WEAKLY TURBULENT PLASMA IN A MAGNETIC FIELD

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Equations describing the behavior of a weakly turbulent plasma in a magnetic field are obtained with mode coupling taken into account. The mode coupling is weak in such a plasma and can be introduced by expanding in powers of the square of the oscillation amplitude. The anomalous diffusion of a plasma in a magnetic field in the presence of fluctuations is discussed.

1. INTRODUCTION

A plasma is capable of many more different forms of turbulent motion than an ordinary fluid. In particular, in addition to the usual strong magnetohydrodynamic turbulence [1,2] a plasma can exhibit a so-called weakly turbulent state (this terminology is due to A. A. Vedenov) characterized by the excitation of many weakly interacting modes of oscillation.

In terms of a statistical description the weakly turbulent state is very similar to a system of weakly interacting particles (i.e., the same plasma viewed thermodynamically) for which one naturally considers an expansion in the ratio of mode interaction energy to total mode energy, this ratio being a small quantity. Vedenov et al [3] and Drummond and Pines [4] have shown that even the quasi-linear approximation, in which mode coupling is neglected, can be used to examine certain aspects of mode excitation and the effect of feedback on the average particle distribution function in a uniform weakly turbulent plasma.

Unfortunately, the quasi-linear approximation is not applicable to a number of problems concerned with anomalous diffusion of a plasma across a magnetic field, particularly those related to the plasma drift instability. [5,6] In problems of this kind one must take account of mode coupling.

In contrast with the dynamic analyses of Drummond and Pines [4] and Sturrock, [7] in which the question of mode coupling was discussed in terms of uniform Langmuir oscillations, in the present work a statistical approach is used from the very beginning; specifically, we form a chain of equations for correlation functions, which are obtained by averaging over a statistical ensemble. In this case the expansion in terms of the small mode interaction energy corresponds to taking account of higher order correlations between modes, that is to say, the number of modes involved in a given elementary interaction process.

For simplicity we limit ourselves here to the first nonvanishing correlation, the third. The neglect of higher correlations is evidently justified in transparent regions, where wave damping or growth is small. More precisely, the amplitude of a given mode must not change appreciably during the characteristic time required for an appreciable phase shift to arise between different modes.

It is also shown here that an analogous chain of equations can be constructed by the Wiener method, [8] which is based on the expansion of an arbitrary random process in powers of the Brownian motion. The results of the two expansions are found to be somewhat different, but this difference can be attributed to the large number of simplifications used in truncating the chain in the Wiener technique.

2. LONGITUDINAL OSCILLATIONS

For reasons of simplicity we start with an analysis of longitudinal waves, in which case the electric field is derivable from a potential: $\mathbf{E} = -\nabla\varphi$. In a strong magnetic field these waves would be the ion acoustic waves, which can be converted into drift waves in an inhomogeneous plasma, [6] and the plasma waves (Langmuir). The distribution function f_j for particles of type j is broken into two parts $f_j = f_0^j + F^j$ where $f_0^j = \langle f_j \rangle$ is the mean value of the distribution function taken over a statistical ensemble.

We assume that the averaged functions f_0^j are slowly varying functions of space and time. We also assume that the wavelength of the high-frequency oscillations excited in the plasma is much smaller than the scale size of the inhomogeneity, so that F^j can be expanded in a Fourier integral:

$$F^j = \int F^j_{\mathbf{k}\omega}(\mathbf{v})\, e^{-i\omega t + i\mathbf{k}\mathbf{r}} d\omega d\mathbf{k};$$

this expression is substituted in the kinetic equation (with self-consistent fields) which is then separated into two equations by the averaging process:

$$\frac{\partial f_0^j}{\partial t} + \mathbf{v}\nabla f_0^j + \frac{e_j}{m_j}\left\{\mathbf{E}_0 + \frac{1}{c}[\mathbf{v}\mathbf{H}_0]\right\}\frac{\partial f_0^j}{\partial \mathbf{v}} = S_j$$

$$\equiv \frac{e_j}{m_j}\frac{\partial}{\partial \mathbf{v}}\int i\mathbf{k}\,\langle \Phi_{-\mathbf{k},-\omega} F^j_{\mathbf{k}\omega}\rangle\, d\mathbf{k}d\omega, \qquad (1)^*$$

$$L_j F^j_{\mathbf{k}\omega} \equiv (-i\omega + i\mathbf{k}\mathbf{v})\,F^j_{\mathbf{k}\omega}$$

$$+ (e_j/m_j)\{\mathbf{E}_0 + c^{-1}[\mathbf{v}\mathbf{H}_0]\}\,\partial F^j_{\mathbf{k}\omega}/\partial \mathbf{v}$$

$$= \frac{e_j}{m_j} i\mathbf{k}\Phi_{\mathbf{k}\omega}\frac{\partial f_0^j}{\partial \mathbf{v}} + \frac{e_j}{m_j}\frac{\partial}{\partial \mathbf{v}}\int i\mathbf{k}'\,(\Phi_{\mathbf{k}'\omega'}F_{\mathbf{k}-\mathbf{k}',\omega-\omega'}$$

$$- \langle \Phi_{\mathbf{k}'\omega'}F_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\rangle)\,d\mathbf{k}'d\omega', \qquad (2)$$

where $\Phi_{\mathbf{k}\omega}$ is the Fourier component of the electric field potential, given by

$$\Phi_{\mathbf{k}\omega} = \sum_j \frac{4\pi e_j}{k^2}\int F^j_{\mathbf{k}\omega}d\mathbf{v}, \qquad (3)$$

while the remaining notation is conventional.

We rewrite (2) in the form

$$F^j_{\mathbf{k}\omega}(\mathbf{v}) = \mathbf{g}^j_{\mathbf{k}\omega}\left\{\mathbf{k}\Phi_{\mathbf{k}\omega}f_0^j + \int \mathbf{k}'\,(\Phi_{\mathbf{k}'\omega'}F_{\mathbf{k}-\mathbf{k}',\omega-\omega'} - \langle \Phi_{\mathbf{k}'\omega'}F_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\rangle)\,d\mathbf{k}'d\omega'\right\}, \qquad (4)$$

where $\mathbf{g}^j_{\mathbf{k}\omega} = iL_j^{-1}(e_j/m_j)\,\partial/\partial\mathbf{v}$ is an operator that operates on the variable $\mathbf{v}$.

Neglecting quadratic terms in (4) we have

$$F^j_{\mathbf{k}\omega}(\mathbf{v}) = k\mu^j_{\mathbf{k}\omega}(\mathbf{v})\,\Phi_{\mathbf{k}\omega},$$

where

$$\mu^j_{\mathbf{k}\omega}(\mathbf{v}) = \mathbf{g}^j_{\mathbf{k}\omega}\frac{\mathbf{k}}{k}f_0^j = iL_j^{-1}\frac{e_j}{m_j}\frac{\mathbf{k}}{k}\frac{\partial f_0^j}{\partial \mathbf{v}}. \qquad (5)$$

Substituting this expression in (3) we obtain the dispersion equation $\epsilon = 0$, where

$$\varepsilon(\mathbf{k},\omega) = 1 - \sum_j \frac{4\pi e_j}{k}\int \mu^j_{\mathbf{k}\omega}(\mathbf{v})\,d\mathbf{v} \qquad (6)$$

is the dielectric constant of the plasma.

In the general case ϵ is complex, corresponding to wave damping or growth. In the unstable case ($\mathrm{Im}\,\omega > 0$) the oscillations obviously grow so long as they do not affect the average distribution function and so long as there is no interaction between modes due to nonlinear effects. When the growth rate is small, i.e., when $\mathrm{Im}\,\omega \ll \mathrm{Re}\,\omega$, one expects that the growth of the oscillations will stop at a rather low level, in which case the nonlinear interaction remains small. Making use of this situation we can expand in the ratio of mode interaction energy to total mode energy, retaining only the first term in the expansion.

We first consider a uniform steady-state situation. In this case all double correlations contain a δ-function; specifically

$$\langle \Phi_{\mathbf{k}'\omega'}\Phi^*_{\mathbf{k}\omega}\rangle = I_{\mathbf{k}\omega}k^{-2}\delta(\omega-\omega')\,\delta(\mathbf{k}-\mathbf{k}');$$

$$\langle F^j_{\mathbf{k}'\omega'}\Phi^*_{\mathbf{k}\omega}\rangle = P^j_{\mathbf{k}\omega}(\mathbf{v})\,\delta(\omega-\omega')\,\delta(\mathbf{k}-\mathbf{k}').$$

Multiplying Eq. (4) by $\Phi^*_{\mathbf{k}\omega}$ and averaging we have

$$P^j_{\mathbf{k}\omega}(\mathbf{v}) = \mu^j_{\mathbf{k}\omega}k^{-1}I_{\mathbf{k}\omega} + \mathbf{g}^j_{\mathbf{k}\omega}\int(\mathbf{k}-\mathbf{k}')\,Q^j_{\mathbf{k}'\omega',\mathbf{k}\omega}\,d\mathbf{k}'d\omega', \qquad (7)$$

where $Q^j_{\mathbf{k}'\omega',\mathbf{k}\omega}(\mathbf{v})$ is the triple correlation function:

$$Q^j_{\mathbf{k}'\omega',\mathbf{k}\omega}\delta(\mathbf{k}-\mathbf{k}'')\,\delta(\omega-\omega'') = \langle F_{\mathbf{k}'\omega'}\Phi_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\Phi^*_{\mathbf{k}''\omega''}\rangle.$$

Multiplying (4) by $\Phi_{\mathbf{k}''-\mathbf{k},\omega-\omega''}\Phi^*_{\mathbf{k}\omega}$ and then averaging over the statistical ensemble we obtain an equation for the triple function. The right side of the equations then contains the quadruple function, for which an additional equation would be required, and so on. Making use of the assumption that the interaction is small we now truncate the chain, neglecting the quadruple correlation of modes characterized by different $\mathbf{k}$ and ω i.e., we equate the quadruple correlation function to the product the pair functions. In this approximation we have

$$Q^j_{\mathbf{k}'\omega',\mathbf{k}\omega} = k\mu^j_{\mathbf{k}'\omega'}q_{\mathbf{k}'\omega',\mathbf{k}\omega} + \mathbf{g}^j_{\mathbf{k}'\omega'}\left\{\frac{\mathbf{k}}{k^2}I_{\mathbf{k}\omega}P^j_{\mathbf{k}'-\mathbf{k},\omega'-\omega} + \frac{\mathbf{k}'-\mathbf{k}}{(\mathbf{k}'-\mathbf{k})^2}I_{\mathbf{k}'-\mathbf{k},\omega'-\omega}P^j_{\mathbf{k}\omega}\right\},$$

$$q_{\mathbf{k}'\omega',\mathbf{k}\omega}\delta(\mathbf{k}'-\mathbf{k}'')\,\delta(\omega'-\omega'') = \langle \Phi_{\mathbf{k}'\omega'}\Phi_{\mathbf{k}-\mathbf{k}',\omega-\omega'}\Phi^*_{\mathbf{k}''\omega''}\rangle. \qquad (8)$$

Using (3), we eliminate q from (8) and find

$$Q^j_{\mathbf{k}'\omega',\mathbf{k}\omega} = \sum_s \mathbf{G}^{js}_{\mathbf{k}'\omega'}\left\{\frac{\mathbf{k}}{k^2}I_{\mathbf{k}\omega}P^s_{\mathbf{k}'-\mathbf{k},\omega'-\omega} + \frac{\mathbf{k}'-\mathbf{k}}{(\mathbf{k}'-\mathbf{k})^2}I_{\mathbf{k}'-\mathbf{k},\omega'-\omega}P^s_{\mathbf{k}\omega}\right\} + q^0_{\mathbf{k}'\omega',\mathbf{k}\omega}. \qquad (9)$$

Here, $\mathbf{G}^{js}_{\mathbf{k}\omega}$ is an operator that operates on an arbitrary function of velocity $\mathbf{y}(\mathbf{v})$ in accordance with the rule

$$\mathbf{G}^{js}_{\mathbf{k}\omega}\mathbf{y}(\mathbf{v}) = \delta_{js}\mathbf{g}^s_{\mathbf{k}\omega}\mathbf{y}(\mathbf{v}) + \frac{4\pi e_s}{k\,\varepsilon_1(\mathbf{k},\omega)}\mu^j_{\mathbf{k}\omega}\int \mathbf{g}^s_{\mathbf{k}\omega}\mathbf{y}(\mathbf{v})\,d\mathbf{v},$$

where by $\epsilon_1(\mathbf{k},\omega)$ we mean $\mathrm{Re}\,\epsilon(\mathbf{k},\omega)$, since the imaginary part of ϵ is small by assumption and can be neglected in the approximation we are using here. However, in view of the possibility that $\mathrm{Re}\,\epsilon$ can vanish we must establish a rule for going around the poles. This rule is established on the basis of the reasonable assumption that taking account of the higher correlations must lead to damping of the free oscillations in $Q \sim \delta(\epsilon)$. Thus, the poles of Q must be traversed, as usual, in the upper half plane of the complex variable ω. In other words, by ϵ_1 we are to understand $\mathrm{Re}\,\epsilon$ with an added imaginary constant that takes account of the damping of Q.

The quantity q^0 in (9) arises by the elimination of q; at this point q^0 is an arbitrary solution of the

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*$[\mathbf{v}\mathbf{H}] = \mathbf{v}\times\mathbf{H}$.

equation $\epsilon_1(\mathbf{k}',\omega')\, q^0_{\mathbf{k}',\omega',\mathbf{k}\omega} = 0$. The quantity q^0 includes the damped initial correlation, which can be neglected, and a stationary part, which can be found from the following considerations. In (8) we have taken account of a small quadratic term in the expression for $F^j_{\mathbf{k}'\omega'}$ and in this case the quantities $\Phi_{\mathbf{k}-\mathbf{k}',\omega-\omega'}$ and $\Phi^*_{\mathbf{k}''\omega''}$ can be taken in the zeroth approximation, that is to say, we assume that they are entirely uncorrelated. Similarly, in the expression for q in $\Phi_{\mathbf{k}'\omega'}$ we keep only the zeroth term, which represents the potential of the free oscillations, so that $\epsilon_1(\mathbf{k}',\omega)\,\Phi_{\mathbf{k}'\omega'} = 0$; but in this case we must take account of the quadratic terms by means of (3) and (4) in $\Phi_{\mathbf{k}-\mathbf{k}',\omega-\omega'}$ and $\Phi^*_{\mathbf{k}''\omega''}$. Again replacing the quadruple correlation function by the product of the pair functions we reduce this additional correlation to the form

$$q^0_{\mathbf{k}'\omega',\,\mathbf{k}\omega} = \sum_s \frac{4\pi e_s}{|\mathbf{k}'-\mathbf{k}|^2 \varepsilon_1(\mathbf{k}-\mathbf{k}',\,\omega-\omega')} \times \int \mathbf{g}^s_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'} \left\{ \frac{\mathbf{k}}{k^2} I_{\mathbf{k}\omega} P^s_{-\mathbf{k}',\,-\omega'} - \frac{\mathbf{k}'}{k'^2} I_{\mathbf{k}'\omega'} P^s_{\mathbf{k}\omega} \right\} d\mathbf{v} + \sum_s \frac{4\pi e_s}{k^2 \varepsilon_1(\mathbf{k},\omega)} \int \mathbf{g}^{s*}_{\mathbf{k}\omega} \left\{ \frac{\mathbf{k}'}{k'^2} I_{\mathbf{k}'\omega'} P^{s*}_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'} + \frac{\mathbf{k}-\mathbf{k}'}{|\mathbf{k}-\mathbf{k}'|^2} I_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'} P^{s*}_{\mathbf{k}'\omega'} \right\} d\mathbf{v}. \tag{10}$$

If Eq. (9), which gives Q, is substituted in (7) taking account of (10) it is evident that $P^j_{\mathbf{k}\omega}$ can be written as a sum of two terms

$$P^j_{\mathbf{k}\omega}(\mathbf{v}) = k^{-1}\tilde{\mu}^j_{\mathbf{k}\omega} I_{\mathbf{k}\omega} + p^j_{\mathbf{k}\omega}(\mathbf{v});$$

the first term is proportional to $I_{\mathbf{k}\omega}$ while the second is a small quantity.

Neglecting the quantity $\rho_{\mathbf{k}\omega}$ in the expression for Q we divide each of the expression in (7) into two parts:

$$\tilde{\mu}^j_{\mathbf{k}\omega} = \mu^j_{\mathbf{k}\omega} + \sum_s \int \left(\frac{\mathbf{k}-\mathbf{k}'}{|\mathbf{k}-\mathbf{k}'|^2} \mathbf{g}^j_{\mathbf{k}\omega} \right) \mathbf{G}^s_{\mathbf{k}'\omega'} \left\{ \frac{\mathbf{k}}{k} \tilde{\mu}^s_{\mathbf{k}'-\mathbf{k},\,\omega'-\omega} + \frac{\mathbf{k}'-\mathbf{k}}{|\mathbf{k}'-\mathbf{k}|} \tilde{\mu}^s_{\mathbf{k}\omega} \right\} I_{\mathbf{k}'-\mathbf{k},\,\omega-\omega'} d\mathbf{k}' d\omega' + \sum_s \int \frac{4\pi e_s}{k'^2 \varepsilon_1(\mathbf{k}',\omega')} (\mathbf{k}'\mathbf{g}^j_{\mathbf{k}\omega} \mu^j_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'}) \times \int \mathbf{g}^s_{\mathbf{k}'\omega'}(\mathbf{v}') \left\{ \frac{\mathbf{k}}{k} \tilde{\mu}^s_{\mathbf{k}'-\mathbf{k},\,\omega'-\omega}(\mathbf{v}') + \frac{\mathbf{k}'-\mathbf{k}}{|\mathbf{k}'-\mathbf{k}|} \tilde{\mu}^s_{\mathbf{k}\omega}(\mathbf{v}') \right\} \times d\mathbf{v}' I_{\mathbf{k}'-\mathbf{k},\,\omega'-\omega} d\mathbf{k}' d\omega', \tag{11}$$

$$p^j_{\mathbf{k}\omega}(\mathbf{v}) = \sum_s \frac{4\pi e_s}{k^2 \varepsilon_1^*(\mathbf{k},\,\omega)} \int \left(\frac{\mathbf{k}-\mathbf{k}'}{|\mathbf{k}-\mathbf{k}'|} \mathbf{g}^j_{\mathbf{k}\omega} \mu^j_{\mathbf{k}'\omega'} \right) \int \mathbf{g}^{s*}_{\mathbf{k}\omega}(\mathbf{v}') \times \left\{ \frac{\mathbf{k}'}{k'} \tilde{\mu}^{s*}_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'}(\mathbf{v}') + \frac{\mathbf{k}-\mathbf{k}'}{|\mathbf{k}-\mathbf{k}'|} \tilde{\mu}^{s*}_{\mathbf{k}'\omega'}(\mathbf{v}') \right\} d\mathbf{v}' I_{\mathbf{k}'\omega'} I_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'} d\mathbf{k}' d\omega'. \tag{11a}$$

We now introduce the effective dielectric constant of the weakly turbulent plasma

$$\tilde{\varepsilon}(\mathbf{k},\,\omega) = 1 - \sum_j \frac{4\pi e_j}{k} \int \tilde{\mu}^j_{\mathbf{k}\omega}(\mathbf{v})\, d\mathbf{v}, \tag{12}$$

using the representation given above for $P_{\mathbf{k}\omega}$ as a sum of two terms, multiply (3) by $\Phi^*_{\mathbf{k}\omega}$, and then average to obtain

$$\tilde{\varepsilon}(\mathbf{k},\,\omega) I_{\mathbf{k}\omega} = \sum_j 4\pi e_j \int p^j_{\mathbf{k}\omega} d\mathbf{v}.$$

Now, replacing $\rho^j_{\mathbf{k}\omega}$ everywhere by the expression in (11a) and making the approximations $\epsilon_1^* \sim \tilde{\epsilon}^*$ and $\mu^s_{\mathbf{k}\omega} \sim \tilde{\mu}^s_{\mathbf{k}\omega}$, which are completely justified within the scope of the treatment used here, we reduce this relation to the form

$$|\tilde{\varepsilon}(\mathbf{k},\,\omega)|^2 I_{\mathbf{k}\omega} = \frac{1}{2} \int \left| \sum_j \frac{4\pi e_j}{k} \int \mathbf{g}^j_{\mathbf{k}\omega} \left\{ \frac{\mathbf{k}/2-\mathbf{k}'}{|\mathbf{k}/2-\mathbf{k}'|} \tilde{\mu}^j_{\mathbf{k}/2+\mathbf{k}',\,\omega/2+\omega'} + \frac{\mathbf{k}/2+\mathbf{k}'}{|\mathbf{k}/2+\mathbf{k}'|} \tilde{\mu}^j_{\mathbf{k}/2-\mathbf{k}',\,\omega/2-\omega'} \right\} \times d\mathbf{v} \right|^2 I_{\mathbf{k}/2+\mathbf{k}',\,\omega/2+\omega'} I_{\mathbf{k}/2-\mathbf{k}',\,\omega/2-\omega'} d\mathbf{k}' d\omega'. \tag{13}$$

It is thus obvious that the electric field fluctuations in a weakly turbulent plasma can be represented as oscillations [in a medium with dielectric constant $\tilde{\epsilon}$] produced by a noise source whose intensity is given by the right side of (13). If the latter is neglected the turbulent motion represents an ensemble of oscillations with characteristic frequencies $\omega_{\mathbf{k}}$ given by the equation $\tilde{\epsilon}(\mathbf{k},\omega_{\mathbf{k}}) = 0$.

3. ARBITRARY OSCILLATIONS

Obviously, the analysis of a plasma in a magnetic field in terms of Langmuir oscillations alone is highly approximate; in a more precise analysis it is necessary to take account of the perturbation of the magnetic field. Turning again to the Fourier representation, in the general case (2) is replaced by

$$L_j F^j_{\mathbf{k}\omega} = -\frac{e_j}{m_j} \left\{ \mathbf{E}_{\mathbf{k}\omega} + \frac{1}{c}[\mathbf{v}\mathbf{H}_{\mathbf{k}\omega}] \right\} \frac{\partial f^j_0}{\partial \mathbf{v}} - \frac{e_j}{m_j} \frac{\partial}{\partial \mathbf{v}} \int \left\{ \left(\mathbf{E}_{\mathbf{k}'\omega'} + \frac{1}{c}[\mathbf{v}\mathbf{H}_{\mathbf{k}'\omega'}] \right) F_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'} - \left\langle \left(\mathbf{E}_{\mathbf{k}'\omega'} + \frac{1}{c}[\mathbf{v}\mathbf{H}_{\mathbf{k}'\omega'}] \right) F_{\mathbf{k}-\mathbf{k}',\,\omega-\omega'} \right\rangle \right\} d\mathbf{k}' d\omega'. \tag{14}$$

Equation (3) is replaced by Maxwell's equations

$$i[\mathbf{k}\mathbf{H}_{\mathbf{k}\omega}] + i\frac{\omega}{c}\mathbf{E}_{\mathbf{k}\omega} = \frac{4\pi}{c}\sum_j e_j \int \mathbf{v} F^j_{\mathbf{k}\omega} d\mathbf{v}, \tag{15}$$

$$i[\mathbf{k}\mathbf{E}_{\mathbf{k}\omega}] - i(\omega/c)\mathbf{H}_{\mathbf{k}\omega} = 0. \tag{16}$$

We first consider a uniform plasma that is stationary on the average. It is assumed that the plasma is in a weakly nonequilibrium state so that waves are excited only in the transparent region, where the linear growth rates are small compared with the frequency. To avoid repeating the calculations given in the earlier section and to demonstrate another means of forming the chain of coupled equations we make use here of the Wiener method of expanding a stationary random process in powers of the Brownian motion.

According to Wiener [8] any stationary random fluctuation (in our case, say the electric field $\mathbf{E}$) can be expanded in an orthogonal functional of the Brownian motion:

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2 + \ldots; \quad \mathbf{E}_0 = \langle \mathbf{E} \rangle,$$

$$\mathbf{E}_1 = \int \mathbf{E}_1(\mathbf{k}, \omega)\, e^{-i\omega t + i\mathbf{k}\mathbf{r}} \rho(\mathbf{k}, \omega, \alpha)\, d\mathbf{k}\, d\omega, \tag{17}$$

$$\mathbf{E}_2 = \frac{1}{2}\int \mathbf{E}_2(\mathbf{k}, \omega; \mathbf{k}', \omega') \times e^{-i(\omega+\omega')t + i(\mathbf{k}+\mathbf{k}')\mathbf{r}} \rho(\mathbf{k}, \omega, \alpha)\, \rho(\mathbf{k}', \omega', \alpha)\, d\mathbf{k}\, d\omega\, d\mathbf{k}'\, d\omega' - \frac{1}{2}\int \mathbf{E}_2(\mathbf{k}, \omega; -\mathbf{k}, -\omega)\, d\mathbf{k}\, d\omega \tag{18}$$

etc. Here α is a variable that enumerates the Brownian trajectories and ρ is the displacement of the Brownian particle, which simulates our random process. The single property of this Brownian motion used in the expansion given above is the correlation relation:

$$\left\langle \rho(\mathbf{k}, \omega, \alpha)\, \rho(\mathbf{k}', \omega', \alpha) \right\rangle \equiv \int \rho(\mathbf{k}, \omega, \alpha)\, \rho(\mathbf{k}', \omega', \alpha)\, d\alpha = \delta(\mathbf{k} + \mathbf{k}')\, \delta(\omega + \omega'). \tag{19}$$

It is evident from Eqs. (17)—(19) that in retaining $\mathbf{E}_1$ only we are assuming that the modes are independent of each other; taking account of second-order terms corresponds to taking account of the first nonvanishing correlation between different modes, i.e., the triple correlation, etc. Thus, in the approximation used in the preceding section we neglect higher terms starting with $\mathbf{E}_3$.

Substituting the expansions for $\mathbf{E}$, $\mathbf{H}$ and F in the kinetic equation, multiplying by $\rho(\mathbf{k}, \omega, \alpha)$, $\rho(\mathbf{k}, \omega, \alpha)\rho(\mathbf{k}', \omega', \alpha)$ etc., and integrating over α, we obtain a chain of equations for $\mathbf{E}_1$, $\mathbf{H}_1$ and F_1:

$$F_1^j(\mathbf{k}, \omega; \mathbf{v}) = i g^j_{\mathbf{k}\omega} f_0^j \left\{ \mathbf{E}_1(\mathbf{k}, \omega) + \frac{1}{c}\left[\mathbf{v}\mathbf{H}_1(\mathbf{k}, \omega)\right]\right\} + i g^j_{\mathbf{k}\omega} \int \left\{ \mathbf{E}_1(-\mathbf{k}', -\omega') + \frac{1}{c}\left[\mathbf{v}\mathbf{H}_1(-\mathbf{k}', -\omega')\right]\right\} F_2^j(\mathbf{k}', \omega'; \mathbf{k}, \omega; \mathbf{v})\, d\mathbf{k}'\, d\omega'$$
$$+ i g^j_{\mathbf{k}\omega} \int \left\{ \mathbf{E}_2(\mathbf{k}', \omega'; \mathbf{k}, \omega) + \frac{1}{c}\left[\mathbf{v}\mathbf{H}_2(\mathbf{k}', \omega'; \mathbf{k}, \omega)\right]\right\} F_1^j(-\mathbf{k}', -\omega'; \mathbf{v})\, d\mathbf{k}'\, d\omega' + \ldots, \tag{20}$$

$$F_2^j(\mathbf{k}', \omega'; \mathbf{k}, \omega; \mathbf{v}) = i g_{\mathbf{k}+\mathbf{k}', \omega+\omega'} f_0^j \left\{ \mathbf{E}_2(\mathbf{k}', \omega'; \mathbf{k}, \omega) + \frac{1}{c}\left[\mathbf{v}\mathbf{H}_2(\mathbf{k}', \omega'; \mathbf{k}, \omega)\right]\right\} + i g_{\mathbf{k}+\mathbf{k}', \omega+\omega'} \left\{ \left(\mathbf{E}_1(\mathbf{k}', \omega') + \frac{1}{c}\left[\mathbf{v}\mathbf{H}_1(\mathbf{k}', \omega')\right]\right) F_1(\mathbf{k}, \omega; \mathbf{v}) + \left(\mathbf{E}_1(\mathbf{k}, \omega) + \frac{1}{c}\left[\mathbf{v}\mathbf{H}_1(\mathbf{k}, \omega)\right]\right) F_1(\mathbf{k}', \omega'; \mathbf{v})\right\} + \ldots, \tag{21}$$

where the dots denote higher order terms that have been neglected. We note, however, that for accuracy to terms of higher order one can replace F_1^j in the last term in the right side of Eq. (20) by $i g^j_{\mathbf{k}\omega} f_0^j \{ \mathbf{E}_1(\mathbf{k}, \omega) + c^{-1}[\mathbf{v} \times \mathbf{H}_1(\mathbf{k}, \omega)]\}$.

If higher terms are neglected in the expression for $\langle |\mathbf{E}^2_{\mathbf{k}\omega}| \rangle$ and it is assumed that $\langle |\mathbf{E}^2_{\mathbf{k}\omega}| \rangle = |\mathbf{E}_1(\mathbf{k}, \omega)|^2$ comparing (20) and (21) with (7) and (8) one notes that the corresponding expansions are not completely identical, although they are very similar. Since the correlation-function method is based on a smaller number of assumptions, it should evidently be given preference. However, the analysis of the structure of the higher order terms of the expansion is facilitated by the Wiener method by virtue of the simplicity of the scheme used for constructing the chain.

We determine F_2, $\mathbf{E}_2$, and $\mathbf{H}_2$ from (21) taking account of Maxwell's equations and substitute the resulting expression in (20), thereby obtaining an equation that represents a generalization of (11) to the case of arbitrary waves:

$$\tilde{\mu}^j_{\mathbf{k}\omega} = \mu^j_{\mathbf{k}\omega} - \sum_s \int (\mathbf{g}^j_{\mathbf{k}\omega} \mathbf{b}_{\mathbf{k}-\mathbf{k}', \omega-\omega'}) \{ (\mathbf{G}^{js}_{\mathbf{k}'\omega'} \mathbf{b}_{\mathbf{k}'-\mathbf{k}, \omega'-\omega}) \tilde{\mu}^s_{\mathbf{k}\omega} + \mathbf{G}^{js}_{\mathbf{k}'\omega'} (\mathbf{b}_{\mathbf{k}'-\mathbf{k}, \omega'-\omega} \tilde{\mu}^s_{\mathbf{k}'-\mathbf{k}, \omega'-\omega}) \}\, I_{\mathbf{k}'-\mathbf{k}, \omega'-\omega}\, d\mathbf{k}'\, d\omega'$$
$$+ i \sum_s \int (\mathbf{g}^j_{\mathbf{k}\omega} A(\mathbf{k}'\omega')\, \mathbf{v}) \frac{4\pi e_s}{\omega'} (\mu^j_{\mathbf{k}'-\mathbf{k}, \omega'-\omega} \mathbf{b}_{\mathbf{k}'-\mathbf{k}, \omega'-\omega}) \times \{ (\mathbf{g}^s_{\mathbf{k}'\omega'} \mathbf{b}_{\mathbf{k}'-\mathbf{k}, \omega'-\omega})\, \tilde{\mu}^s_{\mathbf{k}\omega} + \mathbf{g}^s_{\mathbf{k}'\omega'} (\mathbf{b}_{\mathbf{k}'-\mathbf{k}, \omega'-\omega} \tilde{\mu}^s_{\mathbf{k}'-\mathbf{k}, \omega'-\omega}) \}\, I_{\mathbf{k}'-\mathbf{k}, \omega'-\omega}\, d\mathbf{k}'\, d\omega', \tag{22}$$

where the corresponding quantities are given by the relations

$$\mu^j_{\mathbf{k}\omega} = \mathbf{g}^j_{\mathbf{k}\omega} f_0^j, \tag{23}$$

$$F_1^j(\mathbf{k}, \omega, \mathbf{v}) = i\tilde{\mu}^j_{\mathbf{k}\omega} \{ \mathbf{E}_1(\mathbf{k}, \omega) + c^{-1}[\mathbf{v}\mathbf{H}_1(\mathbf{k}, \omega)]\} = i\tilde{\mu}^j_{\mathbf{k}\omega} \mathbf{b}_{\mathbf{k}\omega} E(\mathbf{k}, \omega), \tag{24}$$

where $\mathbf{b}_{\mathbf{k}\omega} = \mathbf{a}_{\mathbf{k}\omega} + \mathbf{k}(\mathbf{v} \cdot \mathbf{a}_{\mathbf{k}\omega})/\omega - (\mathbf{k} \cdot \mathbf{v})\, \mathbf{a}_{\mathbf{k}\omega}/\omega$ and $\mathbf{a}_{\mathbf{k}\omega}$ is a unit polarization vector

$$\mathbf{E}_{\mathbf{k}\omega} = \mathbf{a}_{\mathbf{k}\omega} E(\mathbf{k}\omega).$$

$$I_{\mathbf{k}\omega}\delta(\mathbf{k}-\mathbf{k}')\,\delta(\omega-\omega')(\mathbf{a}_{\mathbf{k}\omega})_\alpha(\mathbf{a}^*_{\mathbf{k}\omega})_\beta = \langle E_\alpha(\mathbf{k},\omega)\,E^*_\beta(\mathbf{k}',\omega')\rangle.$$

The effect of the operator $\mathbf{G}^{js}_{\mathbf{k}\omega}$ on an arbitrary vector function of velocity $\mathbf{y}(\mathbf{v})$ is given by

$$\mathbf{G}^{js}_{\mathbf{k}\omega}\mathbf{y}(\mathbf{v}) = \delta_{js}\mathbf{g}^{j}_{\mathbf{k}\omega}\mathbf{y}(\mathbf{v}) + \frac{4\pi e_s}{\omega}\left\{\boldsymbol{\mu}^{j}_{\mathbf{k}\omega}\left(1-\frac{\mathbf{kv}}{\omega}\right)+\frac{\mathbf{k}}{\omega}(\boldsymbol{\mu}^{j}_{\mathbf{k}\omega}\mathbf{v})\right\}$$
$$\times A(\mathbf{k},\omega)\int \mathbf{v}\,(\mathbf{g}^{s}_{\mathbf{k}\omega}\mathbf{y}(\mathbf{v}))\,d\mathbf{v}, \tag{25}$$

where the matrix A is defined by $A = \| \epsilon_{\alpha\beta} + \frac{c^2}{\omega^2} k_\alpha k_\beta - \frac{c^2k^2}{\omega^2}\delta_{\alpha\beta} \|^{-1}$, $\epsilon_{\alpha\beta}$ is the dielectric tensor

$$\varepsilon_{\alpha\beta}(\mathbf{k},\omega) = \delta_{\alpha\beta} + \frac{4\pi}{\omega}\sum_j\int e_j v_\alpha \mu^{j}_{\mathbf{k}\omega}\left\{\mathbf{e}_\beta\left(1-\frac{\mathbf{kv}}{\omega}\right)+\frac{\mathbf{k}}{\omega}v_\beta\right\}d\mathbf{v}, \tag{26}$$

and $\mathbf{e}_\beta$ is a unit vector along the β axis (β = x, y, z).

In finding A we again must add an imaginary part to ϵ to take account of the damping of the triple correlation function.

Using the relation (24) between F_1 and $\mathbf{E}_1$ and taking account of Maxwell's equations (15) and (16) it follows that the motion of a weakly turbulent plasma can be represented in the form of an ensemble of waves with electric field given by

$$\sum_\beta\left(k^2\delta_{\alpha\beta} - \frac{\omega^2}{c^2}\tilde{\varepsilon}_{\alpha\beta} - k_\alpha k_\beta\right)E_\beta(\mathbf{k},\omega) = 0, \tag{27}$$

where $\tilde{\epsilon}_{\alpha\beta}$ is the dielectric tensor of the plasma with oscillations taken into account:

$$\tilde{\varepsilon}_{\alpha\beta}(\mathbf{k},\omega) = \delta_{\alpha\beta} + \frac{4\pi}{\omega}\sum_j\int e_j v_\alpha \tilde{\boldsymbol{\mu}}_{\mathbf{k}\omega}\left\{\mathbf{e}_\beta\left(1-\frac{\mathbf{kv}}{\omega}\right)+\frac{\mathbf{k}}{\omega}v_\beta\right\}d\mathbf{v}\,. \tag{28}$$

For a given $\mathbf{k}$ it follows from (27) that modes can only be excited at one of the characteristic frequencies $\tilde{\omega}^i_{\mathbf{k}}$ that satisfy the dispersion equation

$$\det\| k^2\delta_{\alpha\beta} - \omega^2 c^{-2}\tilde{\varepsilon}_{\alpha\beta} - k_\alpha k_\beta \| = 0, \tag{29}$$

with each characteristic frequency corresponding to a completely determined polarization vector, $\mathbf{a}^i(\mathbf{k})$, which is assumed to be real in the transparent region being considered.

Thus, the correlation function $\langle E_\alpha(\mathbf{k},\omega)\,E^*_\beta(\mathbf{k}',\omega')\rangle$ in a stationary uniform plasma can be written in the form

$$\langle E_\alpha(\mathbf{k},\omega)\,E^*_\beta(\mathbf{k}',\omega')\rangle = \sum_i a^i_\alpha a^i_\beta I_i(\mathbf{k})\,\delta(\mathbf{k}-\mathbf{k}')\,\delta(\omega-\omega')\,\delta(\omega-\tilde{\omega}^i_{\mathbf{k}}), \tag{30}$$

where the summation is taken over all characteristic frequencies associated with a given wave vector $\mathbf{k}$.

Absorption at the walls can have a strong effect on the noise amplitude in a transparent region so that it is desirable to take account of any small inhomogeneity of $I(\mathbf{k})$ in space as well as any weak time dependence. The pair correlation function in weakly nonstationary inhomogeneous motion of a plasma differs from (30) by a small "smearing" of the δ-function and, more important, by the addition of small imaginary corrections proportional to the time and space derivatives of $I(\mathbf{k})$. These corrections can obviously be neglected in the expression for $\tilde{\epsilon}$.

We now multiply (15) by $E^*_{\mathbf{k}'\omega'}$, (16) by $H^*_{\mathbf{k}'\omega'}$ and subtract one from the other; from the resulting expression we subtract the analogous complex conjugate relation, interchanging $\mathbf{k}$, $\mathbf{k}'$ and ω, ω'. The final expression is averaged over the statistical ensemble. Taking account of (24) and (28) and the fact that the derivatives of the correlation function $\langle E_\alpha(\mathbf{k},\omega)\,E^*_\beta(\mathbf{k}',\omega')\rangle$ are $\partial/\partial t = i(\omega-\omega')$ and $\nabla = i(\mathbf{k}-\mathbf{k}')$ we obtain an expression for the energy balance

$$\partial W/\partial t + \operatorname{div}\mathbf{S} = -(\omega/4\pi)\operatorname{Im}(\mathbf{a}\tilde{\varepsilon}\mathbf{a})\,I(\mathbf{k}), \tag{31}$$

where the wave energy W is

$$W = \frac{1}{8\pi}\left\{\frac{\partial}{\partial\omega}\langle\omega\mathbf{E}^*\varepsilon\mathbf{E}\rangle + \langle\mathbf{H}^*\mathbf{H}\rangle\right\}$$
$$= \frac{I(\mathbf{k})}{8\pi}\left\{\frac{\partial}{\partial\omega}(\omega\mathbf{a}\tilde{\varepsilon}\mathbf{a}) + \frac{c^2k^2}{\omega^2} - \frac{c^2}{\omega^2}(\mathbf{ka})^2\right\}, \tag{32}$$

while the energy flux $\mathbf{S}$ is given by the relation

$$\mathbf{S} = \left\{\frac{c^2}{\omega}\mathbf{k} - \frac{c^2}{\omega}\mathbf{a}(\mathbf{ka}) - \frac{\partial}{\partial\mathbf{k}}(\omega\mathbf{a}\tilde{\varepsilon}\mathbf{a})\right\}\frac{I(\mathbf{k})}{8\pi}. \tag{33}$$

In the expressions for W and $\mathbf{S}$, $\tilde{\epsilon}$ corresponds to the hermitian part of the dielectric tensor so that the scalar quantity $(\mathbf{a}\cdot\tilde{\epsilon}\mathbf{a})$ can be taken as real while the differentiation with respect to ω and $\mathbf{k}$ is carried out for fixed $\mathbf{a}$.

Substituting in the vector equation (27) $\mathbf{E} = \mathbf{a}E$ and multiplying by $\mathbf{a}$, we have

$$\tilde{D} \equiv \omega^2 c^{-2}(\mathbf{a}\tilde{\varepsilon}\mathbf{a}) + (\mathbf{ka})^2 - k^2 = 0, \tag{34}$$

whence, under the assumption that the antihermitian part of the dielectric tensor is small, we find the growth rate

$$\tilde{\gamma} = -\operatorname{Im}(\mathbf{a}\tilde{\varepsilon}\mathbf{a})\left\{\frac{\partial}{\partial\omega}(\omega\mathbf{a}\tilde{\varepsilon}\mathbf{a}) + \frac{c^2}{\omega^2}k^2 - \frac{c^2}{\omega^2}(\mathbf{ka})^2\right\}^{-1}. \tag{35}$$

Assuming that the energy flux can be written in the form $\mathbf{S} = \mathbf{U}I(\mathbf{k})$, where $\mathbf{U} = -(\partial\tilde{D}/\partial\mathbf{k})(\partial\tilde{D}/\partial\omega)^{-1}$ is the group velocity of the wave, we rewrite (31) in the simpler form:

$$\partial I(\mathbf{k})/\partial t + \mathbf{U}(\mathbf{k})\nabla I(\mathbf{k}) = 2\tilde{\gamma}I(\mathbf{k}). \tag{36}$$

An equation of this type, which plays the role of a wave energy transport equation, should actually be written for each of the characteristic frequencies associated with a given wave vector. Similarly, we could obtain a momentum balance equation from (15) and (16). But this equation is not necessary since the self momentum of the electromagnetic field $(\mathbf{E} \times \mathbf{H})/4\pi c$ is negligibly small compared with the particle momentum (if the plasma density is not low) while the particle momentum can be expressed much more simply in terms of the average distribution function.

Equations (22) and (36) together with the equation for the averaged distribution function represent a closed system of equations that describes a turbulent plasma. It is evident from (36) that a stationary state (on the average) is achieved in an infinite uniform plasma when $\tilde{\gamma} = 0$. A peculiar dynamic equilibrium must obtain under these conditions: γ computed in the linear approximation for a given f_0^j can be nonvanishing for various values of $\mathbf{k}$, corresponding to growth (or damping) of waves by virtue of the acquisition (or loss) of energy from the fundamental (scale-size) turbulence. But in this case each mode exchanges energy with other modes in such a way that its energy is not changed on the average.

We have taken account of only the first nonvanishing mode interaction. The associated processes are the decay of a wave $\mathbf{k}$ into two waves $\mathbf{k}'$ and $\mathbf{k} - \mathbf{k}'$ and the combination of waves $\mathbf{k}'$ and $\mathbf{k} - \mathbf{k}'$ into a wave $\mathbf{k}$. If the noise level is small, in which case the difference between ϵ and $\tilde{\epsilon}$ can be neglected, these equations become the equations of the quasi-linear approximation. [3,4]

In concluding this section we note that the tensor $\tilde{\epsilon}$ introduced above is a real dielectric tensor for a weakly turbulent plasma for arbitrary small oscillations. Any additional wave introduced into the plasma from an external source can be treated as a correction to $I(\mathbf{k})$. The electric field of this wave propagating in the plasma must also satisfy (27); if its intensity is small its contribution in (22) can be neglected and the corresponding dispersion equation is again given by $\tilde{\epsilon}$.

4. EFFECT OF OSCILLATIONS ON THE AVERAGED DISTRIBUTION FUNCTION

We now consider the wave interaction term S_j in (1), which gives the averaged distribution function. Taking account of (24) and the fact that the vector $\mathbf{b}$ can be regarded as real in a weakly nonequilibrium plasma we have

$$S_j = -\frac{e_j}{m_j}\frac{\partial}{\partial \mathbf{v}}\left\langle F^j\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right)\right\rangle$$
$$= -\frac{i}{2}\frac{e_j}{m_j}\frac{\partial}{\partial \mathbf{v}}\int \{\mathbf{b}(\mathbf{k}',\omega')(\tilde{\mu}^j_{\mathbf{k}\omega}\mathbf{b}(\mathbf{k},\omega))$$
$$- \mathbf{b}(\mathbf{k},\omega)(\tilde{\mu}^j_{\mathbf{k}'\omega'}\mathbf{b}(\mathbf{k}',\omega'))^*\}$$
$$\times E(\mathbf{k},\omega)E^*(\mathbf{k}',\omega')\,d\mathbf{k}\,d\mathbf{k}'\,d\omega\,d\omega'$$
$$= -\frac{e_j}{m_j}\frac{\partial}{\partial \mathbf{v}}\int \Big\{-\operatorname{Im}\mathbf{b}(\tilde{\mu}^j_{\mathbf{k}\omega}\mathbf{b})I(\mathbf{k})$$
$$-\frac{1}{2}\operatorname{Re}\left[\frac{\partial I(\mathbf{k})}{\partial t}\frac{\partial}{\partial \omega} - \left(\nabla I(\mathbf{k})\frac{\partial}{\partial \mathbf{k}}\right)\right]\mathbf{b}(\tilde{\mu}^j_{\mathbf{k}\omega}\mathbf{b})$$
$$+\operatorname{Re}(\tilde{\mu}^j_{\mathbf{k}\omega}\mathbf{b})\left[\frac{\partial I(k)}{\partial t}\frac{\partial \mathbf{b}}{\partial \omega} - \left(\nabla I(\mathbf{k})\frac{\partial}{\partial \mathbf{k}}\right)\mathbf{b}\right]\Big\}\,d\mathbf{k}, \qquad (37)$$

where $\omega = \tilde{\omega}_{\mathbf{k}}$.

In the case of arbitrary oscillations we have $\mathbf{b} = \mathbf{k}/k$ and this relation assumes the simpler form:[1)]

$$S_j = \frac{e_j}{m_j}\frac{\partial}{\partial \mathbf{v}}\int \Big\{\frac{\mathbf{k}}{k}\operatorname{Im}\tilde{\mu}^j_{\mathbf{k}\omega}I(\mathbf{k}) + \frac{1}{2k}\nabla I(\mathbf{k})\operatorname{Re}\tilde{\mu}^j_{\mathbf{k}\omega}$$
$$+\frac{1}{2}\frac{\mathbf{k}}{k}\operatorname{Re}\left[\frac{\partial I(\mathbf{k})}{\partial t}\frac{\partial \tilde{\mu}^j_{\mathbf{k}\omega}}{\partial \omega} - \nabla I(\mathbf{k})\frac{\partial \tilde{\mu}^j_{\mathbf{k}\omega}}{\partial \mathbf{k}} - \frac{\mathbf{k}}{k}\tilde{\mu}^j_{\mathbf{k}\omega}\nabla I(\mathbf{k})\right]\Big\}\,d\mathbf{k}, \qquad (38)$$

where $\tilde{\mu}^j_{\mathbf{k}\omega} = (\mathbf{k}\cdot\tilde{\boldsymbol{\mu}}^j_{\mathbf{k}\omega})/k$.

Formally Eq. (1) is in the form of the Boltzmann equation with a collision integral S_j; actually, however, it is much more complicated.

The analysis given above does not contain any fundamental difficulties for generalization to the case of an inhomogeneous plasma. In a weakly inhomogeneous plasma (scale size of variations in temperature and density appreciably greater than the wavelength) we can again expand the field in a Fourier integral in the sense of the quasi-classical approximation. If the inhomogeneity is introduced in the linear approximation the frequency in the expression for ϵ and in the dispersion equation contains corrections proportional to the gradient of the averaged distribution functions. These corrections are very important for drift (convective) waves, which are specifically characteristic of an inhomogeneous plasma, but can evidently be neglected for the usual waves that propagate in a uniform plasma. Furthermore, when the inhomogeneity is introduced the expression for S_j contains additional terms that are proportional to the gradient of f_0^j; to some degree, these

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1)In the quasi-linear approximation for a uniform plasma this expression differs from the corresponding relations in [3] and [4] by the presence of the $\partial I/\partial t$ term; this difference arises because the function f_0^j is defined somewhat differently. In our case f_0^j is simply the mean value of f_j; for example, the wave momentum is taken into account in f_0^j.

terms reflect the nonlocal nature of the wave-particle interaction. Thus, in the general case S_j is an operator that operates on both **v** and **r**. Hence the solution of the kinetic equation involves considerable difficulties and specific approximations of its solution must evidently be chosen to fit a particular problem.

Nevertheless, some general qualitative conclusions can be drawn on the basis of the form of the kinetic equation. In a strong magnetic field [the characteristic time $\widetilde{T}$, in which the wave interaction causes an appreciable change in the averaged particle distribution function, is appreciably greater than the cyclotron frequency $\Omega_j = e_j H/m_j c$] Eq. (1) can be expanded in reciprocal powers of Ω_j. This means, that in describing the averaged motion of particles across the magnetic field we can use equations of the hydrodynamic type; these are obtained from (1) by multiplying by **v** and integrating over velocity. In this case the right side of the equation of motion for a particle of type j will contain a force $\mathbf{R}_j = \int m_j \mathbf{v} S_j d\mathbf{v}$ due to the interaction with the waves. In a two-component plasma, consisting of electrons and ions of one kind, $\mathbf{R}_j$ can be written in the form $\mathbf{R}_j = \mathbf{R}_0/2 \pm \mathbf{R}_{ie}$, where $\mathbf{R}_0$ is the total force acting on a plasma while $\mathbf{R}_{ie}$ is the frictional force exerted on the ions by the electrons by virtue of the waves. The force $\mathbf{R}_0$ is obviously given by $\mathbf{R}_0 = \langle \rho \mathbf{E} + c^{-1}(\mathbf{j} \times \mathbf{H}) \rangle$ where ρ is the charge density and **j** is the current density in the waves. Using Maxwell's equations and neglecting the self momentum of the field this quantity is reduced to the divergence of the Maxwell stress tensor.

As far as the friction force $\mathbf{R}_{ie}$ is concerned we see that its component along the density gradient can easily be balanced by the electric field produced by a small shift of the electrons with respect to the ions and is not important; the component across the density gradient leads to plasma diffusion. Thus, the anomalous diffusion of a weakly turbulent plasma, is caused by the electron-ion "friction," very much in the same way as is the case of collisions in an ordinary plasma. This fact is very important and must always be kept in mind in investigating the effect of any form of oscillation on plasma diffusion.

As an example we consider the particular case in which the gradient I can be neglected in the expression for S_j [and consequently in (36)]. Evidently $\mathbf{R}_0 = 0$ so that the diffusion effect above remains in the equations of motion of the oscillations. To simplify the calculations we limit ourselves to the case of longitudinal oscillations.

From the equation of continuity we have

$$\int (\omega - \mathbf{kv})\, F^j(\mathbf{k}, \omega, \mathbf{v})\, d\mathbf{v} = i \int (\omega - \mathbf{kv})\, \widetilde{\mu}^j_{\mathbf{k}\omega} \mathbf{b}_{\mathbf{k}\omega} E(\mathbf{k},\omega)\, d\mathbf{v} = 0, \tag{39}$$

so that from (38)

$$\mathbf{R}_j = -\frac{1}{8\pi}\int \mathbf{k} \left\{ 2\, \mathrm{Im}\, \widetilde{\varepsilon}_j I(\mathbf{k}) + \frac{\partial I(\mathbf{k})}{\partial t} \mathrm{Re} \frac{\partial \widetilde{\varepsilon}_j}{\partial \omega} \right\} d\mathbf{k}, \tag{40}$$

where

$$\widetilde{\varepsilon}_j = \frac{4\pi}{k} \int e_j \widetilde{\mu}^j_{\mathbf{k}\omega} d\mathbf{v},$$

whence $\widetilde{\epsilon} = 1 + \sum_j \widetilde{\epsilon}_j$. Substituting everywhere in accordance with (36),

$$\frac{\partial I}{\partial t} = 2\widetilde{\gamma} I = -2\, \mathrm{Im}\, \widetilde{\varepsilon}\, \mathrm{Re}\, (\partial \widetilde{\varepsilon}/\partial \omega)^{-1} I,$$

we find for a two-component plasma

$$\mathbf{R}_{ie} = -\mathbf{R}_{ei} = \frac{1}{4\pi}\int \mathbf{k} \left\{ \mathrm{Im}\, \widetilde{\varepsilon}_e \mathrm{Re} \frac{\partial \widetilde{\varepsilon}_i}{\partial \omega} - \mathrm{Im}\, \widetilde{\varepsilon}_i \mathrm{Re} \frac{\partial \widetilde{\varepsilon}_e}{\partial \omega} \right\} \times \mathrm{Re} \left(\frac{\partial \widetilde{\varepsilon}}{\partial \omega} \right)^{-1} I(\mathbf{k})\, d\mathbf{k}. \tag{41}$$

It is evident that these oscillations cause plasma diffusion only if both electrons and ions are involved. For example, if we have high frequency oscillations only (ions assumed fixed so that $\widetilde{\epsilon}_i = 0$) the friction force $\mathbf{R}_{ie}$ and the diffusion flux both vanish. In other words, these oscillations correspond to electron-electron collisions only and can not cause plasma diffusion; on the other hand individual electrons, can be easily shown to diffuse by virtue of these oscillations. [9,10] This example given shows that it is hazardous to extend to a total plasma conclusions that refer only to a single particle, neglecting the correlation of its motion with the motion of the other particles.

5. CONCLUSION

In this paper we derive equations that describe the behavior of a weakly turbulent plasma in a magnetic field, taking account of the nonlinear decay of waves into two waves and the fusion of two waves into one, i.e., the strongest interaction between low-amplitude waves. It is shown that the turbulent motion of such a plasma can be described in terms of an ensemble of waves; the wave dispersion equation in turn depends on the wave spectrum.

The behavior of the averaged distribution function is described by a special collision integral. This collision integral takes account of the finite distance over which the particles interact with waves (this distance is of the order of the mean Larmor radius of the particles) and is an operator that acts on both the velocity and coordinate variables.

To simplify matters we have neglected the thermal Coulomb noise and collisions between particles. In principle, by considering the fluctuation of the distribution function due to the discreteness of the medium it is easy to include thermal noise in the analysis and also to introduce collisions.

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