SOLUTION OF THE BETHE-SALPETER EQUATIONS FOR THE FERMI-YANG MODEL

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We consider bound states of a nucleon and antinucleon, whose interaction may be described by a potential well. Instead of the equations used by Fermi and Yang we solve the Bethe-Salpeter equations. With this change of equations the main qualitative results of the Fermi-Yang solutions, such as the great depth of the well [1] and large number of levels, [2] disappear.

1. Many models have been considered recently in which the mesons are viewed as bound states of a baryon and antibaryon. Therefore there is some sense to calculations which make it possible to obtain the meson mass if some very rough assumptions are made concerning the form of the interaction between the baryons. The first such an attempt was made by Fermi and Yang. [1] They considered the pion as a bound state of the nucleon and antinucleon interacting instantaneously (without retardation) through a square well potential. At that the existence of the vacuum was completely ignored. Recently Vaks [2] has considered other states in such a model.

Below we consider a certain modification of this model. The interaction between the particles is the same, however we solve not the Breit equation used by Fermi and Yang but the Bethe-Salpeter equation. The difference consists in the fact that now the Dirac vacuum is partially taken into account, since for the nucleon under consideration transitions into states with negative energy are forbidden. Formally this manifests itself in treating the pole in the Green's function according to the Feynman prescription. As in the work of Fermi and Yang, we do not take into account the interaction with the vacuum that leads to a modification of the mass of the nucleons, or the interaction between them.

In constructing the full theory one hardly wants to argue that one wrong model is better than another wrong model. However in analogous problems of statistical physics (the theory of superconductivity, the theory of the Fermi liquid) it turns out that it is sufficient for obtaining a qualitative picture to treat the Fermi-filling as described above. All remaining features reduce to a redefinition of constants (effective mass and effective interaction). Moreover, the equations under consideration in contrast to the Fermi-Yang model do not have solutions that give incorrect results in the nonrelativistic case. In any case the results obtained below permit one to assert that the main qualitative features of the Fermi-Yang model, such as the great depth of the well and the large number of levels, are peculiar to that model.

2. We thus assume that the interaction is of the form

$$\bar{\psi}_1\gamma_\mu\psi_1 U(r_1 - r_2)\,\bar{\psi}_2\gamma_\mu\psi_2, \tag{1}$$

where the potential U is instantaneous and depends only on the difference of the spatial coordinates of particles 1 and 2.

The Bethe-Salpeter equation for the bound state of the particle and antiparticle has the usual form

$$K_{\alpha\beta}(p) = \left(\frac{1}{\hat{p}_1 - m}\gamma_r\right)_{\beta\mu}\left(\gamma_r\frac{1}{\hat{p}_2 - m}\right)_{\nu\alpha}\int U(p-p') \times K_{\nu\mu}(p')\,d^4p', \tag{2}$$

where K is the two-particle Green's function, $p_{1,2} = p \pm q/2$, p is the relative four-momentum, q is the total four-momentum [in the center-of-mass frame $q \equiv (\mu, 0, 0, 0)$ where μ is the mass of the meson], and $U(p - p')$ is a Fourier component of the interaction potential. We shall consider bound states with nonzero isotopic spin, for example $p\bar{n}$, and therefore there are no annihilation terms in the equation.

Different states are characterized by the total angular momentum which consists of the relative orbital angular momentum and the spins. It turns out to be more convenient to first dispose of the spinor indices and then decompose according to the orbital angular momentum.

We write K as a superposition of sixteen linearly independent matrices:

$$K_{\alpha\beta} = K_n O^n_{\alpha\beta} \equiv [K^S \cdot 1 + K^P\gamma_5 + K^V_i\gamma_i + K^A_i\gamma_5\gamma_i + K^T_{ik}\,{}^1/_2\,(\gamma_i\gamma_k - \gamma_k\gamma_i)]_{\beta\alpha}. \quad (3)$$

After substitution into Eq. (2) we obtain for the quantities K_n the system of equations

$$K_n(p) = \Pi_{nm}(p) \int U(p-p')\, K_m(p')\, d^4p', \quad (4)$$

$$\Pi_{nm} = \frac{1}{4}\,\mathrm{Sp}\left\{O_n \frac{1}{\hat{p}_1 - m}\gamma_r O_m \gamma_r \frac{1}{\hat{p}_2 - m}\right\}. \quad (5)$$

We note that $\gamma_r(\gamma_i\gamma_k - \gamma_k\gamma_i)\gamma_r = 0$ and therefore K^T does not enter into these equations.

For a nonretarded potential, $U(p-p')$ does not depend on the fourth components of the vectors and therefore both sides of the equation may be integrated over p_0. Introducing the functions $\psi_n(p) = \int K_n(p)\,dp_0$ we obtain

$$\psi_n(p) = \int \Pi_{nm}(p)\, dp_0 \int U(p-p')\,\psi_m(p')\, d^3p'. \quad (6)$$

The integral $\int \Pi_{nm}\,dp_0$ appearing in these equations may be expressed in terms of the two integrals:

$$I = \frac{1}{\pi i}\int \frac{dp_0}{p^2}, \qquad J = \frac{2}{\pi i}\int \frac{dp_0}{p_1^2 p_2^2}.$$

It is precisely in the evaluation of these integrals that the difference between the Fermi-Yang and Bethe-Salpeter equations appears. In the Fermi-Yang model the poles in the integrands are defined by adding $i\delta$ to μ, whereas in the Green's functions the $i\delta$ is added to the energy. As a result we obtain

$$I_{\text{F.Y.}} = 0, \quad J_{\text{F.Y.}} = 2/\mu\,(p^2 + m^2 - \tfrac{1}{4}\mu^2). \quad (7)$$

After substitution the resultant system of equations coincides with the one obtained by Fermi and Yang. Thus, for the pseudoscalar state we obtain

$$(p^2 + m^2 - \tfrac{1}{4}\mu^2)\,\psi^P = \mu V\psi^P - mV\psi_0^A,$$

$$(p^2 + m^2 - \tfrac{1}{4}\mu^2)\,\psi_0^A = 2mV\psi^P - 2\,(p^2 - \tfrac{1}{4}\mu^2)\,V\psi_0^A/\mu. \quad (7')$$

For small μ these equations have two solutions. For one solution as $\mu \to 0$ we have $\psi_0^A \to 0$ and $V \to \infty$, for the other $\psi^P \to 0$ and $V \to 0$. Thus it turns out that in the nonrelativistic limit when $V \ll m$ there exists a solution with binding energy equal to almost 2m. This solution arises because transitions of particle and antiparticle into negative energy states are not forbidden. For a weak interaction one of the particles in the bound state has the energy $+\sqrt{m^2+p^2}$, the other $-\sqrt{m^2+p^2}$, and the total energy is close to zero.

If in the evaluation of the integrals the poles are treated according to the Feynman prescription by replacing m by $m - i\delta$ then we obtain

$$I = (m^2 + p^2)^{-1/2}, \quad J = 1/\sqrt{m^2+p^2}\,(p^2 + m^2 - \tfrac{1}{4}\mu^2). \quad (8)$$

Now $J(p)$ does not grow as $\mu \to 0$; this is related to the fact that transitions of the nucleon into negative energy states are forbidden.

The system of equations determining μ takes on the form

$$J^{-1}\psi^P = 2\,(p^2 + m^2)\int V\psi^P + \frac{1}{2}\mu m \int V\psi_0^A,$$

$$J^{-1}\psi_0^A = -\mu m \int V\psi^P - m^2 \int V\psi_0^A; \quad (9)$$

$$J^{-1}\psi^S = 2p^2 \int V\psi_S + mp_\alpha \int V\psi_\alpha^V,$$

$$J^{-1}\psi_\alpha^V = 2mp_\alpha \int V\psi_S + [(m^2 + p^2)\,\delta_{\alpha\beta} - p_\alpha p_\beta] \int V\psi_\beta^V + \varepsilon_{\alpha\beta\gamma} p_\gamma \frac{\mu}{2}\int V\psi_\beta^A,$$

$$J^{-1}\psi_\alpha^A = -\varepsilon_{\alpha\beta\gamma} p_\gamma \frac{\mu}{2}\int V\psi_\beta^V + (p^2\delta_{\alpha\beta} - p_\alpha p_\beta)\int V\psi_\beta^A. \quad (10)$$

Here $\psi \equiv (\psi_0, \psi_\alpha)$, $\int V\psi \equiv \int V(p-p')\,\psi(p')\,d^3p'$; $J(p)$ is defined by Eq. (8). Equations (9) describe the singlet state (charge-symmetric), and Eqs. (10) the triplet state.

The decomposition in orbital angular momentum proceeds in the usual manner. We introduce the unit vector $\mathbf{n} = \mathbf{p}/p$. Then the dependence of ψ on $\mathbf{n}$ may be extracted. For Eqs. (9) we write

$$\psi^P = p^{-1}\varphi_l Y_{lm}(\mathbf{n}), \quad \psi_0^A = p^{-1}\chi_l(p)\, Y_{lm}(\mathbf{n}).$$

The case $l = 0$ corresponds to a pseudoscalar meson, which will be identified with the pion, $l = 1$ corresponds to a pseudovector meson. Equations (9) turn out to be a system of one-dimensional equations:

$$J^{-1}\varphi_l = 2\,(p^2 + m^2)\int V_l\varphi_l + \frac{1}{2}\mu m \int V_l\chi_l,$$

$$J^{-1}\chi_l = -m\mu \int V_l\varphi_l - m^2 \int V_l\chi_l, \quad (11)$$

where J is defined by Eq. (8),

$$\int V_l\varphi_l \equiv \int_0^\infty V_l(p, p')\,\varphi(p')\,dp',$$

$V_l(p, p')$ is the l-th harmonic of the Fourier component of the potential. For a square well of unit radius and depth V

$$V_l(p, p') = V\,(\sqrt{pp'}/(p^2 - p'^2))\,[p' J_{l+1/2}(p)\, J_{l-1/2}(p') - pJ_{l-1/2}(p)\, J_{l+1/2}(p')]. \quad (12)$$

Equations (10) are more complicated. The functions ψ_α^V and ψ_α^A should be looked for in the form of spherical vectors. However if one limits oneself to states with angular momentum equal to zero and unity then the function ψ may be sought for in the following form:

1) for scalar $\psi^S = p^{-1}\varphi(p)$, $\psi_\alpha^V = (n_\alpha/p)\chi(p)$, $\psi_\alpha^A = 0$

$$J^{-1}\varphi = 2p^2\int V_0\varphi + mp\int V_1\chi, \quad \chi = \frac{m}{p}\varphi; \tag{13}$$

2) for pseudovector $\psi^S = 0$, $\psi_\alpha^A = p^{-1}[S_\alpha - \mathbf{S}\cdot\mathbf{n}n_\alpha]\varphi$, $\psi_\alpha^V = p^{-1}\epsilon_{\alpha\beta\gamma}n_\beta S_\gamma\chi$,

$$J^{-1}\varphi = p^2\int\frac{2V_0+V_2}{3}\varphi + \frac{1}{2}\mu p\int V_1\chi,$$
$$J^{-1}\chi = \frac{1}{2}\mu p\int\frac{2V_0+V_2}{3}\varphi + (p^2+m^2)\int V_1\chi, \tag{14}$$

where S_α is an arbitrary unit vector;

3) for vector $\psi^S = \mathbf{S}\cdot\mathbf{n}\varphi_1$, $\psi_\alpha^V = S_\alpha\varphi_2 + \mathbf{S}\cdot\mathbf{n}n_\alpha\varphi_3$, $\psi_\alpha^A = \epsilon_{\alpha\beta\gamma}n_\beta S_\gamma\varphi_4$,

$$J^{-1}\varphi_1 = 2p^2\int V_1\varphi_1 + mp\left(\int V_0\varphi_2 + \int\frac{V_0+2V_2}{3}\varphi_3\right),$$
$$J^{-1}\varphi_2 = (m^2+p^2)\left(\int V_0\varphi_2 + \int\frac{V_0-V_2}{3}\varphi_3\right) + \frac{1}{2}\mu p\int V_1\varphi_4,$$
$$J^{-1}\varphi_3 = 2mp\int V_1\varphi_1 - p^2\int V_0\varphi_2 + (m^2+p^2)\int V_2\varphi_3 - p^2\int\frac{V_0+2V_2}{3}\varphi_3 - \frac{1}{2}\mu p\int V_1\varphi_4,$$
$$J^{-1}\varphi_4 = \frac{1}{2}\mu p\left(\int V_0\varphi_2 + \int\frac{V_0-V_2}{3}\varphi_3\right) + p^2\int V_1\varphi_4. \tag{15}$$

The notation is the same as in Eq. (11).

3. Unfortunately it is not possible to reduce the resultant equations to a set of differential equations. We therefore restrict ourselves to a discussion of limiting cases and to estimates obtained with the help of a variational method.

Let us show that the well need not be as deep as in the Fermi-Yang model in order that the pion have a small mass when the radius of the well is equal to the inverse of the nucleon mass. We shall assume that the mass of the pion is equal to zero. In any case that assumption results only in an overestimate of the well depth; besides, μ enters the equations in the form of the ratio $(\mu/2m)^2 \sim 1/200$, which can be neglected. Thus, the well depth is given by the smallest eigenvalue of the integral equation

$$\varphi(p) = \frac{2}{\sqrt{p^2+1}}\int_0^\infty V_0(p,p')\varphi(p')dp', \tag{16}$$

where V_0 is determined by Eq. (12).

To obtain an upper estimate to the well depth we make use of the variational principle: for any function f(p) the following inequality is satisfied

$$\int_0^\infty f^2(p)\sqrt{p^2+1}\,dp \geqslant 2\iint_0^\infty dp\,dp' f(p)V_0(p,p')f(p'). \tag{17}$$

We take as the trial function, for example, $f(p) = p/(p^2+1)^2$; then for the well depth V we find

$$V \leqslant 32m/15\pi\,(1-5e^{-2}) = 2\text{ BeV} \tag{18}$$

(in the Fermi-Yang model $V \sim 25$ BeV and tends to infinity as the pion mass tends to zero).

Let us consider now a square well of radius a many times larger than the inverse nucleon mass. In that case in Eqs. (9) $\sqrt{p^2+m^2}$ may be expanded in powers of p^2/m^2, after which they reduce to differential equations in the coordinate representation. Solving these equations we find for the pion mass

$$\mu_\pi^2 = 4(m^2 - mV - 2V^2) + 4(\pi/a)^2(1 - 3V/2m + 2V^2/m^2)$$
$$\approx 6m^2(1-2V/m) + 3(\pi/a)^2. \tag{19}$$

Normalizing the potential to the mass of the pion we find for the masses of the remaining particles —two pseudovector μ_{1A} and μ_{2A}, scalar μ_S and vector μ_V —the following values:

$$\mu_{1A}^2 = 6m^2(1-2V/m) + 3(4.5/a)^2 = \mu_\pi^2 + 30/a^2,$$
$$\mu_{2A}^2 = 4m^2(1-V/m) + 2(4.5/a)^2 = 2m^2 + 30/a^2,$$
$$\mu_S^2 = 4m^2(1-V/m) + (4.5/a)^2 = 2m^2 + 10/a^2,$$
$$\mu_V^2 = 4m^2(1-V/m) + (3.8/a)^2 = 2m^2 + 4.5/a^2. \tag{20}$$

In the other limiting case when the radius of the well is small one may neglect in Eqs. (9) and (10) m and ω in comparison with p. In the zeroth approximation the condition for the existence of a pseudoscalar meson takes the form

$$\varphi = \frac{2}{p}\int_0^\infty V_0(p,p')\varphi(p')dp'. \tag{21}$$

The analogous equation for the other singlet states ($l \neq 0$) will have a solution only for a deeper potential. Since the pion mass does not enter the equation in the zeroth approximation this means that these bound states do not exist if the pions do.

Making use of the inequality $\int fpf\,dp \geq 2\int fV_0f\,dp\,dp'$, which is satisfied for any function f provided that V_0 satisfies Eq. (21), one can show that all triplet states are also absent except for the scalar. For the scalar case Eq. (13) coincides in the zeroth approximation with Eq. (21). Taking into account following terms in the expansion one can show that the mass of the scalar meson satisfies in this limiting case the inequality

$$\mu_\pi^2 + 4m^2 > \mu_S^2 > \mu_\pi^2 + 2m^2. \qquad (22)$$

These last results do not depend on the form of the potential provided only that the effective radius of the interaction is small compared to the nucleon mass.

4. Consequently, it can be said qualitatively that in this model the pseudoscalar meson has the smallest mass regardless of the interaction radius. When the interaction is so chosen that the mass of the pseudoscalar meson is small then the masses of the remaining mesons (if they exist) are of the order of the nucleon mass. The fact that in the case of a wide square well all singlet states have a small mass is, apparently, a peculiarity of this special case.

It would be of great interest to make clear how important is the effect of giving up relativistic invariance. If one does not consider the cut off four-fermion interaction model,[3] then the simplest relativistically invariant model is the one-meson exchange model. Unfortunately this model is beset with some difficulties of principle.[4] The fact that the integral equations are in that case two-dimensional is not significant. In the special case of a zero mass meson they reduce to one-dimensional and even differential equations. But the logarithmic divergence occuring in this theory gives rise to a continuous spectrum.

If it is assumed that the above calculations are not completely divorced from reality then it would be of interest to clarify what would happen if 1) one of the nucleons is replaced by a particle of a different mass, 2) the interaction is assumed to be not of the vector type and 3) the annihilation diagrams are taken into account (this would show how the masses of mesons with different isotopic spin differ in this model).

It follows from the above formulas, for example Eq. (19), that for a sufficiently strong interaction the square of the pion mass will become negative which indicates instability of the vacuum. This would not happen if the change in the nucleon mass due to the interaction were taken into account. With an increasing interaction the binding energy of the nucleon increases but so does its mass, so that the mass of the pseudoscalar meson reaches zero only in the limiting case when the interaction is many times larger than the bare nucleon mass. It would therefore be of interest to calculate the change in the nucleon mass due to the interaction, Eq. (1).

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