26), (24), we find

$$F(0) = \frac{\tau_e^{(r)}}{\tau} + \frac{3}{8},$$

$$g_0(p)|_{p=0} = \frac{3}{8}\hbar\omega_H \frac{\tau}{\tau_e^{(r)}}\int_{-1}^{1} |x|\, N_{11}^0(x)\,dx.$$

Consequently, we get the following expression for the value of $\bar{\epsilon}_\perp(\infty) - \bar{\epsilon}_\perp(0)$, which represents the increase in the mean transverse energy of the electron due to transfer of energy from the photon reservoir to the electron:

$$\bar{\varepsilon}_\perp(\infty) - \bar{\varepsilon}_\perp(0) = \left[\hbar\omega_H \int_{-1}^{1} |x| N_{11}^0(x)\,dx - \bar{\varepsilon}_\perp(0)\right] \Big/ \left[1 + \frac{8}{3}\frac{\tau_e^{(r)}}{\tau}\right]. \qquad (29)$$

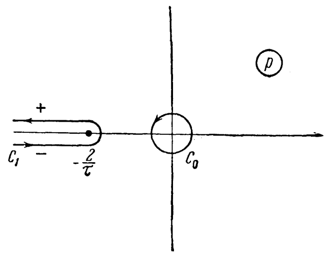
Thus, depending on the sign of the quantity in the numerator of the right hand side of (29), the energy flows from the photon reservoir to the electron reservoir (positive sign) or from the electron reservoir to the photon reservoir (negative sign).

Equations (27) and (28) show that one can put the quantity $\bar{\epsilon}_\perp(t)$ in the form

$$\bar{\varepsilon}_\perp(t) = \bar{\varepsilon}_\perp(\infty) + \frac{1}{2\pi i}\int_{C_1} e^{pt} f(p)\,dp.$$

By denoting the value of the function $f(p)$ on the upper side of the cut by $f_+(p)$, and noting that the value of $f(p)$ on the lower side of the cut $f_-(p) = f_+(p)^*$, we get

$$\bar{\varepsilon}_\perp(t) - \bar{\varepsilon}_\perp(\infty) = -\frac{1}{\pi}\int_{-\infty}^{-2/\tau} dp\, e^{pt}\, \mathrm{Im}\, f_+(p). \qquad (30)$$



The function $F(z)$ on the upper side of the cut is, from (26), equal to

$$F_+(z) = \frac{\tau_e^{(r)}}{\tau} + \frac{3}{8} - \frac{3}{4}z + \frac{3}{8}z^2 \ln|z+2|$$
$$+ \frac{3}{8} z \frac{2 - z^2}{\sqrt{z^2-4}} \ln\left|\frac{z + \sqrt{z^2-4}}{2}\right|$$
$$+ \frac{3\pi}{16} iz \frac{(z + \sqrt{z^2-4})^2}{\sqrt{z^2-4}}, \qquad z \leqslant -2. \qquad (31)$$

If we limit ourselves to a consideration of the isotropic initial photon distribution, then the value of $g_0(p)$ on the upper boundary of the cut is, according to (24), equal to

$$g_{0+}(p) = \hbar\omega_H (\tau/\tau_e^{(r)})\, N_{11}^0\, (F_+(z) - \tau_e^{(r)}/\tau) \qquad (31')$$

and, consequently, Eq. (30) takes the form

$$\bar{\varepsilon}_\perp(t) - \bar{\varepsilon}_\perp(\infty) = \frac{\tau_e^{(r)}}{\tau}(\hbar\omega_H N_{11}^0 - \bar{\varepsilon}_\perp(0)) \int_{-\infty}^{-2/\tau} \frac{e^{pt}}{p} \operatorname{Im}\frac{1}{F_+(\tau p)}\,dp. \quad (32)$$

Since $\tau \leq \tau_e^{(r)}$, then there is sense in considering an asymptotic value for this expression when $t \gg \tau$. In this case, since the values of $p \approx -2/\tau$ play a fundamental role in the integral (32), it is necessary to know only the asymptotic value of $F_+(z)$ for $z \gtrsim -2$, which, according to (31) has the form

$$F_+(z) \approx \tau_e^{(r)}/\tau - \tfrac{3}{2}\pi i/\sqrt{z^2-4}.$$

Substituting this asymptotic expression for the function $F_+(z)$ in Eq. (32) and carrying out simple transformations, we finally obtain

$$\bar{\varepsilon}_\perp(t) - \bar{\varepsilon}_\perp(\infty) = \pi^{-1}\sigma^2(\bar{\varepsilon}_\perp(0) - \hbar\omega_H N_{11}^0)\, e^{-2t/\tau} \times \left\{(\sqrt{\pi}/2\sigma)(\tau/t)^{1/2} - \tfrac{1}{2}\pi e^{t\sigma^2/\tau}\,[1 - \Phi(\sigma\sqrt{t/\tau})]\right\}, \quad t \gg \tau, \quad (33)$$

where $\sigma = 3\pi\tau/4\tau_e^{(r)}$ and $\Phi(y)$ is the error function:

$$\Phi(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-t^2}dt.$$

It is evident from Eq. (33) that even for $t \approx \tau$,

$$\bar{\varepsilon}_\perp(\tau) - \bar{\varepsilon}_\perp(\infty) \approx \tau\bar{\varepsilon}_\perp(0)/\tau_e^{(r)} \ll \bar{\varepsilon}_\perp(0),$$

as was demonstrated in [1].

Finally, we note that one must identify the quantity $\bar{\epsilon}_\perp(\infty)$ with the transverse temperature of the plasma electrons, which is established for $t \gg \tau_e^{(r)}$ (see [1]):

$$\bar{\varepsilon}_\perp(\infty) = T_\perp. \quad (34)$$

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