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PROPAGATION OF WAVES IN A MEDIUM WITH STRONG FLUCTUATION OF THE REFRACTIVE INDEX

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The Green's function is determined for an infinite medium with strong small-scale fluctuations of the refractive index. The solution is derived in a manner similar to the renormalization method in field theory. An expression is deduced for the effective refractive index of the medium and the attenuation coefficient.

INTRODUCTION

THE parameters of dielectrics are measured by transmitting in free space from an antenna (radiator) a radio or sound beam, that passes through the substance, and the amplitude and phase of the transmitted and reflected beams are measured[1-3]. In the reduction of the measurement data, the investigated specimen is assumed to be homogeneous. It is therefore of interest to ascertain the extent to which random inhomogeneities of the specimen influence the measurement results.

The effect of weak inhomogeneities is readily determined by successive approximation; we are interested in strong inhomogeneities such as occur, for example, near the critical point[4]. Measurement of the velocity of sound in this region can apparently yield the singularities of the thermodynamic potentials near the critical point[4-5]. The effect is distorted, however, by strong fluctuations with a correlation law that depends on the singularities of the potentials.

However, the most interesting case of a strongly inhomogeneous dielectric is the non-equilibrium plasma in thermonuclear fusion apparatus[6,7]. Oscillograms of the current discharge or of its magnetic field, obtained with a Rogowski belt, measure integral quantities[6], whereas the microwave beam yields information regarding the processes that occur only on its path, and can detect, by the same token, the regions where the instability arises. A monochromatic signal passing through a non-stationary plasma becomes modulated, with the frequencies and character of the modulation of the transmitted and scattered fields depending on the processes in the plasma.

As seen by the propagating wave, the plasma is an inhomogeneous anisotropic medium with time-varying average parameters and strong fluctuations of these parameters, so that the complete solution of the problem is extremely complicated. In the present paper we consider a simpler problem, that of the propagation of a scalar wave in a homogeneous isotropic medium with slow (quasistatic) fluctuations, which are not small.

The equation for the field ψ has the form

$$\Delta\psi + k^2 [1 + \varepsilon'(\mathbf{r})]\psi = f(\mathbf{r}), \qquad \psi \sim e^{-i\omega t},$$

where $\epsilon'(\mathbf{r})$ denotes the fluctuating part of the dielectric constant ($\overline{\epsilon'} = 0$); the mean value of the dielectric constant is customarily assumed equal to unity, i.e., its deviation from unity is included in k. We assume that the medium has damping, i.e., $\gamma = \operatorname{Im} k > 0$. Generally speaking, the wave equation in an inhomogeneous medium contains the first derivatives of the function ψ, which we have left out, since their inclusion does not change the character of the results.

Putting $k^2\epsilon'(\mathbf{r}) = -\xi(\mathbf{r})$, we write the equation for ψ in the form

$$L_0\psi = (\Delta + k^2)\psi = \xi(\mathbf{r})\psi + f(\mathbf{r}). \tag{1}$$

1. SOLUTION OF EQUATION (1) IN OPERATOR FORM

Let $\mathbf{M}_0$ be the integral operator inverse to L_0:

$$M_0 f(\mathbf{r}) = \int G_0(\mathbf{r} - \mathbf{r}') f(\mathbf{r}') d^3\mathbf{r}';$$

$$G_0(\mathbf{r} - \mathbf{r}') = -\frac{\exp(ik|\mathbf{r} - \mathbf{r}'|)}{4\pi|\mathbf{r} - \mathbf{r}'|}. \tag{2}$$

Then (1) can be represented in the form

$$\psi = M_0\xi\psi + M_0 f. \tag{3}$$

There is a well known solution of (3) in the form of a power series in the operator $\mathbf{M}_0$; it is obtained from the equation $(1 - M_0\xi)\psi = M_0 f$ with the aid of the expansion for $(1 - M_0\xi)^{-1}$:

$$\psi = \sum_0^\infty (M_0\xi)^n M_0 f. \qquad (4)$$

Solution (4) is a known perturbation-theory series, which can be used only in the case of weak fluctuations [8]. It turns out, however, that an expression can be obtained for the mean value of ψ in the form of a series in powers of some 'renormalized' operator M_1, which already takes account of multiple scattering.

Let us put $\psi = \bar{\psi} + \varphi$, $\bar{\varphi} = 0$. In addition to strictly mathematical convenience, the resolution of the field into a regular part $\bar{\psi}$ and a random part φ is consistent with the measurement procedure. The random part describes the modulation of the direct and scattered waves, due to fluctuations in the volume. Averaging (1), we obtain

$$L_0\bar{\psi} = \overline{\xi\varphi} + f, \qquad (5)$$

and after subtracting (5) from (1)

$$L_0\varphi = \xi\bar{\psi} + \widetilde{\xi\varphi}, \qquad (6)$$

where we put $\tilde{F} = F - \bar{F}$. Let us apply the operator M_0 to (6); introducing the random operator K in accordance with the equation

$$KF = M_0\widetilde{\xi F} = M_0(\xi F - \overline{\xi F}), \qquad (7)$$

we obtain

$$\varphi = K\bar{\psi} + K\varphi \qquad (8)$$

(we used the equation $M_0\xi\bar{\psi} = M_0\widetilde{\xi\bar{\psi}} = K\bar{\psi}$).

The introduced operator has an important property: $\overline{K^n f} = 0$ for any function f and for any n. Indeed

$$\overline{K^n f} = \overline{KK^{n-1}f} = M_0\,\overline{[\xi K^{n-1}f - \overline{\xi K^{n-1}f}]} = 0.$$

Since average values are subtracted every time that the operator K is applied, the values $K^n f$ are much smaller than the corresponding values of $(M_0\xi)^n f$.

Multiplying (8) by ξ and averaging, we obtain $\overline{\xi\varphi} = \overline{\xi K\bar{\psi}} + \overline{\xi K\varphi}$. Substituting the obtained value of $\overline{\xi\varphi}$ in (5), we get

$$L_1\bar{\psi} = (L_0 - \overline{\xi K})\bar{\psi} = f + \overline{\xi K\varphi}. \qquad (9)$$

The operator $L_1 = L_0 - \overline{\xi K}$ acting on a nonrandom function $\bar{\psi}$ is equal to $L_0 - \overline{\xi M_0 \xi}$ and is integro-differential, diagonal in the Fourier representation, so that the inverse operator $M_1 = L_1^{-1}$ can be readily constructed. Acting on (9) by means of the operator M_1, we obtain

$$\bar{\psi} = M_1 f + M_1\overline{\xi K\varphi}. \qquad (10)$$

Equations (8) and (10) form a closed system, equivalent to the initial equation (1). Solving this system of equations by successive iterations (starting with $\varphi_0 = 0$), we obtain the series

$$\bar{\psi} = M_1 f + M_1\overline{\xi K^2}M_1 f + M_1\overline{\xi K^3}M_1 f + \ldots + M_1\overline{\xi K^2}M_1\overline{\xi K^2}M_1 f + M_1\overline{\xi K^2}M_1\overline{\xi K^2}M_1\overline{\xi K^2}M_1 f + \ldots, \qquad (11)$$

$$\varphi = KM_1 f + K^2M_1 f + K^3M_1 f + \ldots + KM_1\overline{\xi K^2}M_1 f + K^2M_1\overline{\xi K^2}M_1 f + \ldots + KM_1\overline{\xi K^3}M_1 f + K^2M_1\overline{\xi K^3}M_1 f + \ldots \qquad (12)$$

Unlike the series (4) of perturbation theory, series (11) and (12) are in powers of the operator M_1, and in this connection they have certain advantages, since M_1 already describes multiple scattering.

For the operator M_1 we have

$$M_1 = (L_0 - \overline{\xi M_0 \xi})^{-1} = [L_0(1 - M_0\overline{\xi M_0 \xi})]^{-1} = (1 - M_0\overline{\xi M_0 \xi})^{-1}M_0 = \sum_{n=0}^{\infty}(M_0\overline{\xi M_0 \xi})^n M_0. \qquad (13)$$

Consequently, the operator M_1 sums an infinite number of terms of perturbation-theory series (4). Since we can write the kernel of the operator M_1 in analytic form (see below), each term of (11) is already a sum of an infinite number of components taken from (4). The series (11) is expected to converge when the series (4) diverges (in particular, for Gaussian distributions of ξ), so that the problem can be solved in the case of strong (under certain conditions, even infinite) fluctuations.

The method developed is analogous to the simplest version of the renormalization method of field theory. As applied to problems of our type, it was already used earlier [9,10,12], and the present paper is a development of the work of Bourrett [10]. The latter investigated the homogeneous equation (9) for $\bar{\psi}$ in the case of plane waves. We investigate below the region of applicability of the method, calculate the scattered field, and give a physical interpretation of the result.

The renormalization process can be continued. Iterating (8) N times we obtain

$$\varphi = Q_N\bar{\psi} + K^N\varphi, \qquad (14)$$

where

$$Q_N = \sum_{n=1}^{N} K^n, \qquad \bar{Q}_N = 0$$

is a random operator. Multiplying (14) by ξ and averaging we obtain $\overline{\xi\varphi} = \overline{\xi Q_N} + \overline{\xi K^N\varphi}$. After substitution in (5) this leads to

$$L_N\psi = (L_0 - \overline{\xi Q_N})\,\psi = f + \overline{\xi K^N \varphi}. \qquad (15)$$

L_N is an N-fold renormalized operator, also diagonal in the Fourier representation, so that the inverse operator $M_N = L_N^{-1}$ can be constructed. Applying to (15) the operator M_N we obtain

$$\overline{\psi} = M_N f + M_N \overline{\xi K^N \varphi}. \qquad (16)$$

Equations (14) and (16) form a closed system, the solution of which can be written in the form of series in powers of the operators M_N, Q_N, K, analogous to (11) and (12). With increasing order of the renormalization, even the first term of such a series gives the solution of the problem with sufficient accuracy. The operator M_2 sums an infinite number of terms, taken from the singly-renormalized series (11). It is easy to show that in analogy with (13)

$$M_2 f = M_1 f + M_1 \overline{\xi M_0 \xi M_0 \xi} M_1 f + M_1 \overline{\xi M_0 \xi M_0 \xi M_1 \xi M_0 \xi M_0 \xi} M_1 f + \ldots \qquad (17)$$

M_N can be represented in the form of a series in powers of the operator M_{N-1}. As can be seen from (13), M_1 contain only pair correlations, in which connection the method using M_1 is called bilocal in [10]. In expression (17) for M_2 are contained, in addition to pair correlations, also triple correlations. In exactly the same way, M_N contains correlations of order N + 1, so that this operator takes into account N + 1 point interactions.

2. RENORMALIZED BILOCAL GREEN'S FUNCTION OF EQUATION (1)

Let us find an explicit expression for the operator M_1. For this purpose it is necessary to consider the equation $L_1 g = L_0 g - \overline{\xi M_0 \xi} g = f$ in expanded form

$$\Delta g + k^2 g - k^4 \int G_0(\mathbf{r} - \mathbf{r}')\, \overline{\varepsilon'(\mathbf{r})\, \varepsilon'(\mathbf{r}')}\, g(\mathbf{r}')\, d^3 r' = f(\mathbf{r}). \qquad (18)$$

We assume that the fluctuations of ϵ' are statistically homogeneous, i.e., we assume that $\overline{\epsilon'(\mathbf{r})\,\epsilon'(\mathbf{r}')} = B_\epsilon(\mathbf{r} - \mathbf{r}')$ depends only on the difference in the arguments.

Equation (18) is solved by Fourier transformation. Its solution has the form

$$g(\mathbf{r}) = M_1 f = \int G_1(\mathbf{r} - \mathbf{r}')\, f(\mathbf{r}')\, d^3 r'; \qquad (19)$$

$$G_1(\mathbf{R}) = \frac{1}{8\pi^3} \int e^{i\varkappa \mathbf{R}} \Big[k^2 - \varkappa^2 - k^4 \int G_0(\mathbf{r}_1)\, B_\varepsilon(\mathbf{r}_1)\, e^{-i\varkappa \mathbf{r}_1} d^3 \mathbf{r}_1\Big]^{-1} d^3\varkappa. \qquad (20)$$

In the case when the fluctuations of ϵ are statistically isotropic and $B_\epsilon(\mathbf{r}) = B_\epsilon(r)$, it is possible to carry out in the integrals of (20) the integration over the angle variables, as a result of which we obtain

$$G_1(R) = \frac{1}{2\pi^2 R} \int_0^\infty \varkappa \Big[k^2 - \varkappa^2 + \frac{k^4}{\varkappa} \int_0^\infty B_\varepsilon(r)\, e^{ikr} \sin\varkappa r\, dr\Big]^{-1} \sin\varkappa R\, d\varkappa. \qquad (20a)$$

Formulas (20) and (20a) give an explicit expression for the kernel of the operator M_1. $G_1(R)$ is a nonlinear functional of the correlation function $B_\epsilon(r)$. Specifying, for example, the correlation function $B_\epsilon(r)$ in the form

$$B_\varepsilon(r) = \sigma^2 e^{-r/\alpha a_0}, \qquad (21)$$

where $\alpha = (8\pi)^{-1/3}$ and a_0 is the integral correlation scale, defined in the general case by the equation

$$a_0^3 = B_\varepsilon^{-1}(0) \int B_\varepsilon(r)\, d^3 r, \qquad (22)$$

we obtain for $G_1(R)$ the formula

$$G_1(R) = C_1 R^{-1} e^{i\varkappa_1 R} + C_2 R^{-1} e^{i\varkappa_2 R};$$

$$\varkappa_{1,2} = (\sqrt{2}\alpha a_0)^{-1} \Big[\alpha^2 k^2 a_0^2 - (1 - ik\alpha a_0)^2 \pm \sqrt{(1 - 2ik\alpha a_0)^2 + 4k^4\alpha^4 a_0^4 \sigma^2}\Big]^{1/2},$$

$$C_{1,2} = -(8\pi)^{-1}\Big[1 \pm \Big(1 + \frac{4k^4 a_0^4 \alpha^4 \sigma^2}{(1 - 2ik\alpha a_0)^2}\Big)^{-1/2}\Big]. \qquad (23)$$

For the case $k^4 a_0^4 \sigma^2 \ll 1$, which will be considered in greater detail, we have $|C_2| \ll |C_1|$:

$$C_1 = -1/4\pi, \qquad C_2 = -k^4 a_0^4 \alpha^4 \sigma^2 / 2\pi, \qquad (24)$$

$$\varkappa_1 = k\,[1 + k^2 \alpha^2 a_0^2 \sigma^2 (1 + 2ik\alpha a_0)]^{1/2}, \qquad \varkappa_2 = i/\alpha a_0. \qquad (25)$$

In the case $\sigma = 0$ (no fluctuations), it follows from the general expression (20) that $G_1(\mathbf{R}) = G_0(\mathbf{R})$.

3. LIMITS OF APPLICABILITY OF THE BILOCAL METHOD

Using the bilocal Green's function $G_1(\mathbf{R})$, we can write down the series (11) for $\overline{\psi}$ in the form

$$\overline{\psi}(\mathbf{r}) = \int G_{\text{tot}}(\mathbf{r} - \mathbf{r}')\, f(\mathbf{r}')\, d^3\mathbf{r}; \qquad (26)$$

$$G_{\text{tot}}(\mathbf{R}) = G_1(\mathbf{R}) - k^6 \iiint G_1(\mathbf{R} - \mathbf{r}_1) \times G_0(\mathbf{r}_1 - \mathbf{r}_2)\, G_0(\mathbf{r}_2 - \mathbf{r}_3)\, G_1(\mathbf{r}_3) \times \overline{\varepsilon'(\mathbf{r}_1)\, \varepsilon'(\mathbf{r}_2)\, \varepsilon'(\mathbf{r}_3)}\, d^3\mathbf{r}_1 d^3\mathbf{r}_2 d^3\mathbf{r}_3$$

$$+ k^8 \iiiint G_1(\mathbf{R}-\mathbf{r}_1)\, G_0(\mathbf{r}_1-\mathbf{r}_2) \times G_0(\mathbf{r}_2-\mathbf{r}_3)\, G_0(\mathbf{r}_3-\mathbf{r}_4)\, G_1(\mathbf{r}_4) \times \overline{\widetilde{\varepsilon'(\mathbf{r}_1)\,\varepsilon'(\mathbf{r}_2)}\,\widetilde{\varepsilon'(\mathbf{r}_3)\,\varepsilon'(\mathbf{r}_4)}}\, d^3\mathbf{r}_1 \ldots d^3\mathbf{r}_4 \quad (27)$$

[we have written out only the first term of the series (11)].

Let us stop to calculate formula (27). The mean values of the multiple products $\epsilon'(\mathbf{r})$ contained in this formula reduce, in the case when $\epsilon'(\mathbf{r})$ is a Gaussian random field, to a sum of all possible products of the pair correlations [11]. If two factors are contained in the average product under the $\sim$ sign, the correlation function corresponding to their product does not enter into the sum. In accord with this rule, assuming that $\epsilon'(\mathbf{r})$ is a Gaussian quantity, we have

$$\overline{\varepsilon'(\mathbf{r}_1)\,\varepsilon'(\mathbf{r}_2)\,\varepsilon'(\mathbf{r}_3)} = 0,$$

$$\overline{\widetilde{\varepsilon'(\mathbf{r}_1)\,\varepsilon'(\mathbf{r}_2)}\,\widetilde{\varepsilon'(\mathbf{r}_3)\,\varepsilon'(\mathbf{r}_4)}} = B_\varepsilon(\mathbf{r}_1-\mathbf{r}_3)\, B_\varepsilon(\mathbf{r}_2-\mathbf{r}_4) + B_\varepsilon(\mathbf{r}_1-\mathbf{r}_4)\, B_\varepsilon(\mathbf{r}_2-\mathbf{r}_3). \quad (28)$$

Assuming that B_ϵ and $G_1(R)$ are given by relations (21) and (23), let us estimate the integral containing the fourth moments. We assume here that the following condition is satisfied

$$ka_0 \ll 1, \quad (29)$$

i.e., we assume the fluctuation scale to be small. The estimates make use of inequality (29), by which it can be assumed that the function B_ϵ varies much more rapidly than the functions G_0 and G_1, i.e., the function B_ϵ in (27) is similar to a δ-function which is 'smeared out' over a distance on the order of a_0. (This 'smearing' must be taken into account only in those cases when divergent integrals are obtained if the unsmeared δ-function is substituted).

The requirement that the first nonvanishing correction obtained in this manner be small compared with $G_1(R)$ leads to two limitations

$$|(ka_0)^7 \sigma^4| \ll 1, \quad (30)$$

$$|kR| \ll (1 + k^2 a_0^2 \alpha^2 \sigma^2)^{1/2} / (ka_0)^5 \sigma^4, \quad (31)$$

where R is the distance within which the solution obtained is valid [1)].

Let $\mu \ll 1$ be the relative accuracy with which (30) is satisfied, i.e., $(ka_0)^7\sigma^4 \le \mu$. Then $(ka)^4\sigma^2 \le \sqrt{ka\mu} \ll 1$, but $(ka_0)^3\sigma^2$ and all the more $(ka_0)^2\sigma^2$ need no longer be small. Recognizing that $(ka_0)^4\sigma^2 \ll 1$, we can henceforth use the simplified formulas (24) and (25) for $C_{1,2}$ and $\kappa_{1,2}$, and since $C_2 \ll 1$ we need retain only the first terms of (23).

Conditions (30) and (31) can be satisfied also in the case of infinite fluctuations, when $\sigma^2 \to \infty$, provided only $a_0 \to 0$ at the same time. When $\sigma^2 \to \infty$ we have $(k^2a_0^2) \to \infty$ so that condition (31) yields $kR \ll (ka_0)^{-4}\sigma^{-3}$. In order for the condition $kR \gg 1$ not to be violated as $\sigma \to \infty$, it is necessary to impose on a_0 the limitation

$$a_0^4 \sigma^3 < \text{const}, \quad (32)$$

satisfaction of which permits the transition to the case of infinite fluctuations [condition (30) imposes a weaker limitation on a_0] [2)].

4. ASYMPTOTIC FORM OF THE BILOCAL GREEN'S FUNCTION

Let us find the asymptotic form of the function $G_1(R)$ for large R, without making the form of the function $B_\epsilon(r)$ specific. It follows from (20) that as $R \to \infty$ the asymptotic value of $G_1(R)$ is determined by the behavior of the spectrum of this function as $\kappa \to 0$. Expanding in (20a) $(\sin \kappa r)/\kappa$ in a series and retaining only the first term of this expansion [the next term of the expansion leads upon integration to a small term of order $(ka_0)^4\sigma^2$], we obtain

$$G_1(R) = \frac{1}{2\pi^2 R} \int_0^\infty \varkappa \Big[k^2 - \varkappa^2 + k^4 \int_0^\infty B_\varepsilon(r)\, e^{ikr} r\, dr\Big]^{-1} \sin \varkappa R\, d\varkappa.$$

This expression differs from the spectral expansion of the function $G_0(R)$ (which is obtained from it when $B_\epsilon = 0$) only in the fact that it contains in place of k the quantity

$$k' = k\Big[1 + k^2 \int_0^\infty B_\varepsilon(r)\, e^{ikr} r\, dr\Big]^{1/2}. \quad (33)$$

Consequently, the asymptotic form of $G_1(R)$ coincide with $G_0(R)$, in which k is replaced by k':

$$G_1(R) \cong -e^{ik'R}/4\pi R. \quad (34)$$

A formula such as (33) for a specific correlation function was obtained by Bourrett [11]. We emphasize, however, that the Green's function, as can be seen from formula (23), has a much more compli-

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1)It can be shown that the use of more complicated renormalizations obviates the need for limitation (31).

2)We note that if $\sigma \to \infty$, $a_0 \to 0$, and the relation $\sigma^2 a_0^3 = \text{const}$ is retained, we obtain the δ-correlation case but condition (32) is then violated.

cated form than (34), and this difference is essential at small distances.

The integral in (33), under the condition $ka_0 \ll 1$, is of the order of $\sigma^2 a_0^2$, since the difference between k′ and k, connected with terms of the order of $k^2 a_0^2 \sigma^2$, may not be small.

Let us put $k = p + i\gamma$; $\gamma \ll p$. Recognizing that when $ka_0 \ll 1$ and when $(ka_0)^2\sigma^2$ is arbitrary the following inequality is satisfied

$$\left| k^2 \int_0^\infty \sin pr\, e^{-\gamma r} B_\varepsilon(r)\, r dr \right| \ll \left| 1 + k^2 \int_0^\infty \cos pr\, e^{-\gamma r} B_\varepsilon(r) r dr \right|,$$

we can expand (34) and approximately represent k′ in the form

$$k' = pn_e + i\gamma_e; \quad (35)$$

$$n_e = \left[1 + p^2 \int_0^\infty \cos pr \exp(-\gamma r)\, B_\varepsilon(r)\, r dr\right]^{1/2}. \quad (36)$$

Here n_e is the effective refractive index of the fluctuating medium and γ_e is the effective attenuation

$$\gamma_e = \frac{2n_e^2 - 1}{n_e}\gamma + \frac{p^3}{2n_e}\int_0^\infty \sin pr e^{-\gamma r} B_\varepsilon(r)\, r\, dr. \quad (37)$$

In the case $pa \ll 1$, which we are considering, the integral (37) can be expressed in terms of the scale a_0 defined by formula (22). Indeed, neglecting the factor $e^{-\gamma r}$ and replacing $\sin(pr)$ by pr, we obtain

$$\int_0^\infty \sin pr\, e^{-\gamma r} B_\varepsilon(r)\, r dr \cong p \int_0^\infty B_\varepsilon(r)\, r^2\, dr =. \frac{p\sigma^2 a_0^3}{4\pi},$$

and (37) assumes the form

$$\gamma_e = (2n_e^2 - 1)\,\gamma/n_e + p^4 a_0^3 \sigma^2 / 8\pi n_e. \quad (37a)$$

The effective attenuation γ_e consists of two components. The first is proportional to the absorption and includes the effect whereby the effective absorption path is increased by multiple scattering [the factor $(2n_e^2 - 1)/n_e > 1$]. The second component in (37a) is the effective coefficient of multiple scattering (its connection with the coefficient of single scattering will be considered in the next section) and describes the transition of the energy of the average field into fluctuation energy (the increase in the 'entropy' of the system).

5. CALCULATION OF FIELD FLUCTUATIONS

Let us consider the scattering of waves by the fluctuations of the medium. Assume that a plane wave $A_0 e^{i\mathbf{k}\cdot\mathbf{r}}$ is incident on a certain volume filled with strong small-scale fluctuations of the refractive index. We are interested in the fluctuation field outside of the scattering volume, due to the scattering by the fluctuations inside the volume.

Let the average field inside the scattering volume have the form of a plane wave

$$\bar{\psi}(\mathbf{r}) = A_0 e^{i\mathbf{k}'\mathbf{r}}, \quad (38)$$

where $\mathbf{k}' = k'\mathbf{k}/k$. We employ formula (8), rewritten in the form

$$\varphi = K\bar{\psi} + K^2\bar{\psi} + \ldots = \varphi_1 + \varphi_2 + \ldots;$$

$$\varphi_1 = -\frac{k^2}{4\pi}\int_V \frac{\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\varepsilon'(\mathbf{r}')\,\bar{\psi}(\mathbf{r}')\,d^3\mathbf{r}',$$

$$\varphi_2 = \frac{k^4}{16\pi^2}\int_V\int_V \frac{\exp[ik(|\mathbf{r}-\mathbf{r}'| + |\mathbf{r}'-\mathbf{r}''|)]}{|\mathbf{r}-\mathbf{r}'|\,|\mathbf{r}'-\mathbf{r}''|}$$

$$\times\, \overline{\varepsilon'(\mathbf{r}')\,\varepsilon'(\mathbf{r}'')}\,\bar{\psi}(\mathbf{r}'')\,d^3\mathbf{r}'d^3\mathbf{r}''. \quad (39)$$

We assume that the point of observation is in the Fraunhofer diffraction zone where, as is well known, we can assume that $|\mathbf{r} - \mathbf{r}'| = r - \mathbf{m}\cdot\mathbf{r}'$; $\mathbf{m} = \mathbf{r}/r$, and we can put $|\mathbf{r} - \mathbf{r}'| = r$ in the denominator of (39). Substituting (38) in (39) and then taking the mean square of the modulus of φ_1, we obtain

$$\overline{\varphi_1\varphi_1^*} \cong \frac{k^4 A_0^2}{16\pi^2 r^2}\int_V\int_V \exp\left\{i[k\mathbf{m}(\mathbf{r}''-\mathbf{r}') - \mathbf{k}'(\mathbf{r}''-\mathbf{r}')]\right\}$$

$$\times\, B_\varepsilon(\mathbf{r}'-\mathbf{r}'')\,d^3\mathbf{r}'d^3\mathbf{r}''. \quad (40)$$

Introducing a new integration variable $\boldsymbol{\rho} = \mathbf{r}' - \mathbf{r}''$ and integrating with respect to $\mathbf{r}''$ to obtain the scattering V, we get

$$\overline{\varphi_1\varphi_1^*} = \frac{k^4 A_0^2 V}{16\pi^2 r^2}\int_V \exp[-i(k\mathbf{m} - k'\mathbf{m}_0)\,\boldsymbol{\rho}]\,B_\varepsilon(\boldsymbol{\rho})\,d^3\boldsymbol{\rho}, \quad (41)$$

where $\mathbf{k}' = k'\mathbf{m}_0$ and $\mathbf{m}_0$ is the unit vector of the incoming wave.

The quantities k and k′ in (41) are complex, a manifestation of the change in the incident and scattered wave over the length of the scattering volume. If these effects are small enough to be neglected, we can put $k = p$ and $k' = pn_e$. Since the correlation function B_ϵ decreases rapidly with increasing ρ, the limits of integration in (41) can be extended for real k and k′ to infinity, so that we arrive at

$$\overline{\varphi_1\varphi_1^*} = \tfrac{1}{2}\pi p^4 A_0^2 V r^{-2}\Phi_\varepsilon(p\mathbf{m} - pn_e\mathbf{m}_0), \quad (42)$$

where

$$\Phi_\varepsilon(\varkappa) = \frac{1}{8\pi^3}\int e^{-i\varkappa\rho}B_\varepsilon(\boldsymbol{\rho})\,d^3\boldsymbol{\rho}$$

is the three-dimensional Fourier component of the correlation function.

In the case of statistically isotropic fluctuations we have $\Phi_\epsilon(\kappa) = \Phi_\epsilon(|\kappa|)$ and (42) assumes the form

$$\overline{\varphi_1\varphi_1^*} = \tfrac{1}{2}\pi p^4 A_0^2 V r^{-2} \Phi_\varepsilon \left(p\sqrt{1 - 2n_e\cos\theta + n_e^2}\right), \quad (42a)$$

where θ is the scattering angle and $\cos\theta = \mathbf{m}_0 \cdot \mathbf{m}$.

Expression (42) differs from the analogous formula obtained by the small-perturbation method in the presence of n_e in the argument of Φ_ϵ. As a result of this $\overline{\varphi_1^*\varphi_1}$ is a nonlinear functional of $B_\epsilon(r)$. When the fluctuation intensity changes, the scattering indicatrix changes its form, owing to the change in n_e. The second difference lies in the fact that the forward scattering is determined not by the zeroth component of the spectrum, as in the theory based on the small-perturbation method, but by the spectral component corresponding to

$$\varkappa_{min} = p\,(n_e - 1).$$

The differential effective scattering cross section is found from (42a) to be

$$d\sigma_S(\theta) = \tfrac{1}{2}\pi p^4 V \Phi_\varepsilon \left(p\sqrt{1 - 2n_e\cos\theta + n_e^2}\right) do. \quad (43)$$

In the case when $pa \ll 1$, the spectral density $\Phi_\epsilon(\kappa)$ is approximately equal to $\Phi_\epsilon(0)$ for $\kappa \leq 2\pi/a$. If the maximum wave number $\kappa_{max} = p(n_e + 1)$, which enters in (43), satisfies the condition $\kappa_{max} \leq 2\pi/a$, i.e., if the inequality $pa < 2\pi/(n_e + 1)$ is satisfied, then we can replace the function $\Phi_\epsilon(\kappa)$ in (42) and (43) by $\Phi_\epsilon(0)$, which by definition and by formula (22) is equal to $\Phi_\epsilon(0) = \sigma^2 a_0^3/8\pi^3$. In this case the scattering indicatrix is spherical; for the total diameter we have

$$d\sigma_S = \tfrac{1}{2}(2\pi)^{-3} p^4 a_0^3 \sigma^2 V do, \qquad \sigma_S = p^4 a_0^3 \sigma^2 V/4\pi. \quad (44)$$

In the single scattering approximation the cross section $\beta = \sigma_S/2V$ for backward scattering from a unit of scattering volume is equal to the attenuation coefficient of the wave due to scattering by a fluctuation. This quantity

$$\beta = p^4 a_0^3 \sigma^2/8\pi$$

is n_e times larger than the attenuation coefficient given by the second term of formula (37a). Thus, account of the multiple scattering leads to a decrease in the attenuation coefficient.

In conclusion let us ascertain under what conditions the first term of the series $\varphi = \varphi_1 + \varphi_2 + \cdots$ can be used. Inasmuch as $\overline{\varphi_1{}^*\varphi_2} = 0$, let us estimate $\overline{\varphi_2{}^*\varphi_2}$. By a method analogous to that used above to estimate the corrections of $M_1 f$, we can show that

$$\overline{\varphi_2\varphi_2^*} / \overline{\varphi_1^*\varphi_1} \sim \sigma_S/\sigma_{geom}, \quad (45)$$

where σ_{geom} is the geometric cross section diameter of the scattering volume. This leads to the condition $pL \ll (ka_0)^{-3}\sigma^{-2}$, where L is the linear dimension of the scattering volume. For sufficiently small L we can always use formula (43).

6. CONCLUSION

For a wave propagating in a medium with strong small-scale fluctuations of the refractive index, formulas can be obtained for the real and imaginary parts of the propagation constant of the transmitted wave and for the intensity of the scattered field [see formula (36), (37a), and (44)]. These quantities are expressed in terms of the correlation functions of the fluctuations.

The amplitude-phase relations for the transmitted waves, and also for the intensity of the scattered field, can be measured, so that both the parameters of the correlation wave and the propagation velocity in the homogeneous medium can be determined.

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