THE THEORY OF A FERMI LIQUID CONSISTING OF TWO KINDS OF PARTICLES. APPLICATION TO THE NUCLEUS

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We express the rigidity of a system with respect to changes in density and concentration in terms of the scattering amplitude in a medium of identical or different particles; this allows us to find these amplitudes in a nucleus from the observed values of the rigidity. We have obtained expressions which connect the vertex part and the polarization operator with the scattering amplitudes at the Fermi surface. We have shown that the momenta at the Fermi boundaries for each kind of particle are connected with the densities of these particles through the formula for the density of a perfect Fermi gas. We give an estimate for the effective mass of the quasi-particles in a nucleus.

1. INTRODUCTION

THE only way to obtain quantitative relations between different quantities in the many-body problem when the interparticle interactions are not small is to use a phenomenological approach, i.e., to introduce experimentally determined constants into the theory. A similar situation occurs in the theory of strong interactions between elementary particles when the observed values of masses and charges are introduced into the dispersion relations.

Landau [1,2] has shown, using a uniform system consisting of one type of strongly-interacting fermions as an example, that one can evaluate all important properties of the system if the forward scattering amplitude of the quasi-particles at the Fermi surface is given. This amplitude depends solely on the angle between the directions of the momenta of the particles and is practically determined by the first two or three terms in the expansion in Legendre polynomials. The coefficients of the Legendre polynomials are phenomenological constants which must be determined from a comparison of the theory with experiment.

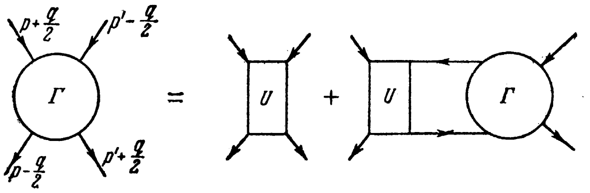
To use such a program for the nucleus one needs first to extend the theory to the case of systems consisting of two kinds of particles. After that one must take into account the finite dimensions of the system and the influence of pair correlations. Some quantities, however, change little when the finite dimensions and the pair correlations are taken into account. Among such quantities we have the rigidity coefficients with respect to a change in the density or the concentration [determined in Sec. 7, Eq. (51)] and the quasi-particle effective mass.

We show in Sec. 7 that these coefficients are expressed in terms of the spherical harmonic of the above-mentioned expansion of the forward neutron-neutron and neutron-proton scattering amplitudes in terms of Legendre polynomials; this allows us to determine these harmonics from the observed values of the rigidity (Sec. 7A).

The effective mass is expressed in terms of the sum of the first harmonics of these two amplitudes (see Secs. 6 and 8B). The interaction with an external field (vertex part) is, as in the case of a system consisting of one kind of particles, expressed in terms of the forward scattering amplitude at the Fermi surface of two identical and two different particles (Sec. 3C). The same is also true of the polarization operator which determines the change in the quasi-particle distribution function under the influence of an external field (Sec. 4). As in the case when there is only one kind of particle [3,4] the momenta at the neutron and proton Fermi boundaries are connected with the average densities of these particles through the formula for the perfect Fermi-gas density (Sec. 5).

2. EQUATION FOR THE SCATTERING AMPLITUDE

A. Derivation of the Landau equation for the amplitude. The scattering amplitude satisfies the equation



(1)

FIG. 1

which in analytical form is

$$\Gamma\,(p, p';\, q) = U\,(p, p';\, q) \;-\; i\int U\,(p,\, p_1;\, q)$$

$$\times\, G\left(p_1 + \tfrac{q}{2}\right) G\left(p_1 - \tfrac{q}{2}\right) \Gamma\,(p_1,\, p';\, q)\,\frac{d^4 p_1}{(2\pi)^4}, \qquad (1)$$

where U indicates all diagrams which do not contain parts joining two horizontal lines. The normalization chosen by us is defined by the fact that when the interaction tends to zero U changes to V_q, the Fourier component of the interaction potential. Every line corresponds to the exact Green's function defined by the expression $G = -i\langle T\psi(x_1)\,\psi^+(x_2)\rangle$

We can split off from the Green function in the momentum representation G(p) a pole term

$$G\,(p) = \frac{a\,(\mathbf{p})}{\varepsilon - \varepsilon_{\mathbf{p}} - i\varkappa} + G_R\,(p). \qquad (2)$$

Here ϵ is the fourth component of p reckoned from the chemical potential μ; ϵ_p the quasi-particle energy, also reckoned from μ; κ the quasi-particle damping coefficient ($\kappa \to \alpha\epsilon|\epsilon|$ as $\epsilon \to 0$); $a(\mathbf{p})$ the renormalization of the Green function:

$$a^{-1}\,(\mathbf{p}) = (\partial G^{-1} / \partial \varepsilon)_{\varepsilon = \varepsilon_{\mathbf{p}}}, \qquad a\,(p_0) \equiv a.$$

As $q \to 0$ the poles of the two Green's functions in the integral of Eq. (1) approach one another and there occurs a maximum in the integrand for $\epsilon_1 = \epsilon_{p_1}$; therefore

$$G\left(p + \tfrac{q}{2}\right) G\left(p - \tfrac{q}{2}\right)$$

$$\approx a^2\,(\mathbf{p})\;\delta\,(\varepsilon - \varepsilon_{\mathbf{p}}) \int G_0\left(p + \tfrac{q}{2}\right) G_0\left(p - \tfrac{q}{2}\right) d\varepsilon + B\,(p,\, q),$$

$$G_0\,(p) = [\varepsilon - \varepsilon_{\mathbf{p}} - i\alpha\varepsilon_{\mathbf{p}}\,|\,\varepsilon\,|]^{-1}.$$

Evaluating the integral we get

$$\int G_0\left(p + \tfrac{q}{2}\right) G_0\left(p - \tfrac{q}{2}\right) d\varepsilon = 2\pi i\,\frac{n_0\,(\mathbf{p} + \mathbf{k}/2) - n_0\,(\mathbf{p} - \mathbf{k}/2)}{\omega - \varepsilon_{\mathbf{p}} + \mathbf{k}/2 + \varepsilon_{\mathbf{p}} - \mathbf{k}/2};$$

$$n_0(p) = \begin{cases} 1, & |\,\mathbf{p}\,| < p_0 \\ 0, & |\,\mathbf{p}\,| > p_0, \end{cases} \qquad (3)$$

where p_0 is the momentum on the Fermi boundary; **k**, ω are the spatial and time components of the vector q. From the last expression we get easily

$$G\left(p + \tfrac{q}{2}\right) G\left(p - \tfrac{q}{2}\right) = i a^2 \delta\,(\varepsilon) \cdot 2\pi\,\frac{\mathbf{kv}}{\omega - \mathbf{kv}}\,\frac{\delta\,(|\,\mathbf{p}\,| - p_0)}{p_0}\,m^*$$

$$+\, B\,(p,\, q) = A + B, \qquad (4)$$

where the function B(p, q) does not contain a δ-function dependence on p and up to terms $\sim k^2/p_0^2$, we may assume ω^2/μ^2 to be independent of q.

Turning now to the scattering amplitude we write Eq. (1) symbolically

$$\Gamma = U + UGG\Gamma = U + \Gamma GGU. \qquad (5)$$

The last equation is immediately obtained from the diagrammatic Eq. (1) if we read it from right to left.

We introduce the amplitude Γ^ω as the limit $\Gamma_{kv/\omega \to 0} \to \Gamma^\omega$. It follows from (4) that $A_{kv/\omega \to 0} \to A^\omega = 0$. Therefore (5) gives

$$\Gamma^\omega = U + UB\Gamma^\omega = U + \Gamma^\omega BU. \qquad (6)$$

Multiplying (5) from the left by $1 + \Gamma^\omega B$ we get

$$\Gamma = \Gamma^\omega + \Gamma^\omega A \Gamma = \Gamma^\omega + \Gamma A \Gamma^\omega. \qquad (7)$$

We get the last equation by multiplying (5) from the right by

$$1 + \Gamma^\omega B.$$

We can get an equation analogous to (7) by introducing the amplitude Γ^k defined by the relation $\Gamma_{\omega/kv \to 0} \to \Gamma^k$. It follows from (7) that

$$\Gamma^k = \Gamma^\omega + \Gamma^\omega A^k \Gamma^k = \Gamma^\omega + \Gamma^k A^k \Gamma^\omega. \qquad (8)$$

Multiplying (7) from the left by $1 + \Gamma^k A^k$ we get

$$\Gamma = \Gamma^k + \Gamma^k A' \Gamma = \Gamma^k + \Gamma A' \Gamma^k,$$

$$A' = A - A^k. \qquad (9)$$

Equations (7) and (9) enable us to find the scattering amplitude on the Fermi surface ($|\mathbf{p}| = |\mathbf{p}'| = p_0$; $\epsilon = \epsilon' = 0$) as a function of **k** and ω, if the functions Γ^ω or Γ^k which depend only on the angle between **p** and **p′** are known.

B. Equation for the amplitude when there are two kinds of particles. One sees easily from Eq. (1) that these results also remain valid when there are two kinds of particles provided we understand by Γ, U, Γ^ω, and Γ^k two-by-two matrices in isotopic space.

Equation (7) thus becomes

$$\Gamma_{aa} = \Gamma_{aa}^\omega + \Gamma_{aa}^\omega A_a \Gamma_{aa} + \Gamma_{ab}^\omega A_b \Gamma_{ba},$$

$$\Gamma_{ba} = \Gamma_{ba}^\omega + \Gamma_{ba}^\omega A_a \Gamma_{aa} + \Gamma_{bb}^\omega A_b \Gamma_{ba}, \qquad (7')$$

where Γ_{aa}, Γ_{aa}^ω are the scattering amplitudes for identical particles; and Γ_{ab}, Γ_{ab}^ω the amplitudes for scattering of particle a by particle b.

We introduce the dimensionless amplitudes

$$F_{aa} = \pi^{-2}\,(p_0 m^* a^2)_a\,\Gamma_{aa}, \qquad F_{bb} = \pi^{-2}\,(p_0 m^* a^2)_b \Gamma_{bb},$$

$$F_{ab} = F_{ba} = \pi^2\,[(p_0 m^* a^2)_a\,(p_0 m^* a^2)_b]^{1/2}\,\Gamma_{ab}. \qquad (10)$$

We get from (4), (7), and (9)

$$F(\mathbf{n}, \mathbf{n}'; q) = F^{\omega}(\mathbf{n}, \mathbf{n}') + \frac{1}{2}\int F^{\omega}(\mathbf{n}, \mathbf{n}_1)\frac{\mathbf{n}_1\mathbf{k}v}{\omega - v\mathbf{k}\mathbf{n}_1}F(\mathbf{n}_1, \mathbf{n}'; q)\frac{do_1}{4\pi},$$

$$F(\mathbf{n}, \mathbf{n}'; q) = F^{k}(\mathbf{n}, \mathbf{n}') + \frac{1}{2}\int F^{k}(\mathbf{n}, \mathbf{n}_1)\frac{\omega}{\omega - v\mathbf{k}\mathbf{n}_1}F(\mathbf{n}_1, \mathbf{n}'; q)\frac{do_1}{4\pi}, \quad (11)$$

where $q = \omega, \mathbf{k}$; $\mathbf{n}$ and $\mathbf{n}'$ are unit vectors along $\mathbf{p}$ and $\mathbf{p}'$, and we must understand by F, F^{ω}, F^{k}, and v the matrices

$$F = \begin{vmatrix} F_{aa}, & F_{ab} \\ F_{ba}, & F_{bb} \end{vmatrix}, \qquad v = \begin{vmatrix} v_a & 0 \\ 0 & v_b \end{vmatrix},$$

where v_a and v_b are the velocities at the Fermi surfaces of the particles a and b respectively.

We obtain a connection between F^k and F^{ω} by going in the first of Eqs. (11) to the limit $\omega/kv \to 0$:

$$F^{k}(\mathbf{n}, \mathbf{n}'; q) = F^{\omega}(\mathbf{n}, \mathbf{n}') - \frac{1}{2}\int F^{\omega}(\mathbf{n}, \mathbf{n}_1)F^{k}(\mathbf{n}_1, \mathbf{n}')\frac{do_1}{4\pi}. \quad (12)$$

C. Dependence on spin variables. Diagonalization of the equation for the amplitude. In the expressions given so far we have omitted the spin indices. In equations (11) and (12) one must besides integrating over the direction of the momentum $\mathbf{p}_1$ of one of the particles also sum over the spin variable of that particle. If we consider the scattering amplitude as an operator in the spin variables, we must take in (11) and (12) the trace over the spin operators acting upon the particle over whose momentum we integrate.

We assume that the dependence of the scattering amplitude on the spin operators has an exchange character:

$$F(\mathbf{n}, \sigma; n', \sigma') = f(\mathbf{n}, \mathbf{n}') + g(\mathbf{n}, \mathbf{n}')\sigma\sigma'. \quad (13)$$

Substituting (13) into (11) and using the equation

$$\frac{1}{2}\mathrm{Sp}_{\sigma_1}[f^{\omega}(\mathbf{n}, \mathbf{n}_1) + g^{\omega}(\mathbf{n}, \mathbf{n}_1)\sigma\sigma_1][f(\mathbf{n}_1, \mathbf{n}') + g(\mathbf{n}_1, \mathbf{n}')\sigma_1\sigma'] = f^{\omega}(\mathbf{n}, \mathbf{n}_1)f^{\omega}(\mathbf{n}_1, \mathbf{n}') + g^{\omega}(\mathbf{n}, \mathbf{n}_1)g^{\omega}(\mathbf{n}_1, \mathbf{n}')\sigma\sigma',$$

we get independent equations for f and g:

$$f = f^{\omega} + \int f^{\omega}\frac{v\mathbf{k}\mathbf{n}_1}{\omega - v\mathbf{k}\mathbf{n}_1}f\frac{do_1}{4\pi},$$

$$g = g^{\omega} + \int g^{\omega}\frac{v\mathbf{k}\mathbf{n}_1}{\omega - v\mathbf{k}\mathbf{n}_1}g\frac{do_1}{4\pi}, \quad (11')$$

which differ from (11) only by a factor 2 in the second term.

Equation (12) leads to

$$f^{k} = f^{\omega} - \int f^{\omega}f^{k}\frac{do_1}{4\pi}, \qquad g^{k} = g^{\omega} - \int g^{\omega}g^{k}\frac{do_1}{4\pi}. \quad (12')$$

One can easily diagonalize Eqs. (11′) and (12′) if we assume that $f_{aa} = f_{bb}$; $v_a = v_b = v$. When applied to the nucleus this means that we neglect the difference between the neutron and the proton velocities on the Fermi surface.

We write

$$\chi = f_{aa} + f_{ab}, \qquad \eta = f_{aa} - f_{ab}.$$

One sees easily that we get for χ and η the independent equations

$$\chi = \chi^{\omega} + \int \chi^{\omega}\frac{v\mathbf{k}\mathbf{n}_1}{\omega - v\mathbf{k}\mathbf{n}_1}\chi\frac{do_1}{4\pi},$$

$$\chi = \chi^{k} + \int \chi^{k}\frac{\omega}{\omega - v\mathbf{k}\mathbf{n}_1}\chi\frac{do_1}{4\pi}. \quad (14)$$

The same equations we get for the quantity η.

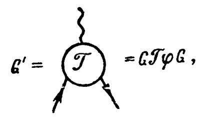
Equations (12′) give

$$\chi_l^{k} = \chi_l^{\omega} - \frac{\chi_l^{\omega}\chi_l^{k}}{2l+1}, \qquad \eta_l^{k} = \eta_l^{\omega} - \frac{\eta_l^{\omega}\eta_l^{k}}{2l+1}, \quad (15)$$

where χ_l and η_l are the coefficients of the expansions of χ and η in a series in Legendre polynomials. We get the same equations also for the quantities $\varphi = g_{aa} + g_{ab}$, $\psi = g_{aa} - g_{ab}$.

3. A SYSTEM IN AN EXTERNAL FIELD

A. Equation for the vertex. The change in the Green's function when an external field is included is determined by the equation



(16)

FIG. 2

where the vertex part $\mathcal{T}$ is determined by all diagrams connecting incoming lines. We can obtain for $\mathcal{T}$ the equation

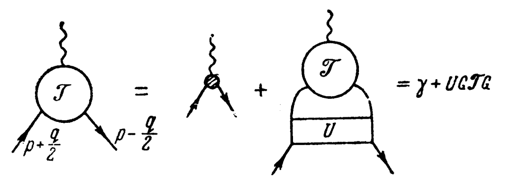


FIG. 3

Here U is the irreducible four-pole introduced in the foregoing, γ the vertex of free particles with respect to the field φ. This equation can easily be derived from considering perturbation theory diagrams by analogy with what one does when deriving the Bethe-Salpeter equation.

The equation for the vertex can also be written in another form:

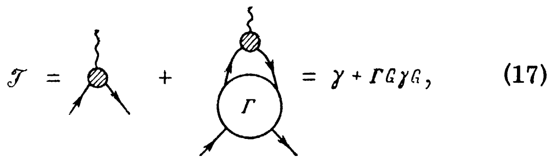


FIG. 4

where Γ is the reducible four-pole which we earlier called the scattering amplitude.

The diagrammatic equations (16) and (17) are in the usual notation of the form

$$G'\left(p+\frac{q}{2}, p-\frac{q}{2}\right) = G\left(p+\frac{q}{2}\right)\mathcal{T}(p; q)\,\varphi(q)\,G\left(p-\frac{q}{2}\right), \qquad (16')$$

$$\mathcal{T}(p, q) = \gamma - i\int \Gamma(p, p'; q)\,G\left(p'+\frac{q}{2}\right)\gamma G\left(p'-\frac{q}{2}\right)\frac{d^4p'}{(2\pi)^4}. \qquad (17')$$

These equations remain the same also in the case of a system with two kinds of fermions provided we understand by $\mathcal{T}$, γ, Γ, and G two-by-two matrices acting in isotopic space. $\mathcal{T}$, γ, and G are then diagonal matrices but Γ has also off-diagonal elements [see (7′)].

In the case of a vector field we get by analogy with (17)

$$\mathcal{T}_\alpha = p_\alpha\gamma + \Gamma p_\alpha\gamma GG, \qquad \alpha = 1, 2, 3. \qquad (18)$$

B. Consequences of gauge invariance. The Lagrangian density for a system which conserves particles has the following property which one can call gauge invariance. The transformation of the quantum operators

$$\Psi' = e^{if(\mathbf{r},\,t)}\Psi \approx (1 + if)\,\Psi \qquad (19)$$

is the same as the change in the Lagrangian density

$$L' = L + j_\alpha \partial f/\partial x_\alpha + \rho\partial f/\partial t, \qquad (19')$$

where j_α, ρ is the current density operator satisfying the operator identity

$$\partial\rho/\partial t + \partial j_\alpha/\partial x_\alpha = 0.$$

This property of the Lagrangian enables us to determine j_α and ρ in terms of quantum operators.

The change in the Green's function under the transformation (19) can be written in the form

$$G' = -i\,(T\Psi'(x)\,\Psi'^{+}(y)) + i\,(T\Psi(x)\,\Psi^{+}(y))$$
$$= -i\,(T\Psi(x)\,\Psi^{+}(y))\,[e^{if(x)-if(y)} - 1].$$

On the other hand, the change of the Green's function under the transformation (19′) is, according to (16) of the form

$$G' = G\mathcal{T}_i(\partial f/\partial x_i)\,G, \qquad i = 1, 2, 3, 4.$$

Changing to the Fourier representation and comparing the two expressions we get easily

$$\mathcal{T}_i q_i = G^{-1}(p + q/2) - G^{-1}(p - q/2). \qquad (20)$$

This is a result well known in quantum electrodynamics. For small q it follows from (20) that

$$\mathcal{T}_i q_i = q_i \partial G^{-1}/\partial p_i. \qquad (21)$$

In quantum electrodynamics one can conclude from (21) that $\mathcal{T}_i = \partial G^{-1}/\partial p_i$ (Ward identity). In the many-body problem $\mathcal{T}_i$ depends on how q (as a function of ω/kv) tends to zero and the Ward identity is thus valid only for a well-defined way of letting q tend to zero which follows from (21), namely,

$$\mathcal{T}_\alpha|_{\omega/kv\to 0} = -\partial G^{-1}/\partial p_\alpha, \qquad \alpha = 1, 2, 3;$$
$$\mathcal{T}|_{kv/\omega\to 0} = \partial G^{-1}/\partial\varepsilon. \qquad (22)$$

If q tends to zero in another way the Ward identity is not valid. For free particles $G^{-1} = G_0^{-1} = \epsilon - \epsilon_p^0$ and thus $\mathcal{T}_\alpha^0 = p_\alpha$ and $\mathcal{T}^0 = 1$ (the free particle mass is chosen to be equal to unity).

From Eq. (17) and the second of Eqs. (22) we get

$$\mathcal{T}^\omega = 1 + \Gamma^\omega (GG)^\omega = 1 + \Gamma^\omega B = \partial G^{-1}/\partial\varepsilon. \qquad (23)$$

Similarly we have from (18)

$$\mathcal{T}_\alpha^k = p_\alpha + \Gamma_\alpha^k p_\alpha (B + A^k) = -\partial G^{-1}/\partial p_\alpha. \qquad (24)$$

Relations (23) and (24) were obtained by Pitaevskiĭ.[4]

We elucidate now to what consequences the gauge invariance leads for a system of two kinds of particles. The fact that the transformations (19) and (19′) are identical is valid for each kind of particle and we could thus obtain relations, introducing two independent fields f_a and f_b. Instead of that it is more convenient to introduce one field f but to ascribe to each kind of particle an arbitrary charge with respect to that field.

Putting in (17)

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \qquad (25)$$

and comparing anew the changes in G caused by the transformations (19) and (19′) we get

$$\mathcal{T}_a^\omega = (1 + \Gamma_{aa}^\omega B_a) + \Gamma_{ab}^\omega B_b \lambda = \partial G_a^{-1}/\partial\varepsilon,$$
$$\mathcal{T}_b^\omega = (1 + \Gamma_{bb}^\omega B_b)\,\lambda + \Gamma_{ba}^\omega B_a = \lambda\partial G_b^{-1}/\partial\varepsilon.$$

As this equation must be satisfied for all values of λ the following identities hold:

$$1 + \Gamma_{aa}^\omega B_a = \partial G_a^{-1}/\partial\varepsilon, \qquad 1 + \Gamma_{bb}^\omega B_b = \partial G_b^{-1}/\partial\varepsilon,$$
$$\Gamma_{ab}^\omega B_b = \Gamma_{ba}^\omega B_a = 0. \qquad (26)$$

Similarly, we get from (18)

$$p_\alpha + \Gamma^k_{aa} p_\alpha (B_a + A^k_a) = -\partial G_a^{-1}/\partial p_\alpha,$$
$$p_\alpha + \Gamma^k_{bb} p_\alpha (B_b + A^k_b) = -\partial G_b^{-1}/\partial p_\alpha,$$
$$\Gamma^k_{ab} p_\alpha (B_b + A^k_b) = \Gamma^k_{ba} p_\alpha (B_a + A^k_a) = 0. \qquad (27)$$

We can write Eqs. (26) and (27) symbolically

$$\gamma \partial G^{-1}/\partial \varepsilon = \gamma + \Gamma^\omega \gamma B, \qquad (26')$$
$$-\gamma \partial G^{-1}/\partial p_\alpha = \gamma p_\alpha + \Gamma^k \gamma p_\alpha (B + A^k). \qquad (27')$$

C. Connection of the vertex with the scattering amplitude near the Fermi surface. Equations (17) and (18) connect the vertex with integrals containing the scattering amplitude, where the integration is over regions both near and far from the Fermi surface. We shall verify that the integrals over the far away regions reduce by means of Eqs. (26) and (27) to factors $\partial G^{-1}/\partial \epsilon$ and $\partial G^{-1}/\partial p$ which are connected with the renormalization of the Green's function and with the quasi-particle effective mass.[1)]

Using the matrix γ introduced earlier through (25) and Eq. (7) we get easily from (17)

$$\mathcal{T} = \gamma \partial G^{-1}/\partial \varepsilon + \Gamma \gamma A \partial G^{-1}/\partial \varepsilon$$

or

$$\mathcal{T}_a = \partial G_a^{-1}/\partial \varepsilon + \Gamma_{aa} A_a \partial G_a^{-1}/\partial \varepsilon + \lambda \Gamma_{ab} A_b \partial G_b^{-1}/\partial \varepsilon,$$
$$\mathcal{T}_b = \lambda \partial G_b^{-1}/\partial \varepsilon + \lambda \Gamma_{bb} A_b \partial G_b^{-1}/\partial \varepsilon + \Gamma_{ba} A_a \partial G_a^{-1}/\partial \varepsilon. \qquad (28)$$

In (28) the vertex near the Fermi surface ($|\mathbf{p}| = p_0$, $\epsilon = 0$) is expressed for arbitrary values of ω/kv in terms of the amplitudes Γ_{aa} and Γ_{ab} on the Fermi surface.

Similarly one can also obtain an expression for the vector vertex $\mathcal{T}_\alpha$. From (18) we get

$$\mathcal{T}_\alpha = \gamma p_\alpha + \Gamma \gamma p_\alpha (A + B); \quad \mathcal{T}^a_\alpha = p_\alpha + \Gamma_{aa} p_\alpha (A + B)_a + \lambda \Gamma_{ab} p_\alpha (A + B)_b.$$

Using Eqs. (8) and (22) we find easily ($A' = A - A^k$)

$$\mathcal{T}_\alpha = -\gamma \partial G^{-1}/\partial p_\alpha - \Gamma \gamma A' \partial G^{-1}/\partial p_\alpha,$$
$$\mathcal{T}^a_\alpha = -\partial G_a^{-1}/\partial p_\alpha - \Gamma_{aa} A'_a \partial G_a^{-1}/\partial p_\alpha - \lambda \Gamma_{ab} A'_b \partial G_b^{-1}/\partial p_\alpha,$$
$$\mathcal{T}^b_\alpha = -\lambda \partial G_b^{-1}/\partial p_\alpha - \lambda \Gamma_{bb} A'_b \partial G_a/\partial p_\alpha - \Gamma_{ba} A'_a \partial G_a^{-1}/\partial p_\alpha. \qquad (29)$$

Introducing the dimensionless amplitude defined in (10) and (13) we get easily for the vertex on the Fermi surface

$$\mathcal{T}\left(\frac{\mathbf{pk}}{k}\right) = \gamma \frac{\partial G^{-1}}{\partial \varepsilon} + \int f(\mathbf{n}, \mathbf{n}_1) \gamma \frac{v k n_1}{\omega - v k n_1} \frac{\partial G^{-1}}{\partial \varepsilon} \frac{do_1}{4\pi}, \qquad (28')$$

and similarly

$$\mathcal{T}_\alpha\left(\frac{\mathbf{pk}}{k}\right) = -\gamma \frac{\partial G^{-1}}{\partial p_\alpha} - \int f(\mathbf{n}, \mathbf{n}_1) \gamma \frac{\omega}{\omega - v k n_1} p_{1\alpha} \frac{\partial G^{-1}}{\partial \varepsilon^0_p} \frac{do_1}{4\pi}, \qquad (29')$$

where $\epsilon^0_p = p^2/2$.

4. POLARIZATION OPERATOR (CORRELATION FUNCTION)

A. Polarization operator for a scalar field. We define the polarization operator $\mathcal{P}_{ik}$ by the equation

$$\langle j_i \rangle = \mathcal{P}_{il} A_l, \qquad (30)$$

where $\langle j_i \rangle$ is the average value of the current arising in the system under the action of the field A_l. We first of all evaluate the component $\mathcal{P}_{44} \equiv \mathcal{P}$ which enables us to find the change in the density under the action of a scalar field

$$n' = \mathcal{P}\varphi. \qquad (31)$$

From the definition of the Green's function

$$n' = -i \int G' \frac{d^4\mathcal{P}}{(2\pi)^4}. \qquad (32)$$

The change in the Green's function G' is determined by Eq. (16) and we can thus write

$$\mathcal{P} = (G\Gamma G) = (A\mathcal{T}) + (B\mathcal{T}). \qquad (33)$$

We shall show that the integrations occurring in (33) over regions far from the Fermi surface can be reduced to renormalization factors and integrals over the Fermi surface.

Equation (28) written symbolically gives after substitution into (33)

$$\mathcal{P} = \left(GG \frac{\partial G^{-1}}{\partial \varepsilon}\right) + \left(GG\Gamma\gamma A \frac{\partial G^{-1}}{\partial \varepsilon}\right).$$

From (17) it follows that the second term can be expressed in terms of the difference $\mathcal{T} - \gamma$:

$$\mathcal{P} = \left(GG \frac{\partial G^{-1}}{\partial \varepsilon}\right) + \left((\mathcal{T} - \gamma) A \frac{\partial G^{-1}}{\partial \varepsilon}\right) = \left(B \frac{\partial G^{-1}}{\partial \varepsilon}\right) + \left(\mathcal{T} A \frac{\partial G^{-1}}{\partial \varepsilon}\right).$$

The first term on the right-hand side is an integral of $\partial G/\partial \epsilon$ and is equal to zero since $G\big|_{\epsilon=\pm\infty} = 0$. [We note that $\mathcal{P}^\omega = ((GG)^\omega \mathcal{T}^\omega) = 0$ for any frequency ω since a field which is uniform in space does not produce any physical changes.] The polarization operator for each kind of particles is thus expressed in terms of $\mathcal{T}$ and $\partial G^{-1}/\partial \epsilon$ on the Fermi surface:

$$\mathcal{P} = (\mathcal{T} A \partial G^{-1}/\partial \varepsilon). \qquad (34)$$

To change from the arbitrary formalism to the usual one we must use Eqs. (32) and (4). We get easily

$$\mathcal{P} = 2 \int_{-1}^{1} \mathcal{T}(x) \frac{kvx}{\omega - kvx} dx \frac{am^* p_0}{4\pi^2}. \qquad (35)$$

[1)]Luttinger and Nozières[5] have done similar calculations for the case of one kind of particles.

When $\mathcal{T}$ is normalized according to (22) ($\mathcal{T}^0 = 1$) the vertex corresponds to the creation of a particle and a hole with total spin zero and with a fixed direction of the particle spin; the factor 2 in Eq. (35) takes into account the two possible z-components of the particle spin.

B. Vector part of the polarization operator. We now evaluate the vector part $\mathcal{P}_{\alpha\beta}$ of the polarization operator. To do this we find the particle current arising under the influence of a field A_α. The current density is connected with the particle distribution function through the equation

$$j_\alpha(q) = \int f^{(1)}_{k,\omega}(\mathbf{p})\, p_\alpha \frac{d\mathbf{p}}{(2\pi)^3} - nA_\alpha(q), \tag{36}$$

where $f^{(1)}_q(\mathbf{p})$ is the Fourier component of the first order correction to the distribution function and n the particle density.

It follows from its definition that the distribution function is connected with G by the equation

$$f_q(\mathbf{p}) = -i\int G\left(p+\frac{q}{2},\ p-\frac{q}{2}\right)\frac{d\varepsilon}{2\pi}, \tag{37}$$

where $G(p_1, p_2)$ is the Green's function in an external field. The correction which is first order in A_α is thus equal to

$$f^{(1)}_q(\mathbf{p}) = -i\int G\left(p+\frac{q}{2}\right)\mathcal{T}_\beta(p, q)\, A_\beta(q)\, G\left(p-\frac{q}{2}\right)\frac{d\varepsilon}{2\pi}. \tag{38}$$

Substituting this expression into (36) we get

$$j_\alpha(q) = \mathcal{P}_{\alpha\beta}A_\alpha,$$
$$\mathcal{P}_{\alpha\beta} = -i\int p_\alpha G\left(p+\frac{q}{2}\right)G\left(p-\frac{q}{2}\right)\mathcal{T}_\beta\frac{d^4p}{(2\pi)^4} - n\delta_{\alpha\beta}. \tag{39}$$

Equation (39) is valid for each kind of particle.

We shall show that the integrals over regions far from the Fermi surface can be reduced by means of (27) to integrals over the Fermi surface. Writing $\mathcal{T}_\alpha$ in the form

$$\mathcal{T}_\alpha = \gamma p_\alpha + \Gamma GG\gamma p_\alpha$$

and using (27) we get easily

$$(p_\alpha GG\mathcal{T}_\beta)_a = -\left(p_\alpha\frac{\partial G^{-1}}{\partial p_\beta}GG\right)_a - \left(p_\alpha GG\Gamma\gamma A'\frac{\partial G^{-1}}{\partial p_\beta}\right)_a$$
$$= -\left(p_\alpha\mathcal{T}_\beta A'\frac{\partial G^{-1}}{\partial\varepsilon^0_p}\right)_a.$$

Hence

$$\mathcal{P}_{\alpha\beta} = -\left(p_\alpha\mathcal{T}_\beta A'\frac{\partial G^{-1}}{\partial\varepsilon^0_p}\right)_a - n_a\delta_{\alpha\beta}. \tag{40}$$

The transition to the usual formalism is performed in the same way as was done to obtain Eq. (35).

5. CONNECTION BETWEEN THE PARTICLE DENSITY AND THE MOMENTA ON THE FERMI SURFACE

One can show[6] that the particle momentum distribution, n(p), in systems with arbitrarily strong interactions, but without Cooper pairing, has a discontinuity for a momentum $p = p_0$. This fact enables us to introduce the Fermi boundary concept also when there are strong interactions between the particles.

It has been shown[3,4] for a system of one kind of fermions that the density of interacting particles is connected with the thus introduced momentum p_0 by the same formula as is valid for the density of free particles:

$$n = 2\,\frac{4\pi}{3}\frac{p_0^3}{(2\pi)^3}. \tag{41}$$

We shall show that the result (41) remains valid also for systems consisting of two kinds of fermions.

Let us consider the change in the density under the influence of a static field φ_k with $k \to 0$. The equilibrium condition gives

$$\mu_a + \varphi = \text{const}, \qquad \mu_b + \lambda\varphi = \text{const}, \tag{42}$$

where μ_a and μ_b are the chemical potentials of the particles a and b. At zero temperature the density n depends only on μ_a and μ_b so that the change in the density of the particles a is equal to

$$n'_a = (\partial n_a/\partial\mu_a + \lambda\,\partial n_a/\partial\mu_b)\,\varphi = \mathcal{P}^k_a\varphi, \tag{43}$$

where, according to (34)

$$\mathcal{P}^k_a = (\mathcal{T}^k_a A^k_a \partial G_a^{-1}/\partial\varepsilon).$$

On the other hand, the vertex is connected with the change in the reciprocal of the Green's function through the relation

$$\mathcal{T}\varphi = \delta G^{-1},$$

since ϵ in G_a is reckoned from μ_a; G_a^{-1} changes only if the change in μ_a and μ_b is taken into account. We find

$$\mathcal{T}^k_a = \partial G_a^{-1}/\partial\mu_a + \lambda\partial G_a^{-1}/\partial\mu_b. \tag{44}$$

We write G^{-1} near $\epsilon = 0$ and $p = p_0$ in the form

$$G^{-1} = [\varepsilon - v(p - p_0)]\,/\,a. \tag{45}$$

We have then for $\epsilon = 0$ and $p = p_0$

$$\left(\frac{\partial G_a^{-1}}{\partial\mu_a}\right) = \frac{v_a}{a}\frac{\partial p_0^a}{\partial\mu_a}, \qquad \left(\frac{\partial G_a^{-1}}{\partial\mu_b}\right)_0 = \frac{v_a}{a}\frac{\partial p_0^a}{\partial\mu_b}, \tag{46}$$

since the derivatives of G^{-1} with respect to a and v vanish in the point $p = p_0$, $\epsilon = 0$.

We find from (35)

$$\mathcal{P}^k = -\mathcal{T}^k m^* a p_0/\pi^2.$$

Using (43), (44), and (46) we get

$$\frac{\partial n_a}{\partial \mu_a} + \lambda \frac{\partial n_a}{\partial \mu_b} = \frac{p_0^2}{\pi^2}\left(\frac{\partial p_0^a}{\partial \mu_a} + \lambda \frac{\partial p_0^a}{\partial \mu_b}\right) = \frac{\partial n_a^0}{\partial \mu_a} + \lambda \frac{\partial n_a^0}{\partial \mu_b}, \quad (47)$$

where expression (41) for the density of free particles with a limiting momentum p_0^a is denoted by n_a^0.

It follows from (47) that $n_a = n_a^0 + C_1$, where C_1 is independent of μ_a and μ_b. Since $n_a = n_a^0$ as $p_0 \to 0$, i.e., in the case of a sufficiently rarefied gas, $C_1 = 0$, and hence

$$n_{a,b} = n_{a,b}^0. \quad (47')$$

6. CONNECTION BETWEEN THE SCATTERING AMPLITUDE AND THE EFFECTIVE MASS

A. Change in the Green's function in a uniform field. If on both kinds of particles the same uniform vector field of arbitrary frequency acts, the system moves as a whole without any internal changes. This fact allows us to obtain yet another identity for the vector vertex when $\lambda = 1$ (the charges of the particles a and b are the same).

The Lagrangian of a system in a uniform field is for $\lambda = 1$ of the form

$$\tilde{L} = L + j_{a\alpha} A_\alpha(t) + j_{b\alpha} A_\alpha(t) = L + \mathbf{PA},$$

where $\mathbf{P}$ is the operator of the total momentum of the system. The operator $\mathbf{P}$ commutes with the Hamiltonian of the system. Since the ground state of a system at rest corresponds to the eigenvalue $\mathbf{P} = 0$ the ground state function of the system will not change under the influence of A_α.

The change in the quantum operators is determined by the formula

$$\tilde{\Psi} = \exp\left(i\mathbf{P}\int_0^t \mathbf{A}\,dt\right)\Psi(\mathbf{r}, t)\exp\left(-i\mathbf{P}\int_0^t \mathbf{A}\,dt\right).$$

The Green's function $\tilde{G}$ in the field A is thus given by the expression

$$\tilde{G}(x_1, x_2) = -i\left(\Phi_0 \Psi(x_1)\exp\left(i\mathbf{P}\int_{t_1}^{t_2}\mathbf{A}\,dt\right)\Psi^+(x_2)\Phi_0\right), \; t_1 > t_2;$$

$$\tilde{G}(x_1, x_2) = i\left(\Phi_0 \Psi^+(x_2)\exp\left(i\mathbf{P}\int_{t_1}^{t_2}\mathbf{A}\,dt\right)\Psi(x_1)\Phi_0\right), \quad t_1 < t_2.$$

Changing from $\mathbf{r}_1 - \mathbf{r}_2$ to the Fourier representation we get

$$\tilde{G}(\mathbf{p}, t_1, t_2) = G(p, \tau)\exp\left(ip\int_{t_1}^{t_2}\mathbf{A}\,dt\right) \quad (48)$$

for both signs of $\tau = t_1 - t_2$ and arbitrary time-dependence $\mathbf{A}(t)$.

We consider $\mathbf{A}$ of the form $\mathbf{A} = \mathbf{A}^0 e^{i\omega t}$ and let $\omega \to 0$. Putting $t_1 = 0$ we get

$$\int_0^\tau \mathbf{A}\,d\tau = \mathbf{A}^0\frac{e^{i\omega\tau} - 1}{i\omega} \to \mathbf{A}^0\tau.$$

We multiply (48) by $e^{i\epsilon\tau}$ and integrate over τ. We find

$$\tilde{G}(\mathbf{p}, \varepsilon) = G(\mathbf{p}, \varepsilon + \mathbf{pA}^0).$$

The vertex which is determined by the relation $\tilde{G}^{-1} - G^{-1} = \mathcal{T}_\alpha A_\alpha$ is thus of the form

$$\mathcal{T}_\alpha^\omega(\lambda = 1) = p_\alpha \partial G^{-1}/\partial\varepsilon \text{ as } A_\alpha \to 0. \quad (49)$$

By a different method this relation was obtained by Pitaevskiĭ[4] for the case of one kind of particles.

We emphasize that (49) in contradistinction to (22) is valid only for the case where the charges are the same for all kinds of particles as far as the field $\mathbf{A}$ is concerned.

We assume now that there is imposed upon the system a field such that the change in the Lagrangian is of the form

$$L' = (\hat{\sigma}_a + \hat{\sigma}_b)_\alpha H_\alpha,$$

where $\hat{\sigma}_a$ and $\hat{\sigma}_b$ are the total spin operators of the particles a and b. If the spin-orbit interaction is small [this was presupposed in Eq. (13)] the total spin operator commutes with the Hamiltonian and, repeating the calculations given above, we get

$$\mathcal{T}_{s\alpha}^\omega H_\alpha \equiv \tilde{G}^{-1} - G^{-1} = \sigma_\alpha H_\alpha \frac{\partial G^{-1}}{\partial\varepsilon}, \quad (50)$$

where $\mathcal{T}_{s\alpha}$ is the spin vertex which is equal to

$$\mathcal{T}_{s\alpha} = \sigma_\alpha + \Gamma G G \sigma_\alpha'.$$

Comparing this with (50) enables us to obtain yet another relation for Γ^ω. Writing $\Gamma^\omega = \Gamma_0^\omega + \Gamma_1^\omega \sigma_\beta \sigma_\beta'$, we get

$$1 + (\Gamma_1^\omega)_{aa} B_a + (\Gamma_1^\omega)_{ab} B_b = \partial G_a^{-1}/\partial\varepsilon. \quad (51)$$

B. The effective mass. Equation (49) combined with (27′) enables us to connect the scattering amplitude with the effective mass. Indeed, we get from (29′)

$$\mathcal{T}_\alpha^\omega = \gamma\frac{\partial G^{-1}}{\partial\varepsilon} v_\alpha + \int f^\omega(\mathbf{n}, \mathbf{n}_1)\gamma v_{1\alpha}\frac{\partial G^{-1}}{\partial\varepsilon}\frac{\partial o_1}{4\pi}. \quad (52)$$

We have used here the relation $-\partial G^{-1}/\partial p_\alpha = (\partial G^{-1}/\partial\epsilon) v_\alpha$ which follows from (45).

From (52) it follows for $\lambda = 1$ that

$$(\mathcal{T}^{\omega}_{a})_a = (\partial G^{-1}/\partial\varepsilon)_a v^a_a \left[1 + \frac{1}{3}\{f^{\omega}_{aa} + f^{\omega}_{ab}\}_1\right],$$

where the index 1 on the curly bracket indicates the first harmonic in the expansion in terms of Legendre polynomials of the dimensionless amplitudes introduced in (10) and (13). Comparing this with (49) we get

$$m^*_a = 1 + \tfrac{1}{3}\{f^{\omega}_{aa} + f^{\omega}_{ab}\}_1 = 1 + \tfrac{1}{3}(\chi^{\omega}_1)_a. \qquad (53)$$

7. DEPENDENCE OF THE ENERGY OF THE SYSTEM ON DENSITY AND CONCENTRATION. CONNECTION OF THE SCATTERING AMPLITUDE WITH THE COMPRESSIBILITY COEFFICIENTS

We assume that the energy density W of the system depends in the following way on the densities n_a and n_b:

$$W = \frac{1}{2}K\frac{[(n_a + n_b) - 2n_0]^2}{2n_0} + \beta\frac{(n_a - n_b)^2}{2n_0}, \qquad (54)$$

where n_0 is the equilibrium density which occurs when $n_a = n_b$ and when there are no external fields. In the case of a nucleus the second term on the right-hand side of (54) leads to the term $\beta(N - Z)^2/A$ in the von Weizsäcker formula for the mass defect of nuclei.

The chemical potentials μ_a and μ_b are by definition equal to

$$\mu_a = \frac{\partial W}{\partial n_a} = \frac{K}{2n_0}(n_a + n_b - 2n_0) + \frac{\beta}{n_0}(n_a - n_b),$$

$$\mu_b = \frac{\partial W}{\partial n_b} = \frac{K}{2n_0}(n_a + n_b - 2n_0) - \frac{\beta}{n_0}(n_a - n_b).$$

We introduce the matrix

$$D = \begin{pmatrix}\partial\mu_a/\partial n_a & \partial\mu_a/\partial n_b\\ \partial\mu_b/\partial n_a & \partial\mu_b/\partial n_b\end{pmatrix} = \frac{1}{2n_0}\begin{pmatrix}K + 2\beta & K - 2\beta\\ K - 2\beta & K + 2\beta\end{pmatrix}. \qquad (55)$$

For the following we need the reciprocal matrix equal to

$$D^{-1} = \begin{pmatrix}\partial n_a/\partial\mu_a & \partial n_a/\partial\mu_b\\ \partial n_b/\partial\mu_a & \partial n_b/\partial\mu_b\end{pmatrix} = \frac{2n_0}{8\beta K}\begin{pmatrix}2\beta + K, & 2\beta - K\\ 2\beta - K, & 2\beta + K\end{pmatrix}. \qquad (56)$$

Equation (44) together with (46) and (56) gives

$$\mathcal{T}^{k}_{a} = \frac{1}{a_a}\frac{v_a}{\partial n^0_a/\partial p^0_a}\left(\frac{\partial n_a}{\partial\mu_a} + \lambda\frac{\partial n_a}{\partial\mu_b}\right)$$

$$= \frac{1}{a_a}\frac{v_a}{\partial n^a_0/\partial p^a_0}\frac{2n_0}{8\beta K}\{2\beta + K + \lambda(2\beta - K)\}. \qquad (57)$$

On the other hand, we get from (28′)

$$\mathcal{T}^{k}_{a} = (\partial G^{-1}/\partial\varepsilon)_a[1 - \{f^k_{aa} + \lambda f^k_{ab}\}_0], \qquad (58)$$

where the index 0 on the curly brackets indicates the zeroth harmonic in the expansion in Legendre polynomials. Comparing (57) and (58) for arbitrary λ we get

$$1 - \{f^k_{aa} + f^k_{ab}\} \equiv 1 - \chi^k_0 = (vp_0/n)_a n_0/3K,$$

$$1 - \{f^k_{aa} - f^k_{ab}\} \equiv 1 - \eta^k_0 = (vp_0/n)_a n_0/6\beta. \qquad (59)$$

8. APPLICATION OF THE THEORY TO THE NUCLEUS

A. Determination of the zeroth harmonics of the scattering amplitude in the nucleus. Equations (56) allow us to find the zeroth harmonics of the scattering amplitude from the experimental values of the rigidities β and K.[7] Putting $\epsilon_0 = vp_0/2 \approx 30$ MeV, $\beta = 25$ MeV, $K \approx 25$ MeV [we note that K as defined by Eq. (51) is ten times less than the value of K introduced in [7]] we get from (56) (for $n_a = n_b = n_0$)

$$\eta^k_0 = 1 - \varepsilon_0/3\beta \approx 0.6, \qquad \chi^k_0 = 1 - 2\varepsilon_0/3K \approx 0.2;$$

$$\{f^k_{aa}\}_0 = 0.4, \quad \{f^k_{ab}\}_0 = -0.2.$$

Using Eq. (15) we can find the quantities χ^{ω}_0 and η^{ω}_0:

$$\chi^{\omega}_0 = \chi^k_0/(1 - \chi^k_0) = 0.25, \qquad \eta^{\omega}_0 = \eta^k_0/(1 - \eta^k_0) = 1.5;$$

$$\{f^{\omega}_{aa}\}_0 = 0.9, \qquad \{f^{\omega}_{ab}\}_0 = -0.6.$$

One can show that the quantity η^k_0 determines the effective charge of the quasi-particles for single-particle low-energy dipole transitions. The large difference between the quantities f^k and f^{ω} shows that the approximation of binary collisions in the nucleus is not at all in accordance with actual facts.

B. Estimate of the higher harmonics. The quasi-particle effective mass in the nucleus is determined by Eq. (52). Since the region where the function $\chi^{\omega}(x)$ changes appreciably is of the order of unity we must expect that

$$\chi^{\omega}_l \sim \chi^{\omega}_0/(2l + 1).$$

We get thus from (52)

$$m^* - 1 \sim \chi^{\omega}_0/9.$$

This estimate gives, of course, only an order of magnitude.

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