ASYMPTOTIC EQUALITY OF THE TOTAL CROSS SECTIONS OF PARTICLE AND ANTIPARTICLE

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The asymptotic equality of the total cross sections $\sigma_+(E)$ and $\sigma_-(E)$ of a particle and its antiparticle is proven under some general assumptions regarding the behavior of the cross sections at high energies. The rate of decrease of the difference $\Delta\sigma(E) = \sigma_+(E) - \sigma_-(E)$ is investigated. The conclusions are based on certain general properties of analytic functions.

1. Pomeranchuk[1] has proved the equality of the total cross sections $\sigma_+(E)$ and $\sigma_-(E)$ for the scattering of a particle and its antiparticle by the same target in the limit as $E \to \infty$, under the assumption that the particle and antiparticle scattering amplitudes $A_\pm(E)$ satisfy the following: a) the cross sections $\sigma_\pm(E) = \operatorname{Im} A_\pm(E)/\sqrt{E^2-\mu^2}$ (where μ is the mass of the particle and E is the energy in the laboratory system) tend to constant limits $\sigma_\pm$ as $E \to \infty$, while the ratios $A_\pm(E)/\sqrt{E^2-\mu^2}$ remain bounded as $E \to \infty$; b) the functions $A_\pm(E)/(E^2-\mu^2)$ satisfy dispersion relations without subtractions. Let us note that it follows from a) that condition b) is simply equivalent to the assumption that the function $A_\pm(E)/(E^2-\mu^2)$ tends to zero on a large circle.[1)]

A number of authors have improved on the result of Pomeranchuk.[2-4] Weinberg[2] has replaced the condition that the $\sigma_\pm(E)$ must tend to finite limits by the condition that the difference $\sigma_+(E) - \sigma_-(E) = \Delta\sigma(E)$ should be of one sign for sufficiently large energies. On the assumption that $\sigma_\pm(E) \sim C_\pm \ln^m E$, $0 < m < 1$, he proves the equality $C_+ = C_-$. Amati, Fierz, and Glaser[3] have remarked that the integral $\int^\infty \Delta\sigma(E)E^{-1}dE$ converges. Lehmann[4] found the asymptotic behavior of $\operatorname{Re} A(E)$ on the assumption that $\operatorname{Im} A(E) \sim aE + bE^{1-\alpha}$, $0 < \alpha \leq 1$. All these results are obtained by studying the asymptotic behavior in the dispersion relations.

We start with certain general properties of analytic functions, formulated below, and make no use of dispersion relations in any form whatsoever.

2. We shall need the following generalization, due to Lindelöf, of the maximum principle.

A. If the function $f(z)$ is regular in the upper half-plane and bounded on the real axis $|f(x)| < M$, then the following alternative holds: either at all points of the upper half-plane $|f(z)| < M$, or $f(z)$ increases faster than a certain exponential, i.e., there exists an $\alpha > 0$, such that $M(R) > e^{\alpha R}$ for sufficiently large R, where $M(R)$ is the maximum of $|f(Re^{i\varphi})|$ on the upper half circle of radius R.[2)]

The second of the theorems that we need makes it possible to judge the behavior of a function in a region from its properties on the boundary of the region.

B. Let $f(z)$ be a function regular in the singly connected region G, and Γ_1 and Γ_2 be two branches of the region boundary that go off to infinity in the negative and positive relative to the region directions, i.e., when moving along Γ_1 the region is left to the right, and when moving along Γ_2 the region is left to the left. Let us denote by $\mathscr{E}_1$ the manifold of the limit values of $f(z)$ as $z \to \infty$ along Γ_1, and by $\mathscr{E}_2$ the manifold of the limit values of $f(z)$ as $z \to \infty$ along Γ_2. Each of these manifolds consists of either one point or a continuum. The theorem asserts that if the manifolds $\mathscr{E}_1$ and $\mathscr{E}_2$ have no points in common, and if one of them does not surround the other, then the function $f(z)$ cannot be bounded in the region G (see [5], p. 67-68).

Of particular importance is the simplest case when the function $f(z)$ tends to definite limits along Γ_1 and Γ_2. In that case each of the manifolds $\mathscr{E}_1$ and $\mathscr{E}_2$ consist of one point. If the limits along Γ_1

1)We may note that if the spectral function $\rho(E)$ along the cuts tends to zero as $E \to \infty$ faster than some negative power $E^{-\mu}$, $\mu > 0$, then the function itself satisfies a dispersion relation without subtractions in, and only in, the trivial case when the function tends to zero on a large circle.

2)See, for example, [5], p. 50 or [6], p. 78. The theorem is also valid for a region obtained from the half-plane by deforming a finite part of its boundary.

and Γ_2 do not coincide then the function $f(z)$ is unbounded in G. If the limits coincide then either $f(z)$ approaches this limit uniformly in G as $z \to \infty$, or $f(z)$ is unbounded in G. If G coincides with the half-plane and along Γ_1 and Γ_2 tends to finite limits, then it follows from Lindelöf's theorem that if $f(z)$ is unbounded it must increase faster than some exponential $e^{\alpha|z|}$. Thus, for example, if it is known that in the half-plane the function increases no faster than some power of z, then $f(z)$ cannot tend to different limits along the positive and negative semi-axes.

Sugawara and Kanazawa[7] have proved this last assertion (they introduce certain superfluous additional requirements) and obtained from it the equality $\lim \sigma_+(E) = \lim \sigma_-(E)$ under the assumption that the limits of $A_\pm(E)/E$ exist as $E \to +\infty$. Our work is conceptually very close to the work of these authors, but we make use of stronger properties of analytic functions.

It follows from the Lindelöf theorem (see first footnote) that if the function $f(z)$ goes to zero like $z^{-\mu}$, $0 < \mu$, as z goes to infinity along the two cuts $(-\infty, C_1)$ and $(C_2, +\infty)$ lying on the real axis, and if $f(z)$ increases slower than an arbitrary exponential at complex infinity, then the function $f(z)$ satisfies a dispersion relation without subtractions.

3. As is known, the scattering amplitudes $A_\pm(E)$ are analytic in the plane with the two cuts $(-\infty, -\mu$ and $(\mu, +\infty)$ along the real axis and have the following properties: 1) $A_+(E) = A_-(E)$, 2) both functions are real in the interval $(-\mu, \mu)$. It therefore follows that along the cuts $A(E - i0) = A^*(E + i0)$.

We assume once and for all that in the E plane the amplitudes $A_\pm(E)$ increase slower than any exponential, i.e., that for arbitrary $\epsilon > 0$ one has $|A(E)| < e^{\epsilon|E|}$ for $|E| > E_0(\epsilon)$. This is a quite liberal requirement. It is satisfied even by as rapidly increasing a function as $e^{\sqrt{E}}$. We also make the additional assumption that along the cuts the function $A(E)/\sqrt{E^2 - \mu^2}$ is bounded as $E \to \pm\infty$.

Let us introduce the function $g(E) = [A_+(E) - A_-(E)]/\sqrt{E^2 - \mu^2}$. According to the optical theorem $\Delta\sigma(E) = \sigma_+(E) - \sigma_-(E) = \operatorname{Im} g(E)$. It is easy to verify that the following relation holds along the cuts:

$$g(-E + i0) = g^*(E + i0),$$
$$g(-E - i0) = -g(E + i0). \qquad (3.1)$$

Let us consider the function $g(E)$ in the upper half-plane. Let $\mathscr{E}_1$ be the manifold of the limiting values of $g(E)$ as $E \to \infty$ along the upper edge of the left cut, and $\mathscr{E}_2$ the manifold of limiting values as $E \to +\infty$ along the upper edge of the right cut. In the complex plane $g = u + iv$ these manifolds are located symmetrically relative to the real axis and are bounded. These manifolds intersect with the real axis since otherwise one of them would have to lie entirely in the upper half-plane and the other entirely in the lower half-plane and they would have no common points. According to the theorems A and B it would then follow that the function $g(E)$ increases faster than some exponential $e^{\alpha|E|}$, which is impossible. And so there exists a real point 2k which belongs to both $\mathscr{E}_1$ and $\mathscr{E}_2$; this means that there exists a sequence of energy values $\{E_n\}$, $E_n \to +\infty$, such that $g(E_n) \to 2k$, and therefore

$$\lim [\sigma_+(E_n) - \sigma_-(E_n)] = \lim \operatorname{Im} g(E_n) = 0. \qquad (3.2)$$

Thus, from the assumption of boundedness of the ratios $A_\pm(E)/E$ as $E \to +\infty$ along the cut it follows that along a certain sequence of energies $\{E_n\}$ the difference of the total cross sections $\Delta\sigma(E_n)$ tends to zero. It therefore follows that if it is assumed that the limit of the difference of cross sections exists, then that limit is equal to zero.

4. These results can be generalized to the case when the cross sections do not tend to constants at high energies but vary according to some law, say increase or decrease as a logarithm to some power.

Let us suppose that

$$\Delta\sigma(E) \sim C\,|\varphi(E)| \qquad (4.1)$$

as $E \to +\infty$ along the cut, where the function $\varphi(E)$ is analytic in the upper half-plane and satisfies the following conditions: 1) for any $\epsilon > 0$ for sufficiently large $|E| > E_0(\epsilon)$ we have $|E|^{-\epsilon} < |\varphi(E)| < |E|^{\epsilon}$; 2) along the real axis $\operatorname{Im} \varphi(E)/|\varphi(E)| \to 0$, and $\varphi(-E)/\varphi(E) \to 1$ as $E \to +\infty$. Any positive or negative power of ln E satisfies these conditions.

If the amplitude is such that the function $g(E)/\varphi(E)$ is bounded as $E \to +\infty$ along the cut, then by applying theorem B to this function we find that $C = 0$. Thus the following assertion is proved.

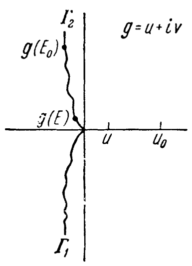
No matter what the function $\varphi(E)$ is, if it satisfies conditions 1) and 2) then the difference of total cross sections $\Delta\sigma(E)$ cannot vary asymptotically according to the law $C|\varphi(E)|$, $C \neq 0$, provided only that the ratio $g(E)/\varphi(E)$ is bounded as $E \to +\infty$ along the cut. In particular, if the total cross sections vary asymptotically according to the law $\sigma_+(E) \sim C_+|\varphi(E)|$ and $\sigma_-(E) \sim C_-|\varphi(E)|$, then $C_+ = C_-$ or $\sigma_+(E)/\sigma_-(E) \to 1$.

If it is known beforehand that the ratio $\operatorname{Re} g(E)/$

Im g(E) is bounded as $E \to \infty$ along the cut, then from the asymptotic behavior $\Delta\sigma(E) \sim C|\varphi(E)|$, $C \neq 0$, automatically follows the boundedness of $g(E)/\varphi(E)$, i.e., the following general assertion is valid.

If the ratio $|\mathrm{Re}\, g(E)|/|\mathrm{Im}\, g(E)|$ is bounded as $E \to +\infty$ along the cut, then the difference of total cross sections $\Delta\sigma(E)$ cannot asymptotically follow the law $C|\varphi(E)|$, $C \neq 0$, with $\varphi(E)$ any function satisfying conditions 1) and 2). Below we shall substantially strengthen this assertion.

5. We give some estimates of the speed with which $\Delta\sigma(E)$ tends to zero (for proofs see Appendix) assuming the existence of a finite limit for the function g(E) as $E \to +\infty$ along the upper edges of the cuts. This limit is equal to some real number 2k which will be assumed to be equal to zero. If that is not the case we would replace A(E) by A(E) − kE. The function g(E) maps the upper edges of the left and right cuts into two curves Γ_1 and Γ_2 in the plane $g = u + iv$, symmetrically located relative to the real axis (see the figure). At that the sufficiently distant upper half-neighborhood of the point $E = \infty$ goes over into a certain neighborhood U (perhaps many-sheeted) of the point $g = 0$, bounded by Γ_1 and Γ_2. The following inequalities are valid.



1) If the neighborhood U makes an opening angle $2\alpha\pi \leq \pi$ with the vertex at the origin of the coordinates then, starting with some E_0,

$$\left|\frac{\Delta\sigma(E)}{\Delta\sigma(E_0)}\right| < C\left(\frac{E_0}{E}\right)^{\alpha/2}. \qquad (5.1)$$

The condition 1) is certainly satisfied if, as $E \to \infty$

$$|\mathrm{Re}\, g(E)|/|\mathrm{Im}\, g(E)| \leqslant \cot\alpha.$$

2) If the neighborhood U contains a tip whose boundaries are defined by the equation $|v| = |u|^{\nu}$, $\nu > 1$, then starting with some E_0,

$$\left|\frac{\Delta\sigma(E)}{\Delta\sigma(E_0)}\right| \leqslant \left(1 + \frac{\nu-1}{4}\ln\frac{E_0}{E}\right)^{-\nu/(\nu-1)}. \qquad (5.2)$$

The condition 2) is certainly satisfied if, as $E \to \infty$

$$|\mathrm{Im}\, g(E)| \geqslant |\mathrm{Re}\, g(E)|^{\nu}.$$

If it is known that the curves Γ_1 and Γ_2 have at the point zero tangents that form the angle 2φ then it can be shown that $\Delta\sigma(E)$ decreases like

$$\psi(E)\, E^{-\beta}, \quad \beta = \frac{2}{\pi}\tan^{-1}\varphi$$

where $\psi(E)$ is a function which decreases or increases slower than any positive or negative power of E.

APPENDIX

We limit ourselves to the proof of inequality (5.1); at that it is sufficient to consider $\alpha = 1/2$ since the substitution $g = g^{1/2\alpha}$ will reduce to this case. Let $g(E_0)$ and $g(E)$ be two points on Γ; let $\Gamma(E, E_0)$ be a chord with ends in these two points and let $\rho(u)$ be the distance from the real point u to the chord $\Gamma(E, E_0)$. The following inequality holds (see, for example, [5], pp. 85-86)

$$\int_{u_E}^{u_0} \frac{du}{\rho(u)} > \frac{1}{4}\ln\frac{E}{E_0},$$

where u_0 is the nearest point of intersection of the map of the semicircle of radius E_0 with the positive semiaxis $u \geq 0$, and u_E is the farthest point of intersection of the map of the circle of radius E with the same semiaxis. It is obvious that for $\alpha = 1/2$ we have $\rho(u) \geq \sqrt{u^2 + |g(E)|^2}$, and therefore

$$\int_0^{u_0} \frac{du}{\sqrt{u^2 + |g(E)|^2}} = \ln\frac{u_0 + \sqrt{u_0^2 + |g(E)|^2}}{|g(E)|} > \frac{1}{4}\ln\frac{E}{E_0},$$

or

$$\frac{g(E)}{u_0 + \sqrt{u_0^2 + |g(E)|^2}} < \left(\frac{E_0}{E}\right)^{1/4}.$$

Since $|\Delta\sigma(E)| \leq |g(E)|$, the inequality (5.1) follows.

[1] I. Ya. Pomeranchuk, JETP **34**, 725 (1958), Soviet Phys. JETP **7**, 499 (1958).

[2] S. Weinberg, Phys. Rev. **124**, 2049 (1961).

[3] Amati, Fierz, and Glaser, Phys. Rev. Lett. **4**, 89 (1960).

[4] H. Lehmann, Nucl. Phys. **29**, 300 (1962).

[5] R. Nevanlinna, Odnoznachnye analiticheskiye funktsii (Single-valued Analytic Functions), OGIZ, 1941.

[6] N. G. Chebotarev and N. N. Meĭman, Trudy, Math. Inst. Acad. Sci., 1949.

[7] M. Sugawara and A. Kanazawa, Phys. Rev. **123**, 1895 (1961).

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