ON THE THEORY OF RELATIVISTIC COULOMB SCATTERING. II

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Formulas for the elastic scattering cross section of electrons by nuclei have been derived by taking into account screening and the finite size of the nucleus. The Laplace transformation for the scattering charge density and expansion of the matrix elements in the transformation parameter were employed.

1. INTRODUCTION

THE exact determination of the amplitude of relativistic scattering of charged particles by a distributed charge is a complicated computational problem. In this connection, the development of approximate methods for calculation of scattering cross sections is very useful. One of these methods is the method of the Born approximation. This method has been applied not only for the calculation of scattering cross sections with account of screening, [1] but also for obtaining the cross sections for scattering by the distributed charge of the nucleus. [2,3] However, the comparatively poor convergence of the Born theories on the one hand, and the difficulty of calculation of higher Born approximations on the other, have reduced the value of the method. Moreover, all calculations by this method have always been connected with the choice of definite models of the charge distribution.

In the present research, which is a continuation of the previous work of the author [4] (cited below as I), the potential created by the atomic shell and by the deviation of the charge distribution of the nucleus from that of a point charge is taken into account as a perturbation. The generalized functions of Furry-Sommerfeld-Maue in the Coulomb field of the nucleus were used as the unperturbed wave functions [I, (6)]. Application of the Laplace transformation for the density of the nucleus and of the electron shell of the atom makes it possible to separate the additional small parameter. In the expansion in this parameter, the scattering cross section is expressed in terms of the mean value of the determined characteristics of the charge distribution without reference to concrete models.

2. GENERAL FORM OF THE SCATTERING CROSS SECTION

Consider a charge Z isotropically distributed in space with density $\rho(r)$. Applying the Laplace transformation, we get

$$r\rho(r) = \frac{Z}{4\pi}\int_0^\infty \chi(\lambda)\,\lambda^2 e^{-\lambda r}\,d\lambda. \tag{1}$$

The form factor $F(q)$ of the given charge distribution $\rho(r)$ in this case takes the form

$$F(q) = \frac{1}{Z}\int \rho(r)\,e^{-i\mathbf{qr}}\,d^3r = \int_0^\infty \frac{\chi(\lambda)\,\lambda^2\,d\lambda}{q^2+\lambda^2}$$
$$= 1 - q^2\int_0^\infty \frac{\chi(\lambda)}{q^2+\lambda^2}\,d\lambda. \tag{2}$$

The potential energy of the electron in the field of the charge (1) can be represented in momentum space by the following expression:

$$V(q) = -\frac{\alpha Z}{2\pi^2}\frac{F(q)}{q^2} = -\frac{\alpha Z}{2\pi^2}\int_0^\infty d\lambda\,\{\delta(\lambda) - \chi(\lambda)\}\,\frac{1}{q^2+\lambda^2}. \tag{3}$$

We note the following set of relations which follow from (1):

$$\int_0^\infty \chi(\lambda)\,\lambda^{-n}\,d\lambda = \frac{\langle r^n\rangle}{(n+1)!}, \qquad n \geqslant -1;$$
$$\int_0^\infty \chi(\lambda)\,\lambda^{n+2}\,d\lambda = \left\{\left(-\frac{\partial}{\partial r}\right)^n r\rho(r)\right\}_{r\to 0}, \quad n \geqslant 0, \tag{4}$$

where $\langle A\rangle$ denotes the average value of the quantity A.

The scattering of a charged particle by an atom can be considered as the scattering by a nuclear charge density $\rho_n(r)$ and by an electron shell charge density $\rho_e(\mathbf{r})$ with the total potential energy of the form

$$V(q) = -\frac{\alpha Z}{2\pi^2}\int_0^\infty d\lambda\,\{[\delta(\lambda) - \chi_{\text{nuc}}(\lambda)] - [\delta(\lambda) - \chi_e(\lambda)]\}\frac{1}{q^2+\lambda^2} = V_1 + V_2; \quad (5)$$

$$V_1 = -V(0), \qquad V_2 = -\gamma_s V(\lambda_s),$$

$$\gamma_s = -\int_0^\infty d\lambda_s\,[\delta(\lambda_s) - \chi_e(\lambda_s)] - \int_0^\infty d\lambda_s \chi_{\text{nuc}}(\lambda_s),$$

$$V(\lambda) \equiv \frac{\alpha Z}{2\pi^2(q^2+\lambda^2)}. \quad (6)$$

The term with screening in (6) can also be represented in the form of a finite number of terms I, (11); in this case, the operator γ_s in (6) takes the form [see I, (14b)]

$$\gamma_s = \sum_{s=1}^{4} a_s - \int_0^\infty d\lambda_s \chi_{\text{nuc}}(\lambda_s). \quad (7)$$

As was shown in I, the amplitude of scattering by the two potentials V_1 and V_2 can be represented in the following fashion:

$$F(\mathbf{k}, \mathbf{p}) = \bar{u}_k f(\mathbf{k}, \mathbf{p}) u_p,$$

$$f(\mathbf{k}, \mathbf{p}) = f_1(\mathbf{k}, \mathbf{p}) + f_2(\mathbf{k}, \mathbf{p}) \quad (\hbar = c = 1); \quad (8)$$

$$f_1(\mathbf{k}, \mathbf{p}) = 2\pi^2\{\langle \mathbf{k}|\hat{V}_1|\varphi_p^0\rangle + \langle\varphi_k^0|\hat{V}_1|\varphi_p^{\mathrm{I}}\rangle + \langle\varphi_k^{\mathrm{I}}|\hat{V}_1 R_1|\varphi_p^{\mathrm{I}}\rangle\}; \quad (9)$$

$$f_2(\mathbf{k}, \mathbf{p}) = 2\pi^2\{\langle\varphi_k^0|\hat{V}_2|\varphi_p^0\rangle + \langle\varphi_k^0|\hat{V}_2|\varphi_p^{\mathrm{I}}\rangle + \langle\varphi_k^{\mathrm{I}}|\hat{V}_2|\varphi_p^0\rangle + \langle\varphi_k^{\mathrm{I}}|\hat{V}_2|\varphi_p^{\mathrm{I}}\rangle + \langle\varphi_k^0|\hat{V}_2 G\hat{V}_1|\varphi_p^{\mathrm{I}}\rangle + \langle\varphi_k^{\mathrm{I}}|\hat{V}_1 G\hat{V}_2|\varphi_p^0\rangle + \langle\varphi_k^0|\hat{V}_2 G\hat{V}_2|\varphi_p^0\rangle + o(V_1V_2^2, V_2^3)\}. \quad (10)$$

In these formulas, $\mathbf{p}$ and $\mathbf{k}$ are the momenta of the incoming and outgoing electrons, $|\varphi_p^s\rangle$ are the wave functions of the electron in the potential V_1, which are determined in I, (6). The formula (10) is obtained from I, (5b) with the help of I, (6b).

As was shown in (I), (18)—(19), the matrix element of (9) and the first four matrix elements of (10) can be expressed in the form

$$S_1 = 2\pi^2\langle\varphi_k^0|\hat{V}(\lambda)|\varphi_p^0\rangle = \gamma_4 \alpha Z K(\mathbf{q}, \mathbf{k}, \mathbf{p}, \lambda) M e^{i\varphi}, \quad (11)$$

$$S_2 = 2\pi^2\langle\varphi_k^0|\hat{V}(\lambda)|\varphi_p^{\mathrm{I}}\rangle = \gamma_4(\alpha Z)^2\frac{\tilde{\nabla}_p}{-2i\xi}\int_\lambda^\infty d\eta K(\mathbf{q}, \mathbf{k}, \mathbf{p}, \eta) M e^{i\varphi}, \quad (12)$$

$$S_3 = 2\pi^2\langle\varphi_k^{\mathrm{I}}|\hat{V}(\lambda)|\varphi_p^{\mathrm{I}}\rangle = \gamma_4(\alpha Z)^3\frac{\tilde{\nabla}_p}{(-2i\xi)}\frac{\tilde{\nabla}_k}{(-2i\xi)} \times \int_\lambda^\infty d\eta \int_\eta^\infty d\eta' K(\mathbf{q}, \mathbf{k}, \mathbf{p}, \eta') M e^{i\varphi}, \quad (13)$$

$$M = e^{\pi\xi}|\Gamma(1-i\xi)|^2, \qquad e^{i\varphi} = \frac{\Gamma(1-i\xi)}{\Gamma(1+i\xi)},$$

$$\tilde{\nabla} = \boldsymbol{\alpha}\nabla, \qquad \xi = \frac{\alpha Z E}{p}, \quad (14)$$

where α_l are the Dirac matrices, $K(\mathbf{q}, \mathbf{k}, \mathbf{p}, \eta)$ is determined by formulas I, (19a), (19b). Making use of the results of Nordsieck,[5] we can represent $K(\mathbf{q}, \mathbf{k}, \mathbf{p}, \eta)$ in the form

$$K(\mathbf{q}, \mathbf{k}, \mathbf{p}, \eta) = \frac{1}{a}\left(\frac{a}{\alpha_1}\right)^{i\xi}\left(\frac{a}{\alpha_2}\right)^{i\xi} {}_2F_1(i\xi, i\xi; 1; z); \quad (15)$$

$$a = q^2 + \eta^2, \qquad \alpha_1 = (\mathbf{q}+\mathbf{p})^2 - (p + i\eta)^2,$$

$$\alpha_2 = (\mathbf{q}-\mathbf{k})^2 - (k + i\eta)^2,$$

$$z = [2a(kp + \mathbf{kp}) + (a-\alpha_1)(a-\alpha_2)]/\alpha_1\alpha_2,$$

$$\mathbf{q} = \mathbf{k} - \mathbf{p}. \quad (16)$$

After calculation of the gradients in (12)—(13), which do not act on $\mathbf{q}$, we can replace $\mathbf{q}$ by $\mathbf{k}-\mathbf{p}$ in (16); we then obtain

$$\alpha_1 = \alpha_2 = -i\eta(2p + i\eta), \qquad z = -q^2/\eta^2. \quad (17)$$

The expression for (15) at large values of Z (small values of η) can be obtained by using an expansion of the hypergeometric functions in inverse powers of z:[6]

$${}_2F_1(i\xi, i\xi; 1; z) = (-z)^{-i\xi}\frac{1}{|\Gamma(1-i\xi)|^2} \times \left\{1 + i\xi\ln(-z) + \xi^2 L_2\left(\frac{1}{z}\right) + o\left(\frac{\xi^3}{z}\right)\right\}. \quad (18)$$

Substituting (18) in (15), and using (17), we get

$$K(\mathbf{q}, \mathbf{k}, \mathbf{p}, \lambda) = \frac{1}{M}\frac{1}{q^2}\frac{\varepsilon^2}{\varepsilon^2+\mu^2}\left\{\frac{\varepsilon^2+\mu^2}{\varepsilon(1+i\mu)}\right\}^{2i\xi} \times \left\{1 + i\xi\ln\frac{\varepsilon^2}{\mu^2} + \xi^2 L_2\left(-\frac{\mu^2}{\varepsilon^2}\right) + o(\mu^2\xi^3)\right\},$$

$$\mu = \lambda/2p, \qquad \varepsilon = q/2p. \quad (19)$$

As is seen from (19), the value of K is proportional to 1/M for small $\mu(\lambda)$. Since the principal contribution to the integrals (12) and (13) is made, for small λ, in the region close to the lower limit of integration, M is practically eliminated from all three matrix elements (11)—(13). This statement will be valid also for the remaining terms of (10). The case of large λ will be considered [in Sec. (5)] only in the region where $E/p \sim 1$ and $\xi \sim \alpha Z$. Therefore, we shall everywhere expand M in terms of ξ in what follows.

Making use of the formulas of the appendix and discarding terms of order $(\alpha Z)^4$ and the phase factor $e^{i\varphi}$, we get the following expressions for the matrix elements appearing in (9) and (10):[1)]

$$S_0 = 2\pi^2 \langle \mathbf{k} \mid \hat{V}_1 \mid \varphi_p^0 \rangle = \gamma_4 \alpha Z q^{-2} \exp(i\xi \ln \varepsilon^2), \quad (20)$$

$$S_1 = \gamma_4 \frac{\alpha Z}{q^2} \frac{\varepsilon^2}{\varepsilon^2+\mu^2}\Big\{1 + i\xi 2 \ln \frac{\varepsilon^2+\mu^2}{\mu(1+i\mu)}\Big| - \xi^2\Big[2\ln^2 \frac{\varepsilon^2+\mu^2}{\varepsilon(1+i\mu)} + 4\ln\frac{\varepsilon^2+\mu^2}{\varepsilon(1+i\mu)}\ln\frac{\varepsilon}{\mu} - L_2\Big(-\frac{\mu^2}{\varepsilon^2}\Big)\Big]\Big\}, \quad (21)$$

$$S_2 = 2\pi^2 \langle \varphi_k^0 | \hat{V}(\lambda) | \varphi_p^{\mathrm{I}} \rangle = 2\pi^2 \langle \varphi_k^{\mathrm{I}} | \hat{V}(\lambda) | \varphi_p^0 \rangle = \gamma_4 \frac{(\alpha Z)^2}{q^2} \frac{E - \gamma_4 m}{p} \Big\{K_1(\mu) + 2\alpha Z \frac{E}{p}\Big[K_2(\mu) + \frac{\pi}{2}K_1(\mu)\Big]\Big\}, \quad (22)$$

$$S_3 = \gamma_4 \frac{(\alpha Z)^3}{q^2} \frac{m(\gamma_4 E - m)}{p^2} K_3(\mu), \quad (23)$$

$$S_4 = 2\pi^2 \langle \varphi_k^0 | \hat{V}(\lambda)\, G\hat{V}_1 | \varphi_p^{\mathrm{I}} \rangle = 2\pi^2 \langle \varphi_k^{\mathrm{I}} | \hat{V}_1 G\hat{V}(\lambda) | \varphi_p^0 \rangle = \gamma_4 \frac{(\alpha Z)^3}{q^2} \Big\{\frac{E}{p}\frac{E-\gamma_4 m}{p} K_2^1(\mu) + \frac{m(\gamma_4 E - m)}{p} K_3^1(\mu)\Big\}, \quad (24)$$

$$S_5 = 2\pi^2 \langle \varphi_k^0 | \hat{V}(\lambda_1)\, G\hat{V}(\lambda_2) | \varphi_p^0 \rangle = -\gamma_4 \frac{(\alpha Z)^2}{q^2}\Big\{\frac{E}{p} K_4(\mu_1, \mu_2) + \frac{E-\gamma_4 m}{p} K_5(\mu_1, \mu_2)\Big\}. \quad (25)$$

The functions $K_s(\mu)$ entering into (22)—(25) are determined in the appendix.

Substituting (20)—(25) in (9) and (10), we get an expression for the transverse scattering cross section in the presence of the initial polarization ζ [see I, (30)]:

$$\sigma(\vartheta) = \sigma_R\, Q\,\{1 + S\mathbf{n}\zeta\}, \qquad \sigma_R = \Big(\frac{\xi}{2p}\Big)^2 \frac{1}{\varepsilon^4}, \qquad \mathbf{n} = \frac{[\mathbf{kp}]}{kp \sin\vartheta}; \quad (26)^*$$

$$\varepsilon = \sin\frac{\vartheta}{2}, \qquad \cos\vartheta = \frac{\mathbf{kp}}{kp}, \qquad Q = Q_1 + Q_2, \qquad S = \frac{R}{Q}, \qquad R = R_1 + R_2, \quad (27)$$

where Q_1 and R_1 are determined by the scattering on a point nucleus and are given by formulas I, (31) and I, (32) after expansion of M(14) in these formulas in powers of ξ. The expressions for Q_2 and R_2 have the form [2)]

$$Q_2 = (1 - \beta^2\varepsilon^2)\mathcal{A}(2 + \mathcal{A}) + 2\alpha Z \,\mathrm{Re}\,\{\beta(1-\varepsilon^2) \times [\mathcal{C} + \mathcal{A}(\mathcal{C}_0 + \mathcal{C})] + \beta^{-1}(1-\beta^2\varepsilon^2)[\mathcal{B} + \mathcal{A} \times (\mathcal{B}_0 + \mathcal{B})]\} + 2(\alpha Z)^2 \,\mathrm{Re}\,\{(1-\varepsilon^2) \times [2\mathcal{D} - \mathcal{E} + \mathcal{A}(2\mathcal{D}_0 - \mathcal{E}_0) + \mathcal{B}\mathcal{C}_0^* + \mathcal{C}\mathcal{B}_0^* + \beta^2\mathcal{C}\mathcal{C}_0^*] + (1-\beta^2\varepsilon^2)[\mathcal{A}\mathcal{E}_0 + \mathcal{E} + \beta^{-2}(\mathcal{B}\mathcal{B}_0^* - \mathcal{A}\mathcal{F} - \mathcal{F})]\}, \quad (28)$$

$$R_2 = -2\alpha Z\varepsilon(1-\beta^2)^{1/2}(1-\varepsilon^2)^{1/2}\,\mathrm{Im}\,\{\beta[\mathcal{C} + \mathcal{A}(\mathcal{C}_0 + \mathcal{C})] + \alpha Z[2\mathcal{D} - \mathcal{E} + \mathcal{A}(2\mathcal{D}_0 - \mathcal{E}_0) + \mathcal{C}\mathcal{B}_0^* + \mathcal{C}_0\mathcal{B}^*]\}; \quad (29)$$

$$\mathcal{B}_0 = i\ln\varepsilon^2, \qquad \mathcal{C}_0 = K_1(0), \qquad \mathcal{D}_0 = K_2(0) + \tfrac{1}{2}\pi K_1(0),$$

$$\mathcal{E}_0 = K_3^{\varepsilon}(0), \qquad \mathcal{F}_0 = -\tfrac{1}{2}\ln^2\varepsilon^2, \qquad \mathcal{A} = \gamma_s\frac{\varepsilon^2}{\varepsilon^2+\mu_s^2},$$

$$\mathcal{B} = 2i\gamma_s\frac{\varepsilon^2}{\varepsilon^2+\mu_s^2}\ln\frac{\varepsilon^2+\mu_s^2}{\mu_s(1+i\mu_s)} + \gamma_s\gamma_r K_4(\mu_s, \mu_r),$$

$$\mathcal{C} = 2\gamma_s K_1(\mu_s) + \gamma_s\gamma_r K_5(\mu_s, \mu_r),$$

$$\mathcal{D} = \gamma_s[2K_2(\mu_s) + \pi K_1(\mu_s) + K_2^1(\mu_s)],$$

$$\mathcal{E} = \gamma_s[K_3(\mu_s) + 2K_3^1(\mu_s)],$$

$$\mathcal{F} = -\gamma_s\Big[2\ln^2\frac{\varepsilon^2+\mu_s^2}{\varepsilon(1+i\mu_s)} + 4\ln\frac{\varepsilon^2+\mu_s^2}{\varepsilon(1+i\mu_s)}\ln\frac{\varepsilon}{\mu_s} - L_2\Big(-\frac{\mu_s^2}{\varepsilon^2}\Big)\Big]. \quad (30)$$

3. SCATTERING AMPLITUDES IN THE POTENTIAL $e^{-\lambda r}/r$ FOR SMALL λ

The parameter μ in (30), as will be shown in Secs. 4 and 5, can in certain cases be regarded as a small quantity and expansion can be carried out in terms of it. However, before undertaking to obtain the corresponding formulas, we shall make clear certain properties of the scattering amplitude in the potential $V(\lambda)$ (6) for small λ.

The scattering amplitude in this potential can be represented in the form (9), by replacing V_1 in (9) by $V(\lambda)$. We shall first consider the term in (9) corresponding to nonrelativistic scattering. By representing it in the form of a perturbation series in powers of αZ, and carrying out certain transformations treated in detail in Sec. 2 and [7] (in what follows we shall denote this paper by the symbol II), we obtain

$$f(q, \lambda) = 2\pi^2 \langle \mathbf{k} | \hat{V}(\lambda) | \psi_p^0 \rangle = \gamma_4 \alpha Z B(i\xi, q, a)\, A(i\xi, \eta, a), \qquad \eta = \lambda/p, \qquad \eta \ll a \ll 1, \quad (31)$$

where $A(i\xi, \eta, a)$ is identical with the corresponding function in II and is given by Eq. II, (15). The value of $B(i\xi, q, a)$ is determined by the series II, (9a) with the coefficients B_k, which are equal to

$$B_k = \int_a^1 \frac{dx_1}{(1-x_1)\Lambda_1}\int_{x_1}^1 \frac{dx_2}{(1-x_2)\Lambda_2}\cdots\int_{x_{k-1}}^1 \frac{dx_k}{(1-x_k)\Lambda_k}F(x_k),$$

$$B_0 = \frac{1}{q^2+\lambda^2}; \quad (32)$$

[1)]The first equation in (22) and (24) is satisfied only under the condition that the matrix elements are enclosed between the bispinors $\bar{u}_k$ and u_p [see (8)].

*[kp] = k × p.

[2)]We note that Eqs. I, (33) and I, (34) contain misprints corrected in (28), (29) and (30).

$$\Lambda_k^2 = x_k^2 - \eta^2(1-x_k) + (2i\eta\Lambda_1 - \eta^2)\frac{1-x_k}{1-x_1} + \ldots + (2i\eta\Lambda_{k-1} - \eta^2)\frac{1-x_k}{1-x_{k-1}} + i\varepsilon x_k,$$

$$F(x_k) = \frac{1-x_k}{(\mathbf{q}+\mathbf{p}x_k)^2 - (p\Lambda_k + i\lambda)^2} = \frac{1}{q^2+\lambda^2-p^2\left[\frac{2i\eta\Lambda_1-\eta^2}{1-x_1}+\ldots+\frac{2i\eta\Lambda_k-\eta^2}{1-x_k}\right]}. \qquad (33)$$

In contrast with II, it is not possible to neglect λ in (32) because all the B_k diverge at the upper limit for $\lambda = 0$ as a consequence of the equality $k = p$. To clarify the actual behavior of B_k for small λ, we divide all the integrals into two parts with $x_k < 1-b$ and $x_k > 1-b$, where $\eta \ll b \ll 1$; we then get

$$B_k = B_0 \sum_{n=0}^{k} D_n C_{k-n}; \qquad (34)$$

$$C_n = \int_a^{1-b}\frac{dx_1}{(1-x_1)x_1}\int_{x_1}^{1-b}\frac{dx_2}{(1-x_2)x_2}\cdots\int_{x_{n-1}}^{1-b}\frac{dx_n}{(1-x_n)x_n} = \frac{1}{n!}\left\{\int_a^{1-b}\frac{dx}{(1-x)x}\right\}^n = \frac{1}{n!}\ln^n\frac{1}{ab}; \qquad (35)$$

$$D_n = \int_0^b\frac{dy_1}{y_1}\int_0^{y_1}\frac{dy_2}{y_2}\cdots\int_0^{y_{n-1}}\frac{dy_n}{y_nS_n},$$

$$y_k = 1-x_k, \qquad S_k = 1+\sigma\left(\frac{1}{y_1}+\ldots+\frac{1}{y_k}\right) = S_{k-1}+\frac{\sigma}{y_k},$$

$$\sigma z_k = \frac{-i\lambda(2p+i\lambda)}{q^2+\lambda^2} = \frac{-i\mu(1+i\mu)}{\varepsilon^2+\mu^2}, \qquad \mu = \frac{\lambda}{2p}, \qquad \varepsilon = \frac{q}{2p}. \qquad (36)$$

Making the change of variables $\sigma z_k = y_k S_k$, $dy_k/y_k S_k = dz_k/S_{k-1}z_k$, we get

$$D_k = D_k(z_0) = \int_1^{z_0}\frac{dz_1}{z_1}\int_1^{z_1+1}\frac{dz_2}{z_2}\cdots\int_1^{z_{k-1}+1}\frac{dz_k}{z_k} = \int_1^{z_0}\frac{dz}{z}D_{k-1}(z+1), \qquad z_0 = \frac{b+\sigma}{\sigma} \gg \sigma. \qquad (37)$$

In its structure, (37) is identical with Eq. II, (13) for A_k. Therefore, by making the same transformation as in II, we get

$$D_k = \sum_{n=0}^{k}\frac{1}{n!}\ln^n\frac{b}{\sigma}d_{k-n}, \qquad \sigma = \frac{-i\mu(1+i\mu)}{\varepsilon^2+\mu^2}, \qquad (38)$$

$$d_k = d_k(x=1) = \int_0^{x=1}\frac{dt}{t}\left\{\sum_{n=0}^{k-1}\frac{1}{n!}\ln^n(1+t)\,d_{k-n-1}\left(\frac{1+t}{t}\right) - d_k\left(\frac{1}{t}\right)\right\} \quad \left(t = \frac{1}{z}\right), \qquad (39)$$

$$d_0 = 1, \qquad d_1 = 0, \qquad d_2 = \pi^2/12. \qquad (40)$$

Using (34), (35), and (38), and also II, (9a), II, (15) and II, (20), we get the following expression for (31):

$$f(q,\lambda) = \gamma_4\frac{\alpha Z}{q^2}\frac{\varepsilon^2}{\varepsilon^2+\mu^2}\left\{\frac{\varepsilon^2+\mu^2}{1+i\mu}\right\}^{i\xi}b(i\xi)\,d(i\xi)\,e^{-2i\xi\ln\mu} + o(\mu); \qquad (41)$$

$$b(\alpha) = e^{-\alpha C}/\Gamma(1+\alpha), \qquad d(\alpha) = \sum_{k=0}^{\infty}\alpha^k d_k. \qquad (42)$$

It is not difficult to verify that $d(\alpha)$ has a finite radius of convergence. As $\mu \to 0$, the amplitude (41) must go over into (20); therefore $d(i\xi)$ can differ from $b^{-1}(i\xi)$ only in its phase factor. By comparing the first few coefficients of the expansion of b [II, (14c)] and d [Eq. (40)], we find that $b(i\xi)d(i\xi) = 1$. Thus we finally get

$$f(q,\lambda) = \gamma_4\frac{\alpha Z}{q^2}g(\varepsilon,\mu)\,e^{-2i\xi\ln\mu}, \qquad g(\varepsilon,0) = e^{i\xi\ln\varepsilon^2}. \qquad (43)$$

We note that the infrared phase factor for the amplitude (31) has been shown to be equal to the square of the infrared phase factor for the wave function II, (21).

We have considered only the nonrelativistic scattering amplitude. In exactly the same way, it can be shown that the infrared phase factor (43) is separated out also for the relativistic amplitude.

4. SCATTERING AT E < 1 MeV

For energies less than 1 MeV, the form factor of the nucleus (2) can be regarded as equal to unity; therefore the operator γ_s in (6) and (7) can be replaced by

$$\gamma_s = -\int_0^\infty d\lambda_s[\delta(\lambda_s) - \chi_e(\lambda_s)] = \sum_{s=1}^{4}a_s, \qquad \gamma_s\cdot 1 = 0. \qquad (44)$$

As follows from I, (11), the parameter μ in (30) is found in this case to be equal to

$$\mu_s = \frac{m}{2p}\frac{Z^{1/3}}{121}b_s, \qquad (45)$$

where b_s are quantities of the order of unity and were determined in I, (11b). Over a rather wide range of energies and charges of the nucleus Z, the parameter (45) is a small quantity; therefore, we shall expand the expression for the cross section (26)—(29) in terms of the parameter (45).

We note that, by virtue of the last equation in (44), we have[3)]

$$\gamma_s F(\mu_s) = (\gamma_s\mu_s)F'(0) + \tfrac{1}{2}(\gamma_s\mu_s^2)F''(0) + \ldots \qquad (\gamma_s F(0) = 0). \qquad (46)$$

3)The expansion of the amplitude (10) in terms of μ will also contain terms proportional to $\ln\mu$; however these terms, as was shown in the previous section, are an expansion of the phase factor and should not enter into the formula for the cross section (28) and (29).

It follows from (46) that all the terms containing V_2^n are proportional to $(\mu\alpha Z)^n$, and consequently the terms neglected in (10) are of the order $(\alpha Z)^3\mu^2$.

Expanding (30) in terms of μ, we get

$$\mathcal{B} - 2i(\gamma_s \ln \mu_s) + 2(\gamma_s\mu_s) + i(\gamma_s\mu_s^2) - \frac{2(1-\ln\varepsilon^2)-\varepsilon^2}{\varepsilon^2} + o(\mu^3), \tag{47a}$$

$$\mathcal{C} = -2(\gamma_s\mu_s) + i(\gamma_s\mu_s^2) + o(\mu^3), \tag{47b}$$

$$\mathcal{D} = (\gamma_s\mu_s)\left\{\frac{1-\varepsilon^2}{\varepsilon^2}\mathcal{C}_0 - 2i(1+\ln\varepsilon^2)\right\} + 2i(\gamma_s\mu_s\ln\mu_s) - i\mathcal{C}_0(\gamma_s\ln\mu_s) + o(\mu^2), \tag{47c}$$

$$\mathcal{E} = -4i(\gamma_s\mu_s)(1+\ln\varepsilon) + 4i(\gamma_s\mu_s\ln\mu_s) + o(\mu^2), \tag{47d}$$

$$\mathcal{F} = 4i(\gamma_s\mu_s)\ln\varepsilon^2 + 2\ln\varepsilon^2(\gamma_s\ln\mu_s) - 4i(\gamma_s\mu_s\ln\mu_s) + o(\mu^2). \tag{47e}$$

Substituting (47a), (47b) in (28) and (29), we have

$$Q_2 = \mathcal{A}\{2(1-\beta^2\varepsilon^2) + \pi\alpha Z\beta\varepsilon(1-\varepsilon)\} + 2\alpha Z(\gamma_s\mu_s) \times\left\{2\frac{1-\beta^2}{\beta} + \pi\alpha Z(1-\varepsilon^2)\left[\frac{1-\varepsilon}{\varepsilon} + (1-\beta^2)\frac{\varepsilon}{1+\varepsilon}\right]\right\} + o(\mu^2\alpha^2Z^2), \tag{48}$$

$$R_2 = -2\alpha Z\varepsilon\left(\frac{1-\beta^2}{1-\varepsilon^2}\right)^{1/2}\{\mathcal{A}\beta\varepsilon^2\ln\varepsilon + \beta(1-\varepsilon^2)(\gamma_s\mu_s^2) + 2\alpha Z(\gamma_s\mu_s)\ln\varepsilon\} + o(\mu^2\alpha^2Z^2),$$
$$\mathcal{A} = F(q) - 1 = \gamma_s\frac{\varepsilon^2}{\varepsilon^2+\mu_s^2} = -\frac{1}{\varepsilon^2}(\gamma_s\mu_s^2) + o\left(\frac{\mu^4}{\varepsilon^4}\right). \tag{49}$$

Using (4), (44), and I, (11a), (11b), we get

$$\gamma_s\mu_s = \left\langle\frac{1}{r}\right\rangle_e = 1.3\nu, \quad \gamma_s\mu_s^2 = [r\rho_e(r)]_{r\to 0} = 4.5\nu^2, \quad \nu = \frac{m}{2p}\frac{Z^{1/3}}{121}. \tag{50}$$

We note that the terms proportional to $\ln\mu$ in (47a)—(47e) are eliminated in the expressions for the cross section (48) and (49), we also eliminate the terms proportional to $\mu\ln\mu$; this makes it possible to confirm the fact that the expansion $g(\epsilon,\mu)$ in (43) in terms of μ does not contain terms proportional to $\mu\ln\mu$.

In the expansion in (30) in powers of μ, the quantity μ/ϵ appears as an expansion parameter in addition to μ; therefore, (48) and (49) are applicable only for large angles, where $\mu/\epsilon \ll 1$. However, in the zero terms in αZ of (48) and (49), the parameter μ^2/ϵ^2 appears only from the expansion of $\mathcal{A}$. Therefore, we have not expanded $\mathcal{A}$ in terms of μ, assuming that the exact calculation of $\mathcal{A}$ does not present any difficulties. Thanks to this fact, (48) and (49) can be used for not very small energies for $\theta = 60°$ and even 30°.

We note that the first term in (48) and (49) is proportional to μ^2, as a consequence of which the principal contribution will be made by terms proportional to αZ. The last term, which is proportional to $(\alpha Z)^2$, is of the same order as the first.

The quantities $Q = \sigma_{\xi=0}/\sigma_R$ and $S = R/Q$ are plotted in the drawings as a function of the angle for the electron and the positron at various values of the kinetic energy of the electrons W and charge of the nucleus Z [for the positron in (48), (49), only the sign of Z changes but not ν]; Q_2 and R_2 are computed from the formulas (48) and (49). Since terms of the order of $(\mu\alpha Z)^2$ are discarded in (48) and (49), these formulas are more exact than the corresponding formulas I, (31) and I, (32) for Q_1 and R_1. Therefore, we have substituted the exact values of Doggett and Spencer for Q_1 in (26).[8] The values of R_1 and Q_1 in S were determined from I, (31), (32) with account of the expansion of M in terms of ξ because the deviation of the approximate values of I, (31), (32) for $S_1 = R_1/Q_1$ from the exact values is smaller than for Q_1 (see [9]).

5. SCATTERING AT E > 1 MeV

In the case in which the energy of the incident electron E > 1 MeV, the effect of the electron shell can be neglected.[4)] Then the operator γ_S in (6) takes the form

$$\gamma_s = \gamma_\lambda = -\int_0^\infty d\lambda\chi(\lambda); \quad \gamma_\lambda\cdot 1 = -1. \tag{51}$$

If the energy in this case is not very large, so that $2pR \ll 1$, where R is the radius of the nucleus, then it is appropriate to expand Eq. (30) in inverse powers of μ. This leads to an expansion of the cross section in the quantity $X_n = \langle(2pR)^n\rangle/(n+1)!$ on the basis of (4). The terms of (10) which are proportional to $(\alpha Z)^3$ are shown in this case to be of the order of the neglected terms; therefore we shall not take them into account and shall write (29) and (28) up to terms of order αZ [5)]

[4)]As is well known, for a kinetic energy of the electrons $W = E - m$ lying in the range 0.2 MeV < W < 5 MeV, the effect of the electron shell and the finite dimensions of the nucleus does not exceed 1%. This also follows from Eqs. (48), (49), and (52), (53).

[5)]The functions $K_4(\mu,\nu)$ and $K_5(\mu,\nu)$ in (30) are not represented in the form $\Sigma_{m,n} a_{m,n}\mu^{-m}\nu^{-n}$ in the expansions in terms of $1/\mu$ and $1/\nu$; however, as a consequence of the fact that the principal contribution in the interval (51) is the region close to $\lambda \sim 1/R$, one can write approximately $\gamma_\mu\gamma_\nu K(\mu,\nu) = \gamma_\mu\gamma_\nu K(\mu,\mu) = -\gamma_\mu K(\mu,\mu)$. We also note that the term of (52) containing X_2 was obtained by Lewis.[3]

$$Q_2 = -\varepsilon^2 X_2\{2(1-\beta^2\varepsilon^2) + \pi\alpha Z\beta\varepsilon(1-\varepsilon)\} + 5\alpha Z\varepsilon^2 X_3\left[(1-\beta^2\varepsilon^2)/\beta - \tfrac{1}{3}\beta(1-\varepsilon^2)\right]; \quad (52)$$

$$R_2 = -2\alpha Z\beta\varepsilon^3\left(\frac{1-\beta^2}{1-\varepsilon^2}\right)^{1/2} X_2\{1-\varepsilon^2(1+\ln\varepsilon)\},$$

$$X_n = \frac{\langle(2pR)^n\rangle}{(n+1)!}. \quad (53)$$

For 2pR > 1, it is again possible to make an expansion in μ. However, in this case, it is convenient not to separate the Coulomb term in (5). The potential energy of the nucleus in this case is described in the following fashion:

$$V(q) \equiv V = \Gamma_\lambda V(\lambda), \quad \Gamma_\lambda = \Gamma_\mu = \int_0^\infty d\lambda\{\delta(\lambda) - \chi_{\text{nuc}}(\lambda)\}, \quad (54)$$

$$\Gamma_\mu F(0) = \int_0^\infty d\lambda\{\delta(\lambda) - \chi_{\text{nuc}}(\lambda)\}F(0) = 0. \quad (55)$$

We represent the amplitude of the scattering (8) in the form of a Born series in powers of (54):

$$f(\mathbf{k}, \mathbf{p}) = 2\pi^2\langle\mathbf{k}|\hat{V}|\psi_p\rangle = 2\pi^2\{\langle\mathbf{k}|\hat{V}|\mathbf{p}\rangle + \langle\mathbf{k}|\hat{V}\hat{G}\hat{V}|\mathbf{p}\rangle + \langle\mathbf{k}|\hat{V}\hat{G}\hat{V}\hat{G}\hat{V}|\mathbf{p}\rangle + \ldots\}. \quad (56)$$

By virtue of (55) [see (46)], each term of (56) is seen to be of the order $(\mu\alpha Z)^n$, where n is of the order of the number of terms in the series. The last term expressed in (56) is proportional to $(\mu\alpha Z)^3$, and we shall not take it into consideration. Using the expansion (A.8) and (A.9) in μ, and neglecting m/E in comparison with unity, while taking Eq. (4) into account, we obtain the following expression for the scattering amplitude (26):[6)]

$$\sigma(\vartheta) = \sigma_R(1-\varepsilon)^2\left\{[r\rho_{\text{nuc}}(r)|_{r\to 0}]\frac{1}{4p^2}\right\}\left(1 - 8\alpha Z\left\langle\frac{1}{2pR}\right\rangle\right). \quad (57)$$

We note that if the charge distribution in the nucleus is described by the Fermi function, which is practically constant close to zero, then the contribution to the amplitude will be given only by terms containing μ and μ^3, because on the basis of (4) we have

$$\Gamma_\mu\mu = -\left\langle\frac{1}{2pR}\right\rangle, \quad (2p)^3\Gamma_\mu\mu^3 = \frac{\partial}{\partial r}r\rho_{\text{nuc}}(r)|_{r\to 0} \approx \rho_{\text{nuc}}(0),$$

$$(2p)^{n+2}\Gamma_\mu\mu^{n+2} = (-1)^n\left\{r\frac{\partial^n\rho_{\text{nuc}}}{\partial r^n} + n\frac{\partial^{n-1}\rho_{\text{nuc}}}{\partial r^{n-1}}\right\}_{r\to 0} \to 0 \quad (n \neq -1.1). \quad (58)$$

Because of the absence in (A.8) and (A.9) of the terms of the expansion containing $\mu_S\mu_T$, $\mu_S\mu_T(\mu_S^2 + \mu_T^2)$ and $\mu_S^3\mu_T^3$, the contribution to the amplitude in this case can be made only by the third and subsequent terms in (56). In principle, this fact can serve as a proof of the validity of the particular model of the nucleus.

6. CONCLUSIONS

Application of the Laplace transform has made it possible to separate the small parameters μ_S in explicit form. Expansion in terms of these parameters has made the resultant formulas (48), (49), (52), (53), and (57) simple and convenient for computation. The fundamental difficulty in the method under consideration is the appearance of logarithmic terms ($\ln\mu$) which at first glance make a meaningless expression for the scattering amplitude. However, as was shown in Sec. 3, these terms group themselves into a phase factor and do not enter into the scattering cross section.

It is evident from Eqs. (48) and (49) and Figs. 1 and 2 that the screening effect at not very small Z increases the scattering cross section for electrons somewhat and decreases it for positrons. This fact is in agreement with the result of the researches of Mitra and Tietz.[1]

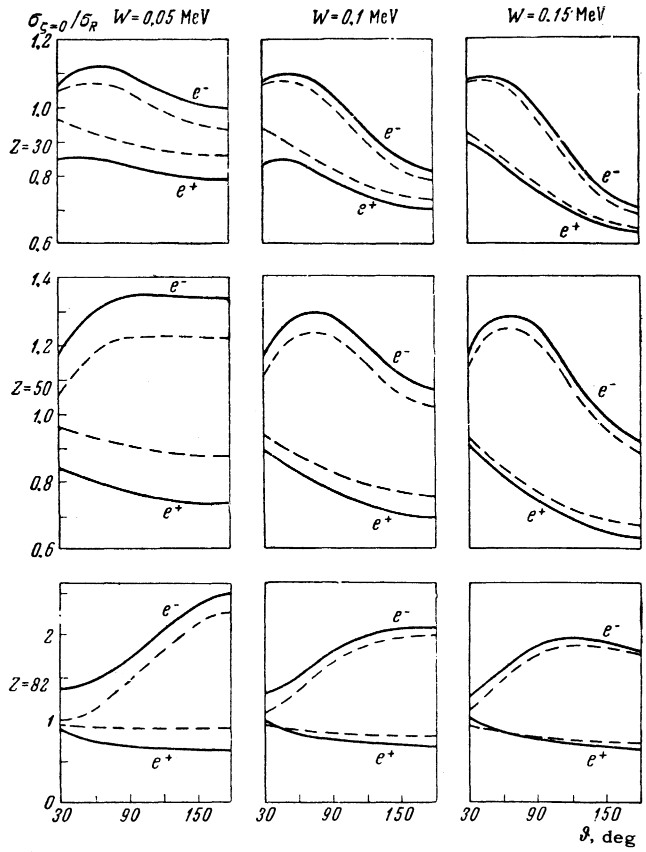

FIG. 1. Angular dependence of the quantity $Q(\vartheta) = Q_1(\vartheta) + Q_2(\vartheta)$ for electrons and protons. The dashed curves give $Q_1(\vartheta)$, constructed from the results of Doggett and Spenser.[8]

6)Equation (57) was obtained in a somewhat different form by Lewis.[3]

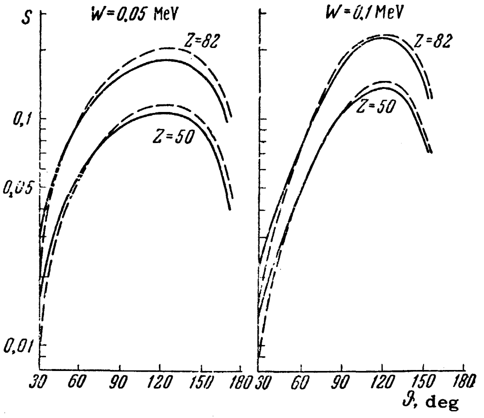

FIG. 2. Angular dependence of the quantity $S(\vartheta) = [R_1(\vartheta) + R_2(\vartheta)]/[Q_1(\vartheta) + Q_2(\vartheta)]$. The dashed curves give $S_1(\vartheta) = R_1(\vartheta)/Q_1(\vartheta)$. The quantities $R_1(\vartheta)$ and $Q_1(\vartheta)$ are calculated from Eqs. I, (31) and I, (32).

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APPENDIX

The last term in (10) has the form

$$S_5 = 2\pi^2 \langle \varphi_k^0 | \hat{V}(\lambda_1) G\hat{V}(\lambda_2) | \varphi_p^0 \rangle$$

$$= 2\pi^2 \langle \mathbf{k} | \hat{V}(\lambda_1) G\hat{V}(\lambda_2) | \mathbf{p} \rangle + o(V_1 V_2^2)$$

$$= \frac{(\alpha Z)^2}{2\pi^2} \int \frac{\gamma_4 (i\hat{s} - m)\gamma_4 d^3 s}{(\mathbf{q}_{ks}^2 + \lambda_1^2)(s^2 - p^2 - i\varepsilon)(\mathbf{q}_{sp}^2 + \lambda_2^2)};$$

$$\mathbf{q}_{fs} = \mathbf{f} - \mathbf{s}. \qquad \text{(A.1)}$$

Recalling that the bispinor u_p stands on the right hand side of all the matrix elements [see (8)], and taking into account the identity

$$(m - i\hat{s})\gamma_4 u_p = (2E + \tilde{q}_{sp}) u_p,$$

we represent (A.1) in the form

$$S_5 = S_5^1 + S_5^2, \quad S_5^1 = -2\pi^2 \langle \mathbf{k} | \hat{V}(\lambda_1) G_0 \hat{V}(\lambda_2) | \mathbf{p} \rangle,$$

$$S_5^2 = -2\pi^2 \langle \mathbf{k} | \hat{V}(\lambda_1) G_{\mathrm{I}} \hat{V}(\lambda_2) | \mathbf{p} \rangle; \qquad \text{(A.2)}$$

$$G_0 = 2E/(s^2 - p^2 - i\varepsilon) \qquad G_{\mathrm{I}} = \tilde{q}_{sp}/(s^2 - p^2 - i\varepsilon). \qquad \text{(A.3)}$$

By using Eq. II, (3), we get

$$S_5^1 = -\gamma_4 (\alpha Z)^2 iE \int_0^1 \frac{dy}{\Lambda} \frac{1}{(\mathbf{k} - \mathbf{B})^2 - (\Lambda + i\lambda_1)^2}, \qquad \text{(A.4)}$$

$$S_5^2 = -\gamma_4 (\alpha Z)^2 \frac{1}{2} \int_0^1 dy \left(\tilde{\nabla}_B \int_{\lambda_1}^{\infty} d\eta - i \frac{\tilde{p}y}{\Lambda} \bigg|_{\eta = \lambda_1} \right) \qquad \text{(A.5)}$$

$$\times \frac{1}{(\mathbf{k} - \mathbf{B})^2 - (\Lambda + i\eta)^2},$$

$$\Lambda^2 = p^2 y^2 - \lambda_2^2 (1 - y), \qquad \mathbf{B} = \mathbf{p} - \mathbf{p}y. \qquad \text{(A.6)}$$

Integration over y in (A.4) and (A.5) is easily carried out by means of the substitution $\Lambda + py = 2pt$. By then calculating the gradient and the integral over η in (A.5), and taking I, (21a) into account, we get

$$S_5^1 = -\gamma_4 \frac{(\alpha Z)^2}{q^2} \frac{E}{p} K_4(\mu_1, \mu_2),$$

$$S_5^2 = -\gamma_4 \frac{(\alpha Z)^2}{q^2} \frac{\tilde{p}}{p} K_5(\mu_1, \mu_2); \qquad \text{(A.7)}$$

$$K_4(\mu_1, \mu_2) = \frac{i}{P} \ln \frac{2\varepsilon\alpha\beta(\alpha + \beta) - i\{1 + (\alpha + \beta)^2 + P\}}{2\varepsilon\alpha\beta(\alpha + \beta) - i\{1 + (\alpha + \beta)^2 - P\}} = i \ln \varepsilon^2$$

$$- i(\ln \mu_1 + \ln \mu_2) + \mu_1 + \mu_2 + i \frac{2 - \varepsilon^2}{2\varepsilon^2} (\mu_1^2 + \mu_2^2)$$

$$- \frac{3 + \varepsilon^2}{3\varepsilon^2} (\mu_1^3 + \mu_2^3) - \frac{3}{\varepsilon^2} \mu_1 \mu_2 (\mu_1 + \mu_2) + o(\mu^4); \qquad \text{(A.8)}$$

$$K_5(\mu_1, \mu_2) = \frac{1}{2} \frac{\varepsilon^2}{1 - \varepsilon^2} \Big\{ (1 + \alpha^2 + \beta^2) K_4(\mu_1, \mu_2)$$

$$+ i\left(\frac{1}{\varepsilon} \ln \frac{1 + i\alpha + i\beta}{-1 + i\alpha + i\beta} - \ln \frac{1 + i\varepsilon\alpha}{i\varepsilon\alpha} - \ln \frac{1 + i\varepsilon\beta}{i\varepsilon\beta} \right) \Big\}$$

$$= K_1(0) - \mu_1 - \mu_2 + \frac{i}{2} (\mu_1^2 + \mu_2^2) + \frac{1}{3} \frac{1 + \varepsilon^2}{\varepsilon^2} (\mu_1^3 + \mu_2^3)$$

$$+ \frac{1}{\varepsilon^2} \mu_1 \mu_2 (\mu_1 + \mu_2) + o(\mu^4), \qquad \text{(A.9)}$$

where

$$P = \{(1 + \alpha^2 + \beta^2)^2 - 4\alpha^2\beta^2(1 - \varepsilon^2)\}^{1/2},$$

$$\alpha = \mu_1/\varepsilon, \quad \beta = \mu_2/\varepsilon,$$

$$\varepsilon = q/2p, \qquad \mu = \lambda/2p, \qquad \mathbf{q} = \mathbf{k} - \mathbf{p}. \qquad \text{(A.10)}$$

Equations (A.8) and (A.9) were obtained by another method.[3] However, we need the expressions (A.4) and (A.5) for calculation of the matrix element in (10). In this case, we initially consider the following quantity:

$$S(\lambda_1, \lambda_2) = -2\pi^2 \langle \varphi_k^{\mathrm{I}} | \hat{V}(\lambda_1) G\hat{V}(\lambda_2) | \varphi_p^0 \rangle$$

$$= S^1(\lambda_1, \lambda_2) + S^2(\lambda_1, \lambda_2), \qquad \text{(A.11)}$$

$$S^1(\lambda_1, \lambda_2) = -\int \langle \varphi_k^{\mathrm{I}} | \mathbf{f} \rangle 2\pi^2 \langle \mathbf{f} | \hat{V}(\lambda_1) G_0 V(\lambda_2) | \mathbf{p} \rangle d^3 f$$

$$+ o(\alpha^4 Z^4), \qquad \text{(A.12)}$$

$$S^2(\lambda_1, \lambda_2) = -\int \langle \varphi_k^{\mathrm{I}} | \mathbf{f} \rangle 2\pi^2 \langle \mathbf{f} | \hat{V}(\lambda_1) G_{\mathrm{I}} V(\lambda_2) | \mathbf{p} \rangle + o(\alpha^4 Z^4). \qquad \text{(A.13)}$$

Substituting the functions (A.12) and (A.13) in the integrand in place of the second factor of the expression (A.4) and (A.5), and using $\langle \varphi_k^{\mathrm{I}} | \mathbf{f} \rangle$ in I, (17b) in explicit form, we carry out the integration over f by means of I, (A.1). We then get

$$S^1(\lambda_1, \lambda_2) = i\gamma_4 (\alpha Z)^3 E \int_{\lambda_1}^{\infty} d\eta \int_0^1 dy \frac{1}{\Lambda} \tilde{J}(\eta). \qquad \text{(A.14)}$$

$$S^2(\lambda_1, \lambda_2) = \gamma_4 (\alpha Z)^3 \int_{\lambda_1}^{\infty} d\eta \frac{1}{2} \int_0^1 dy \tilde{J}(\eta')$$

$$\times \left(\overleftrightarrow{\tilde{\nabla}}_B \int_{\eta}^{\infty} d\eta' - i\tilde{p} \frac{y}{\Lambda} \bigg|_{\eta' = \eta} \right). \qquad \text{(A.15)}$$

$$\mathbf{J}(\eta)=\frac{\nabla_k}{-2i\xi}\frac{1}{2\pi i}\oint^{(0+1+)}\frac{dx}{x}\left(\frac{-x}{1-x}\right)^{i\xi}$$

$$\times\frac{1}{(\mathbf{n}-\mathbf{B}-\mathbf{k}x)^2-(kx+\Lambda+i\eta)^2}\Big|_{\mathbf{n}=\mathbf{k}}$$

$$=\frac{\mathbf{k}-\mathbf{B}+\mathbf{k}(\Lambda+i\eta)/k}{[(\mathbf{k}-\mathbf{B})^2-(\Lambda+i\eta)^2][B^2-(k+\Lambda+i\eta)^2]}. \quad (A.16)$$

In (A.15) the operator with the arrow pointing to the left acts on the left. By means of a direct verification, we can establish the fact that

$$\tilde{J}(\eta)\overleftarrow{\nabla}_B=\frac{\partial}{\partial i\eta}\frac{\tilde{k}}{k}\tilde{J}(\eta),$$

therefore

$$S^2(\lambda_1,\lambda_2)=\gamma_4(\alpha Z)^3 i\int_{\lambda_1}^{\infty}d\eta\frac{1}{2}\int_0^1 dy\left\{\frac{\tilde{k}}{k}\tilde{J}(\eta)-\frac{y}{\Lambda}\tilde{J}(\eta)\tilde{p}\right\}. \quad (A.17)$$

The integral over y in (A.14) and (A.17) is easily carried out with the aid of the latter substitution. The integral over η can be expressed in terms of elementary functions only after expansion of the integrand function in a series in λ_2. Discarding terms of the order λ_2^2, we get, by using I, (21a):

$$S^1(\lambda_1,\lambda_2)=\gamma_4\frac{(\alpha Z)^3}{q^2}\frac{E}{p}\frac{\tilde{p}}{p}$$

$$\times\int_{\mu_1}^{\infty}d\eta\left\{\frac{1}{(1+i\eta)(\varepsilon^2+\eta^2)}\left(\ln\frac{\varepsilon^2+\eta^2}{-i\eta(1+i\eta)}-\ln i\mu_2\right)\right.$$

$$\left.-i\mu_2\frac{2\varepsilon^2+i\eta(\varepsilon^2-\eta^2)}{(\varepsilon^2+\eta^2)^2(1+i\eta)^2}\right\}+o(\mu_2^2), \quad (A.18)$$

$$S^2(\lambda_1,\lambda_2)=\gamma_4\frac{(\alpha Z)^3}{q^2}\int_{\mu_1}^{\infty}d\eta\left\{\left[\frac{i}{1+i\eta}\ln\frac{\varepsilon^2+\eta^2}{\eta^2}-\frac{i}{1+i\eta}\ln\frac{\eta+\mu_2}{\eta}\right.\right.$$

$$\left.+\mu_2\frac{\varepsilon^2+i\eta}{(\varepsilon^2+\eta^2)(1+i\eta)}\right]\frac{1-\tilde{k}\tilde{p}/kp}{2}$$

$$\left.-\mu_2\frac{\varepsilon^2}{(\varepsilon^2+\eta^2)(1+i\eta)}\frac{1+\tilde{k}\tilde{p}/kp}{2}\right\}+o(\mu_2^2). \quad (A.19)$$

Further, noting that

$$S_3=\langle\varphi_k^{\mathrm{I}}|\hat{V}(\lambda)|\varphi_p^{\mathrm{I}}\rangle=-\langle\varphi_k^{\mathrm{I}}|\hat{V}(\lambda)G_{\mathrm{I}}V(0)|\mathbf{p}\rangle$$

$$+o(\alpha^4Z^4)=S^2(\lambda,0), \quad (A.20)$$

$$S_4=\langle\varphi_k^{\mathrm{I}}|\hat{V}_1G\hat{V}(\lambda)|\varphi_p^0\rangle=-\langle\varphi_k^{\mathrm{I}}|\hat{V}(0)G\hat{V}(\lambda)|\mathbf{p}\rangle$$

$$+o(\alpha^4Z^4)=S(0,\lambda) \quad (A.21)$$

and taking into account the identity I, (21a), (21b), we get

$$S_3=\gamma_4\frac{(\alpha Z)^3}{q^2}\frac{m(\gamma_4E-m)}{p^2}K_3(\mu), \quad (A.22)$$

$$K_3(\mu)=i\int_{\mu}^{\infty}\frac{d\eta}{1+i\eta}\ln\frac{\varepsilon^2+\eta^2}{\eta^2}=K_3(0)$$

$$+2i\mu(\ln\mu-\ln\varepsilon-1)+o(\mu^2), \quad (A.23)$$

$$S_4=\gamma_4\frac{(\alpha Z)^3}{q^2}\left\{\frac{E}{p}\frac{E-\gamma_4m}{p_4}K_2^1(\mu)+\frac{m(\gamma_4E-m)}{p^2}K_3^1(\mu)\right\}, \quad (A.24)$$

$$K_2^1(\mu)=K_2(0)+\frac{\pi}{2}K_1(0)$$

$$-iK_1(0)\ln\mu+\mu\frac{1-\varepsilon^2}{\varepsilon^2}K_1(0)+o(\mu^2), \quad (A.25)$$

$$K_3^1(\mu)=K_3(0)+i\mu(\ln\mu-\ln\varepsilon-1)+o(\mu^2), \quad (A.26)$$

$$K_1(\mu)=\int_{\mu}^{\infty}\frac{\varepsilon^2d\eta}{(\varepsilon^2+\eta^2)(1+i\eta)}=K_1(0)-\mu+\frac{i}{2}\mu^2+o(\mu^3), \quad (A.27)$$

$$K_1(0)\equiv\mathscr{C}_0=\frac{\pi}{2}\frac{\varepsilon}{1+\varepsilon}+i\frac{\varepsilon^2}{1-\varepsilon^2}\ln\varepsilon, \quad (A.28)$$

$$K_2(\mu)=i\int_{\mu}^{\infty}\frac{\varepsilon^2d\eta}{(\varepsilon^2+\eta^2)(1+i\eta)}\ln\frac{\varepsilon^2+\eta^2}{-i\eta(1+i\eta)}$$

$$=K_2(0)+\mu\frac{\pi}{2}+i\mu(\ln\mu-\ln\varepsilon^2-1)+o(\mu^2). \quad (A.29)$$

The exact values of (A.27) and (A.29) were obtained in I and are given by the formulas I, (24) and I, (25).

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