THE ROLE OF SHORT-RANGE THREE-PARTICLE FORCES IN PROCESSES WITH THE GENERATION OF THREE PARTICLES NEAR THRESHOLD

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The role of three-particle short-range forces is considered in reactions with the formation of three particles near the threshold of the reaction on the condition that the pair forces are also short-range. It is shown that when all pair amplitudes of zero-energy are $a_{ik} \sim r_0$, where r_0 is the radius of action of the forces, or when one of the amplitudes, for instance $|a_{23}| \geq r_0$, whereas $a_{12} \sim r_0$ and $a_{13} \sim r_0$, the three-particle forces can, under certain conditions, appreciably alter the energy dependence of the amplitude of the reaction.

INTRODUCTION

THE three body problem with short-range forces has been considered up to now under the assumption that the principal role is played by pair interactions between particles [1-3]. The contribution due to particle interaction in the region where the distances between all the particles are on the order of the effective range of the forces was usually neglected, owing to the small dimensions of this region. It will be shown in the present paper, however, that three-particle forces can lead to the presence of a pole in the scattering amplitude, regarded as an analytic function of the system energy E, near $E = 0$. In this case the three-particle forces play an essential role and can change appreciably the energy dependence of the scatteri ng amplitude. The pole of the amplitude can lie either on the physical sheet at $E < 0$ or on the non-physical sheet. In the former case it corresponds obviously to the bound three-particle state. We consider two cases in this paper.

In the first section we determine the energy dependence of the reaction $A + B \rightarrow A' + B' + C$ under the condition that the pair amplitudes at zero energy are $a_{ik} \sim r_0$, where r_0 is the effective range of the forces. It is shown that three-particle forces with range $R_0 \lesssim r_0$ play an important role if $a(0) \gg r_0^4$, where $a(0)$ is the amplitude of the transformation of three particles into three at zero energy [the normalization of $a(0)$ will be defined below].

Three-particle forces lead to a narrow resonance of width on the order of $\Delta E \sim R_0^2\hbar^2/Ma(0)$, where M is the sum of the particle mass, in the amplitude of the reaction that occurs with production of three particles. In the resonance region, the pair interactions can be neglected. The scattering amplitude of the three particles can be expressed in this region in terms of three arbitrary parameters: the three-particle amplitude $a(0)$ and the two radii r_1 and r_2, of which one is contained in the answer logarithmically. Three-particle forces can also lead to the existence of a bound state of the three particles.

In the second section we consider the case when one of the pair amplitudes, say a_{23} is much larger than r_0, with $a_{12} \sim r_0$ and $a_{13} \sim r_0$. Three-particle forces with range $R_0 \lesssim r_0$ are significant if $a(0) \gg r_0^2 a_{23}^2$. The width of the resonance is of order $\Delta E \sim a_{23}^2\hbar^2/Ma(0)$. If we neglect the nonresonant pair interactions of particle 1 with particles 2 and 3, then the three-particle scattering amplitude can be expressed in terms of three parameters, namely $a(0)$, a_{23}, and a certain length $\rho^2 \sim a_{23}^2 \times \ln^{-1}(a_{23}/r_0)^2$. In the present work we determine the energy dependence in the reaction $A + B \rightarrow A' + B' + C$ and upon scattering of particle 1 by the bound state of particles 2 and 3.

We plan to consider in a future paper the role of three-particle forces under the condition that all paired amplitudes $a_{ik} \gg r_0$.

1. ROLE OF THREE PARTICLE FORCES WHEN $a_{ik} \sim r_0$

We consider a reaction with production of three particles, for example the reaction of the type

$$A + B \rightarrow A' + B' + C \tag{1}$$

near threshold. The spins of the particles will for simplicity be assumed equal to zero. Reactions of this type were considered in several papers [1,2] under the assumption that the three-particle interactions can be neglected. In the present section we consider the case when the three-particle forces play an appreciable role, so that $a(0) \gg r_0^4$, in spite of the fact that $a_{ik} \sim r_0$. We shall assume that the range of the three-particle interaction is $R_0 \lesssim r_0$.

If the particles A′, B′, and C are produced in a small volume $\sim r_0^3$, then the dependence of the amplitude of reaction (1) on the energy and on the other quantum numbers of the final state is determined, generally speaking, by the wave function of the particles A′, B′, and C in a region where the distances between the particles are $\lesssim r_0$ [1]. In this region the wave functions for different energies are proportional to one another. In the absence of three particle forces the proportionality of the wave functions was proved by Gribov [1]. In the case of interest to us this statement will be proved below. Therefore the amplitude of reaction (1) $\langle A'B'C \mid AB \rangle_E$ near threshold can be written in the form

$$\langle A'B'C|AB\rangle_E = A(E) \langle A'B'C|AB\rangle_0, \tag{2}$$

where $\langle A'B'C \mid AB\rangle_0$ is the amplitude at threshold energy.

The quantity A(E) depends on the energy and the other quantum numbers of the final state and is the complex conjugate of the factor of proportionality between wave functions whose asymptotic form is a plane incident wave plus a convergent one, and which are determined for a complex-conjugate potential (if the interaction potential between the particles A′, B′, and C is complex, that is, in the case of decay particles).

Starting from the equations written down below for the wave function of the particles A′, B′, and C, it is easy to verify that in all the cases of interest to us the quantity A(E) simply coincides with the coefficient of the proportionality of the wave functions with asymptotic form consisting of a plane incident plus diverging wave.

In determining the wave function of the particles A′, B′, and C we neglect the contribution from the pair interactions. (The conditions under which this approximation is valid will be obtained below.) Then the energy dependence of the amplitude of reaction (1), and consequently also of the quantity A(E), can be readily obtained by summing the diagrams shown in the figure.

For what follows, however, it is more convenient to obtain A(E) directly from the Schrödinger equation. In the c.m.s. of particles A′, B′, and C the wave function is a function of $\mathbf{r}_{23}$ and $\boldsymbol{\rho}_1$, where

$$\mathbf{r}_{23} = \mathbf{r}_2 - \mathbf{r}_3, \ \boldsymbol{\rho}_1 = \mathbf{r}_1 - (m_2\mathbf{r}_2 + m_3\mathbf{r}_3) / (m_2 + m_3),$$

with $\mathbf{r}_j$ and m_j the coordinate and mass of the j-th particle. For brevity we write the wave function in the form $\Psi_E(\mathbf{R})$, where $\mathbf{R}$ is the radius vector in six dimensional space constructed on the vectors $\sqrt{\mu_{23}/M}\,\mathbf{r}_{23}$ and $\sqrt{\mu_1/M}\,\boldsymbol{\rho}_1$, where

$$\mu_{23}^{-1} = m_2^{-1} + m_3^{-1}, \quad \mu_1^{-1} = m_1^{-1} + (m_2 + m_3)^{-1},$$
$$M = m_1 + m_2 + m_3.$$

The wave function $\Psi_E(\mathbf{R})$ satisfies the equation [1] ($\hbar = 1$)

$$\Psi_E(\mathbf{R}) = e^{i\mathbf{k}\mathbf{R}} + \int G_E(|\mathbf{R} - \mathbf{R}'|) V(\mathbf{R}') \Psi_E(\mathbf{R}') d^6R'. \tag{3}$$

Here $\mathbf{k}$ is a six-vector ($\sqrt{M/\mu_{23}}\,\mathbf{k}_{23}$, $\sqrt{M/\mu_1}\,\mathbf{k}_1$, where $\mathbf{k}_{23}$ is the wave vector of particles 2 and 3 in their center-of-mass system, and $\mathbf{k}_1$ is the wave vector of particle 1 relative to the center of mass of particles 2 and 3.) The quantities k_{23}^2 and k_1^2 are related by the equation $k_{23}^2/2\mu_{23} + k_1^2/2\mu_1 = E$. The function $V(\mathbf{R})$ in formula (3) is the particle interaction potential, while $G_E(R)$ is the Green's function

$$G_E(R) = -i\frac{M^2E}{8\pi^2}\gamma\frac{H_2^{(1)}(\sqrt{2ME}\,R)}{R^2}, \tag{4}$$

where $H_2^{(1)}(z)$ is the Hankel function of the first kind and $\gamma = (m_1m_2m_3/M^3)^{3/2}$.

We write the transition amplitude $\langle \mathbf{k}_{23}', \mathbf{k}_1' | \mathbf{k}_{23}, \mathbf{k}_1\rangle$ in the following fashion:

$$\langle \mathbf{k}_{23}', \mathbf{k}_1' | \mathbf{k}_{23}, \mathbf{k}_1\rangle = iM^{-1}a(\mathbf{k}_{23}', \mathbf{k}_1', \mathbf{k}_{23}, \mathbf{k}_1)\,\delta(E - E').$$

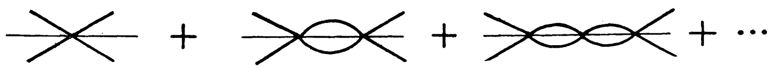
The number of transitions dN per unit time in a unit volume, under the condition that the initial particle densities are equal to unity, is determined by the formula

$$dN = (2\pi)^{-1}\frac{|a(\mathbf{k}_{23}', \mathbf{k}_1', \mathbf{k}_{23}, \mathbf{k}_1)|^2}{M^2}\delta(E - E')\frac{d^3k_{23}'}{(2\pi)^3}\frac{d^3k_1'}{(2\pi)^3}. \tag{5}$$

From an examination of the diagrams describing the scattering of the three particles, it follows that the amplitude $a(\mathbf{k}_{23}', \mathbf{k}_1', \mathbf{k}_{23}, \mathbf{k}_1)$ is expressed in terms of $\Psi_E(\mathbf{R})$ by means of the formula

$$a(\mathbf{k}_{23}', \mathbf{k}_1', \mathbf{k}_{23}, \mathbf{k}_1) = -\int e^{i\mathbf{k}'\mathbf{R}} V(\mathbf{R})\Psi_E(\mathbf{R})\,d^6R. \tag{6}$$

We shall not prove formula (6) here, since the exact normalization of the quantity $a(\mathbf{k}_{23}', \mathbf{k}_1', \mathbf{k}_{23}, \mathbf{k}_1)$ is not very important to us.



We solve Eq. (3) under the assumption that the pair interaction can be neglected. Then the potential $V(\mathbf{R})$ differs from zero only when $R \lesssim R_0 \sim r_0$. Therefore when $k_{ij}r_0 \ll 1$ the free term in (3) can be replaced by unity, and the Green's function (4) can be expanded in its argument up to terms that tend to zero as $k_{ij}r_0 \to 0$. The equation obtained in this manner can be solved by the method developed in detail in [4]. We do not present the corresponding derivations here, since they are completely analogous to those given earlier [4], except that now the solution must be sought in the form of a series in the eigenfunctions of the equation

$$\varphi_n(\mathbf{R}) = \lambda_n \int G_0(|\mathbf{R} - \mathbf{R}'|)\, V(\mathbf{R}')\, \varphi_n(\mathbf{R}')\, d^6R'. \quad (7)$$

Just as before [4], it turns out that the inequality $a(0) \gg r_0^4$ can be satisfied only if one of the values $\lambda_n = \lambda$ is close to unity: $|\lambda - 1| \sim r_0^4/a(0)$. Then, accurate to terms $\sim k_{ij}^2 r_0^2$ the wave functions for different energies in the region $R \lesssim R_0$ are proportional to one another:

$$\Psi_E(\mathbf{R}) = A(E)\, \Psi_0(\mathbf{R}); \quad (8)$$

$$A(E) = [1 - a(0)\, \zeta(E)]^{-1}, \quad (9)$$

$$\zeta(E) = -\frac{ME\gamma}{8\pi^2}\left[-\frac{8\pi^2}{r_1^2} + \frac{ME}{2\pi}\left(\ln\frac{\sqrt{2ME}\, r_2}{2} - \frac{\pi i}{2}\right)\right]. \quad (10)$$

Here r_1 and $r_2 > 0$ are real parameters $\sim r_0$.

The amplitude $a(\mathbf{k}_{23}', \mathbf{k}_1', \mathbf{k}_{23}, \mathbf{k}_1)$ can be determined from formula (6), in which we substitute expression (8) and replace the exponential under the integral sign by unity. Then $a(\mathbf{k}_{23}', \mathbf{k}_1', \mathbf{k}_{23}, \mathbf{k}_1)$ turns out to depend only on E: $a(\mathbf{k}_{23}', \mathbf{k}_1', \mathbf{k}_{23}, \mathbf{k}_1) \equiv a(E)$, and its value is

$$a(E) = A(E)\, a(0). \quad (11)$$

From the results [5] concerning the connection between the number of zeros and poles of the amplitude it follows that on the physical sheet $0 \le \arg E \le 2\pi$ the amplitude has three poles if $a(0) < 0$ and two poles when $a(0) > 0$.

The position of the pole at the energy $E = E_0$, satisfying the condition $ME_0R_0^2 \ll 1$, can be readily determined from formulas (9) and (10), if it is noted that under this condition the first term in formula (10) is much larger than all the remaining terms. Neglecting these terms, we obtain an equation for E_0:

$$1 - ME_0\gamma r_1^{-2}\, a(0) = 0, \quad (12)$$

from which it follows that

$$E_0 \approx r_1^2/M\gamma a(0). \quad (13)$$

If $a(0) < 0$, then also $E_0 < 0$. In this case there exists a bound state of the three particles, with energy E_0. The remaining two poles for $a(0) < 0$ obviously lie outside the region where the theory is valid. It follows from (11), (10), and (9) that in the case when $a(0) < 0$ the amplitude $a(E) \approx a(0)$ if $E \lesssim -r_1^2/M\gamma a(0)$ and $a(E) \approx r_1^2/M\gamma E \ll a(0)$ when $E \gg -r_1^2/M\gamma a(0)_1$. Thus, in the case when $a(0) < 0$ the three-particle forces lead to a narrow resonance in the amplitude, the width of which is

$$\Delta E \sim |r_1^2/M\gamma a(0)|, \quad (14)$$

whereas in the region of resonance $a(E) \approx a(0)$. The three particle forces lead also to the existence of the bound state of the three particles, with energy E_0, given by formula (13). When $a(0) > 0$, it follows from (13) that $E_0 > 0$. However, in the region $E > 0$ there can be no strictly real poles in $a(E)$ and consequently E_0 should have a small imaginary part. The magnitude of this imaginary part can be determined by substituting in (9) for E_0 the quantity $E_0 = |E_0|\, E^{i\varphi}$, where $|E_0|$ is determined by formula (13), and the argument φ is a small quantity. It is found then that

$$\varphi \approx -r_1^4/\gamma a(0). \quad (15)$$

Since $\varphi < 0$, the pole E_0 for $a(0) > 0$ lies on the nonphysical sheet, because $0 \le \varphi \le 2\pi$ on the physical sheet. On the physical sheet, in the case $a(0) > 0$, the amplitude $a(E)$ has no poles in the region where the theory is valid.

When $E = |E_0|$ in the case when $a(0) > 0$ the amplitude $a(E)$, as follows from (11), (10), (9), and (13), has an order of magnitude

$$a(|E_0|) \sim a(0)\, (a(0)/r_1^4) \gg a(0), \quad (16)$$

and the width of the interval ΔE, in which relation (16) holds true is given by the formula

$$\Delta E \sim r_1^6/M\gamma^2 a^2(0). \quad (17)$$

Thus, when $a(0) > 0$ there is no three-particle bound state, and the amplitude $a(E)$ depends on the energy as follows:

in the region where

$$|1 - M\gamma Ea(0)/r_1^2| \ll 1, \qquad |a(E)| \gg a(0);$$

in the region where

$$|1 - M\gamma a(0)/r_1^2| \sim 1, \quad a(E) \sim a(0);$$

finally, when

$$M\gamma Ea(0)/r_1^2 \gg 1, \quad a(E) \sim r_1^2 (M\gamma E)^{-1} \ll a(0).$$

Let us ascertain now at what energies we can neglect the contribution of the pair interactions in the amplitude $a(E)$. The pair interactions lead to terms $\sim k_{ij}a_{ij} \sim \sqrt{ME}\, r_0$ in the scattering amplitude, whereas the triple forces produce a contribution $\sim a(0)MEr_0^{-2}$ in the resonance region. The pair forces can obviously be neglected if $\sqrt{ME}\, r_0 \ll MEa(0)r_0^{-2} \lesssim 1$, from which it follows that

$$r_0^6 a^{-2}(0) M^{-1} \ll E \lesssim r_0^2 a^{-1}(0) M^{-1}. \qquad (18)$$

If $a(0) < 0$, then for $E > 0$ the inclusion of terms $\sim E^2$ in $A(E)$ is obviously in excess of the accuracy, since the contribution from these terms is much smaller in this region than the contribution from the pair interactions. Therefore when $a(0) < 0$ the formula for $A(E)$ in region (18) should be written in the form

$$A(E) = (1 - \gamma a(0) ME/r_1^2)^{-1}. \qquad (19)$$

When $a(0) > 0$ it is meaningful to include the term $\sim E^2$ in formula (10) only in the region where the inequality $|a(E)| \gg a(0)$ holds true. If $a(E) \sim a(0)$, it is necessary to use formula (19) for $A(E)$.

The energy dependence of reaction (1) in the region of the triple-force resonance is determined by formula (2), in which $A(E)$ is determined in turn by formulas (9) and (10) or by formula (19).

The method developed is applicable also when A′, B′, and C are decay particles. Therefore all the results of this section are valid also for decay particles, but the quantities $a(0)$, r_1, and r_2 are in this case, generally speaking, complex.

2. THREE PARTICLE FORCES UNDER THE CONDITION THAT $a_{23} \gg r_0$, $a_{12} \sim r_0$, AND $a_{13} \sim r_0$

Let us assume that $a_{23} \gg r_0$, with $a_{12} \sim r_0$, and $a_{13} \sim r_0$, and let us determine the wave function of the particles A′, B′, and C, neglecting the nonresonant interaction between particle 1 and particles 2 and 3, under the condition that a resonant three-particle interaction with range $R_0 \lesssim r_0$ exists between the particles. For simplicity we shall assume first that A′, B′, and C are not decay particles, although the results of this section, as will be explained below, are valid also for decay particles.

The c.m.s. wave function of the particles satisfies the equation

$$\Psi_E(\mathbf{k}_{23}, \mathbf{k}_1; \mathbf{R}) = \varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23}) e^{i\mathbf{k}_1\rho_1} + \int G_E(\mathbf{R}, \mathbf{R}') V(\mathbf{R}') \Psi_E(\mathbf{k}_{23}, \mathbf{k}_1; \mathbf{R}') d^6R'. \qquad (20)$$

Here $G_E(\mathbf{R}, \mathbf{R}')$ is the Green's function of the Schrödinger equation for three particles with potential $V_{23}(r_{23})$, which is the potential of the interaction between particles 2 and 3, while $\varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23})$ is the wave function of particles 2 and 3 with relative-motion energy $E_{23} = k_{23}^2/2\mu_{23}$, satisfying the equation

$$(-\Delta_{23}/2\mu_{23} + V_{23}(r_{23}) - E_{23})\varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23}) = 0. \qquad (21)$$

For large r_{23} the function $\varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23})$ has an asymptotic form

$$\varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23}) = e^{i\mathbf{k}_{23}\mathbf{r}_{23}} + a_{23}(k_{23}) r_{23}^{-1} e^{ik_{23}r_{23}},$$

with scattering amplitude $a_{23}(k_{23}) = a_{23}(1 - ik_{23}a_{23})^{-1}$.

If $a_{23} < 0$, then particles 2 and 3 can form a bound state. In this case the scattering of particle 1 by the bound state of particles 2 and 3 can be solved by choosing as the function $\varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23})$ the particle bound state function $\varphi_d(\mathbf{r}_{23})$.

The wave function describing the scattering of particle 1 by the bound state of the two others will be denoted by $\Psi_E(\mathbf{k}_1; \mathbf{R})$. From now on, for brevity, we shall denote the functions $\Psi_E(\mathbf{k}_{23}, \mathbf{k}_1; \mathbf{R})$ and $\Psi_E(\mathbf{k}_1; \mathbf{R})$ by $\Psi_E(\mathbf{R})$ whenever no misunderstanding will result.

The Green's function in (20) is

$$G_E(\mathbf{R}, \mathbf{R}') = -\frac{1}{(2\pi)^6} \times \int \frac{\varphi_{\mathbf{k}'_{23}}(\mathbf{r}_{23})\varphi^*_{\mathbf{k}'_{23}}(\mathbf{r}'_{23}) \exp\{i\mathbf{k}'_1(\rho_1 - \rho'_1)\}}{k'^2_{23}/2\mu_{23} + k'^2_1/2\mu_1 - E - i\delta} d^3k'_{23} d^3k'_1 + G_E^{(s)}(\mathbf{R}, \mathbf{R}'), \qquad (22)$$

where the term $G_E^{(s)}(\mathbf{R}, \mathbf{R}')$ appears only in the presence of a bound state of particles 2 and 3:

$$G_E^{(s)}(\mathbf{R}, \mathbf{R}') = -\frac{\theta(-a_{23})}{(2\pi)^3}\varphi_d(r_{23})\varphi_d(r'_{23}) \times \int \frac{\exp\{i\mathbf{k}'_1(\rho_1 - \rho'_1)\} d^3k'_1}{k'^2_1/2\mu_1 - \alpha^2/2\mu_{23} - E - i\delta} \qquad (23)$$

where $-\alpha^2/2\mu_{23}$ is the energy of the bound state $\alpha = a_{23}^{-1}$.

Equation (20) can be solved by the method detailed in [4]. It turns out here that the wave functions at different energies and at different values of $\mathbf{k}_{23}$ and $\mathbf{k}_1$ are proportional to one another in the range of the forces. We assume this fact here without proof, since it is quite natural, and determine the proportionality factor directly from the condition of orthogonality of the wave functions:

$$\int \Psi_0^*(0, 0; \mathbf{R}) \Psi_E(\mathbf{R}) d^6R = 0 \quad (E \neq 0). \qquad (24)$$

The zero-energy function $\Psi_0(0, 0; \mathbf{R})$ is obviously complex if particles 2 and 3 can form a bound state.

To determine the coefficient of proportionality between the functions $\Psi_E(\mathbf{R})$ and $\Psi_0(0, 0; \mathbf{R})$ in the range of action of the forces, we substitute the formula (20) in (24) in place of the functions $\Psi_E(\mathbf{R})$ and $\Psi_0(0, 0; \mathbf{R})$ and carry out integration with respect to R, using for the Green's function $G_E(\mathbf{R}, \mathbf{R}')$ and $G_0(\mathbf{R}, \mathbf{R}')$ the formulas (22) and (23). We then obtain in place of (24) the equality

$$\int \varphi_0(r_{23}) V(\mathbf{R}) \Psi_E(\mathbf{R}) d^6R - \int \varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23}) e^{i\mathbf{k}_1\rho_1} V(\mathbf{R}) \Psi_0^*(0, 0; \mathbf{R}) d^6R + \int \Psi_0^*(0, 0; \mathbf{R}') V(\mathbf{R}') [G_0^*(\mathbf{R}', \mathbf{R}'') - G_E(\mathbf{R}', \mathbf{R}'')] V(\mathbf{R}'') \Psi_E(\mathbf{R}'') d^6R' d^6R''. \quad (25)$$

The function $\Psi_E(\mathbf{R})$ enters into (25) only in the range of the triple forces. As noted above, it can be shown that it has then the form

$$\Psi_E(\mathbf{R}) = A(k_{23}^2, E) \Psi_0(0, 0; \mathbf{R}). \quad (26)$$

Substituting (26) into (25) and replacing the exponential under the integral sign in the second term in (25) by unity, we obtain an equation for $A(k_{23}^2, E)$, from which it follows that

$$A(k_{23}^2, E) = N(k_{23}^2) C(E), \quad (27)$$

where $N(k_{23}^2)$ is the coefficient of proportionality between the functions $\varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23})$ and $\varphi_0(r_{23})$ when $r_{23} \lesssim r_0$:

$$\varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23}) = N(k_{23}^2) \varphi_0(r_{23}), \quad (28)$$

while the value of $C(E)$ is

$$C(E) = a^*(0)/a(0)(1 - a(0) f(E)), \quad (29)$$

$$a(0) = -M \int \varphi_0(r_{23}) V(\mathbf{R}) \Psi_0(0, 0; \mathbf{R}) d^6R, \quad (30)$$

$$f(E) = \frac{M}{a^2(0)} \int \Psi_0^*(0, 0; \mathbf{R}') V(\mathbf{R}') [G_0^*(\mathbf{R}', \mathbf{R}'') - G_E(\mathbf{R}', \mathbf{R}'')] V(\mathbf{R}'') \Psi_0(0, 0; \mathbf{R}'') d^6R' d^6R''. \quad (31)$$

We have

$$N(k_{23}^2) = (1 - ik_{23}a_{23})^{-1}, \quad (32)$$

if $\varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23})$ is a function of the continuous spectrum, and

$$N(-\alpha^2) = -(\alpha^3/2\pi)^{1/2} \quad (33)$$

for a bound-state function.

If we write the particle scattering amplitude in the form (4), then, in analogy with (6), the amplitude $a(\mathbf{k}_{23}', \mathbf{k}_1', \mathbf{k}_{23}, \mathbf{k}_1)$ is expressed in the case of interest to us in terms of the wave function

$$a(\mathbf{k}_{23}, \mathbf{k}_1', \mathbf{k}_{23}, \mathbf{k}_1) = -M \int \varphi_{\mathbf{k}_{23}'}(\mathbf{r}_{23}) V(\mathbf{R}) \Psi_E(\mathbf{k}_{23}, \mathbf{k}_1; \mathbf{R}) d^6R. \quad (34)$$

When $E = 0$ and $k_{23}' = k_{23} = 0$, the amplitude $a(\mathbf{k}_{23}', \mathbf{k}_1', \mathbf{k}_{23}, \mathbf{k}_1)$ is obviously equal to $a(0)$.

To determine $C(E)$ it is necessary to calculate the integral (31). For this purpose it is first necessary to integrate with respect to d^3k_1' in expressions (22) and (23) for the Green's function. In the resultant expression it is possible to separate in the integral with respect to d^3k_{23}' the term that does not depend on the energy, in which large $k_{23}' \sim r_0^{-1}$ are significant in the integration. This part of the integral is a certain constant which does not depend on the energy. The integral of the remaining part converges when $k_{23}' \sim a_{23}^{-1}$, and can therefore be calculated by using formula (28) for $\varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23})$ and expanding the integrand in powers of $|\rho_1 - \rho_1'|$, discarding the terms that tend to zero when $|\rho_1 - \rho_1'| \to 0$, after which the integration with respect to d^3k_{23}' can be carried out in explicit form. A detailed calculation of the integral (31) is given in the Appendix. We give here only the results.

Let us consider first the case when $a_{23} > 0$, that is, particles 2 and 3 cannot form a bound state. Then the amplitude $a(0)$ is real. The function $A(k_{23}^2, E)$, as shown in the Appendix, is equal to

$$A(k_{23}^2, E) = \frac{1}{1 - ik_{23}a_{23}} \frac{1}{1 - a(0) f_1(E)}; \quad (35)$$

$$f_1(E) = \frac{ME\gamma}{\rho^2} - \frac{1}{(2\pi)^3} \frac{2\mu_1}{M} \left(\frac{\mu_1}{\mu_{23}}\right)^{1/2} \frac{1}{a_{23}^4} \Big[(1 + \mu_{23}Ea_{23}^2) \times \left(\ln \frac{\sqrt{2\mu_{23}Ea_{23}^2}}{2} - \frac{\pi i}{2}\right) - \frac{\mu_{23}Ea_{23}^2}{2} + \frac{\sqrt{1 + 2\mu_{23}Ea_{23}^2}}{2} \left(\ln \frac{\sqrt{1 + 2\mu_{23}Ea_{23}^2} + 1}{\sqrt{1 + 2\mu_{23}Ea_{23}^2} - 1} + \pi i\right)\Big], \quad (36)$$

where $\rho^2 > 0$ is an arbitrary parameter. ρ^{-2} can be written in the form $(r_1 \sim r_0)$

$$\rho^{-2} = \frac{\sqrt{2\mu_{23}}(2\mu_1)^{3/2}}{4(2\pi)^3} a_{23}^{-2} \ln \frac{a_{23}^2}{r_1^2}. \quad (37)$$

To determine the energy dependence of the reaction (1) it is necessary to substitute in formula (2) in place of $A(E)$ the quantity $A(k_{23}^2, E)$, in accordance with formulas (35) and (36). It follows from formulas (35) and (36) that the energy dependence of reaction (1) has a rather complicated character if $a(0) f_1(E) \sim 1$. Since $f(E) \sim MEa_{23}^{-2}$, in accordance with (36), the conditions $a(0) f_1(E) \sim 1$ and $MEr_0^2 \ll 1$ are compatible only if $a(0) \gg r_0^2 a_{23}^2$. In the opposite case the three-particle forces lead only to an insignificant change in the energy dependence of the amplitude. If $|2\mu_{23}Ea_{23}^2| \ll 1$, then, as can be readily verified by direct calculation, formula (35) goes over into formula (9), which was derived in the preceding section.

In the region $0 < E \lesssim a_{23}^4 r_0^2 a^{-2}(0) M^{-1}$ one cannot neglect the nonresonant pair forces, and allowance for the energy dependence in the second term of $A(k_{23}^2, E)$ is in excess of the accuracy.

Let us assume now that $a_{23} < 0$, so that a bound state of particles 2 and 3 is possible. In this case the amplitude a(0) is complex. Therefore in place of the proportionality factor with respect to the wave function $\Psi_0(0, 0; \mathbf{R})$ it is more convenient to introduce a proportionality factor $\widetilde{A}(k_{23}^2, E)$ with respect to the wave function $\Psi(0; \mathbf{R})$ at energy $E = -\alpha^2/2\mu_{23}$, and to introduce in place of the parameter a(0) the real amplitude $a_d(0)$ of the scattering of particle 1 by the bound state of particles 2 and 3 at a system energy $E = -\alpha^2/2\mu_{23}$. The elastic scattering amplitude $a_d(\mathbf{k}_1', \mathbf{k}_1)$ and inelastic scattering amplitude $a_d(\mathbf{k}_1', \mathbf{k}_{23}', \mathbf{k}_1)$ are determined by formulas similar to (34), and do not depend on the scattering angle, so that

$$a_d(\mathbf{k}_1', \mathbf{k}_1) \equiv a_d(k_1), \quad a_d(\mathbf{k}_1', \mathbf{k}_{23}', \mathbf{k}_1) \equiv a_d(k_{23}', k_1);$$

$$a_d(k_1) = -\frac{2\mu_1}{4\pi}\int \varphi_d(r_{23})\, V(\mathbf{R})\, \Psi_E(\mathbf{k}_1; \mathbf{R})\, d^6R, \tag{38}$$

$$a_d(k_{23}', k_1) = -\frac{2\mu_1}{4\pi}\int \varphi_{\mathbf{k}_{23}'}(\mathbf{r}_{23})\, V(\mathbf{R})\, \Psi_E(\mathbf{k}_1; \mathbf{R})\, d^6R. \tag{39}$$

From (38), (39), (36), and (33) it follows that

$$a_d(k_{23}', k_1) = -\left(\frac{2\pi}{\alpha^3}\right)^{1/2} \frac{1}{1 - ik_{23}' a_{23}}\, a_d(k_1). \tag{40}$$

The coefficient of proportionality between the wave functions, as shown in the Appendix, is

$$\widetilde{A}(k_{23}^2, E) = -(2\pi/\alpha^3)^{1/2} N(k_{23}^2)\, \widetilde{C}(E); \tag{41}$$

$$\widetilde{C}(E) = [1 - a_d(0)(\eta(E) - \eta(-\alpha^2/2\mu_{23})) - ia_d(0)\sqrt{2\mu_1}(\alpha^2/2\mu_{23} + E)^{1/2}]^{-1}, \tag{42}$$

$$\eta(E) = (4\pi^2 M/\alpha^3\mu_1)\, f_1(E). \tag{43}$$

To determine the energy distribution of reaction (1) it is necessary to substitute in formula (2) in place of the function A(E) the function $\widetilde{A}(k_{23}^2, E)$, in accordance with formulas (41) and (42). The factor $N(k_{23}^2)$ is determined by formula (32) if three particles are produced during the reaction, and by formula (33) if a bound state of particles 2 and 3 is produced.

From (39) and (41) it follows that the amplitude for the elastic scattering of particle 1 on the bound state of the two others has the form

$$a_d(k_1) = a_d(0)\, \widetilde{C}(E). \tag{44}$$

The inelastic scattering amplitude is determined by (40). In scattering on the bound state of particles 2 and 3, or upon formation of a bound state in reaction (1), the energy E is expressed in terms of k_1^2 by means of the formula $E = k_1^2/2\mu_1 - \alpha^2/2\mu_{23}$. Therefore in this case the last term in formula (42) is obviously equal to $ia_d(0)k_1$. If the energy E is negative and $|E| \ll \alpha^2/2\mu_{23}$, then we can neglect in the denominator of (42) the second term compared with the first and third. Then, as expected, the quantity $\widetilde{C}(E)$ is

$$\widetilde{C}(E) = 1/(1 - ia_d(0)\, k_1). \tag{45}$$

Let us ascertain now under what conditions the particles 1, 2, and 3 can form a bound state. Based on the results of Ansel'm et al [5] it can be stated that the quantities $A(k_{23}^2, E)$ and $\widetilde{A}(k_{23}^2, E)$ have two poles on the complex plane E on the physical sheet where $0 \le \arg E \le 2\pi$, $0 \le \arg(E + \alpha^2/2\mu_{23}) \le 2\pi$, if $a(0) < 0$ or respectively $a_d(0) < 0$, and one pole in the opposite case. In both cases, one of the poles is located in the region of large negative energies. In order to determine the position of this pole, it is possible to neglect in the formula (35) for $A(k_{23}^2, E)$ or, respectively, in formula (42) for $\widetilde{A}(k_{23}^2, E)$, all the terms in the denominator which grow at large energies more slowly than E. From the equation obtained it follows that this pole lies at $E = E_0 \sim -\mu_{23}^{-1} r_0^{-2}$.

Thus, the pole E_0 lies outside the region where the theory is valid. Inasmuch as there are no more poles on the physical sheet when $a(0) > 0$ or $a_d(0) > 0$, particles 1, 2, and 3 cannot form a bound state if $a(0) > 0$ or $a_d(0) > 0$. When $a(0) < 0$, or, respectively, $a_d(0) < 0$, there is one more pole on the complex E plane. It should be located on the real axis when $E < 0$ and corresponds to the bound state of the three particles.

The results of this section are valid also for the decay case. In the case of decay particles, it is necessary to use in place of (24)

$$\int \overline{\Psi}_0^*(0, 0; \mathbf{R})\, \Psi_E(\mathbf{R})\, d^6R = 0, \quad E \neq 0, \tag{46}$$

where $\overline{\Psi}_0(0, 0, \mathbf{R})$ is the wave function for $E = 0$, determined for the complex-conjugate potential. In the Green's function (22), the product $\varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23})\varphi^*_{\mathbf{k}_{23}}(\mathbf{r}_{23}')$ must be replaced by the product $\varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23})\overset{*}{\varphi}_{\mathbf{k}_{23}}(\mathbf{r}_{23}')$, where the function $\varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23})$ should be calculated with a potential that is the complex conjugate of the potential $V_{23}(r_{23})$. The formulas for $A(k_{23}^2, E)$ and $\widetilde{A}(k_{23}^2, E)$ remain in this case the same as before, but the parameters $a(0)$, $a_d(0)$, a_{23}, and ρ^2 are generally speaking, complex.

Thus, if $|a_{23}| \gg r_0$, the energy dependence of the amplitude (1) is determined by formula (2), in which one should substitute in place of A(E) the quantities $A(k_{23}^2, E)$ or $\tilde{A}(k_{23}^2, E)$, determined by formulas (35) and (42). The elastic scattering amplitude of particle 1 on the bound state is determined by formula (44), while the inelastic scattering amplitude is determined by formula (40).

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APPENDIX

Let us integrate with respect to d^3k_1' in formulas (22) and (23) for the Green's function $G_E(\mathbf{R}, \mathbf{R}')$. Then the Green's function is written in the form

$$G_E(\mathbf{R}, \mathbf{R}') = -\frac{1}{(2\pi)^3}\frac{2\mu_1}{4\pi}\int d^3k_{23}' \frac{\varphi_{\mathbf{k}_{23}'}(\mathbf{r}_{23})\varphi^*_{\mathbf{k}_{23}'}(\mathbf{r}_{23}')}{|\rho_1-\rho_1'|}$$
$$\times \exp\{-\gamma_E(k_{23}')\,|\rho_1-\rho_1'|\} + G_E^{(s)}(\mathbf{R}, \mathbf{R}'); \qquad (A.1)$$

$$G_E^{(s)}(\mathbf{R}, \mathbf{R}') = -\theta(-a_{23})\frac{2\mu_1}{4\pi}\varphi_d(r_{23})\varphi_d(r_{23}')$$
$$\times \frac{\exp\{i\gamma_E(\alpha)\,|\rho_1-\rho_1'|\}}{|\rho_1-\rho_1'|}; \qquad (A.2)$$

$$\gamma_E(k_{23}) = \sqrt{2\mu_1}\,(k_{23}^2/2\mu_{23}-E)^{1/2}, \text{ where } \gamma_E(k_{23})$$
$$= -i\,|\gamma_E(k_{23})|, \text{ when } \gamma_E^2(k_{23}) < 0; \qquad (A.3)$$

$$\gamma_E(\alpha) = \sqrt{2\mu_1}\,(\alpha^2/2\mu_{23}+E)^{1/2}. \qquad (A.4)$$

We now substitute (A.1) in formula (31). Then the expression for $G_E^{(S)}(\mathbf{R}, \mathbf{R}')$ and $G_0^{(S)}(\mathbf{R}, \mathbf{R}')$ can be expanded in powers of $|\rho_1-\rho_1'|$, discarding the terms that tend to zero when $|\rho_1-\rho_1'| \to 0$, and use formula (33) for the functions $\varphi_d(r_{23})$. We then obtain

$$f(E) = \theta(-a_{23})\frac{\alpha^3}{8\pi^2}\frac{a^*(0)}{a(0)}\frac{2\mu_1}{M}\,i\left[\left(\frac{\alpha^2}{2\mu_{23}}+E\right)^{1/2}\right.$$
$$\left.+\left(\frac{\alpha^2}{2\mu_{23}}\right)^{1/2}\right] + \tilde{f}(E); \qquad (A.5)$$

$$\tilde{f}(E) = -\frac{M}{a^2(0)}\int \Psi_0^*(0, 0; \mathbf{R}')\,V(\mathbf{R}')\,\Sigma(\mathbf{R}', \mathbf{R}'')$$
$$\times V(\mathbf{R}'')\,\Psi_0(0, 0; \mathbf{R}'')\,d^6R'\,d^6R''. \qquad (A.6)$$

The function $\Sigma(\mathbf{R}', \mathbf{R}'')$ is in turn equal to

$$\Sigma(\mathbf{R}', \mathbf{R}'') = -\frac{1}{(2\pi)^3}\frac{2\mu_1}{4\pi}\int \varphi_{\mathbf{k}_{23}'}(\mathbf{r}_{23}')\varphi^*_{\mathbf{k}_{23}'}(\mathbf{r}_{23}'')$$
$$\times \frac{\exp\{-\gamma_0(k_{23}')\,|\rho_1'-\rho_1''|\}}{|\rho_1'-\rho_1''|}$$
$$\times [1-\exp\{-[\gamma_E(k_{23}')-\gamma_0(k_{23}')]\,|\rho_1'-\rho_1''|\}]\,d^3k_{23}'. \qquad (A.7)$$

Inasmuch as $|\gamma_E(k_{23}') - \gamma_0(k_{23}')|\,|\rho_1'-\rho_1''| \ll 1$ for all k_{23}', the exponential in the last term of (A.7) can be expanded in powers of $|\rho_1'-\rho_1''|$, until decreasing terms are reached, after which we obtain for $\Sigma(\mathbf{R}', \mathbf{R}'')$ the formula

$$\Sigma(\mathbf{R}', \mathbf{R}'') = -\frac{1}{(2\pi)^3}\frac{2\mu_1}{4\pi}\int \varphi_{\mathbf{k}_{23}'}(\mathbf{r}_{23}')\varphi^*_{\mathbf{k}_{23}'}(\mathbf{r}_{23}'')$$
$$\times \exp\{-\gamma_0(k_{23}')\,|\rho_1'-\rho_1''|\}$$
$$\times [\gamma_E(k_{23}')-\gamma_0(k_{23}')]\,d^3k_{23}'. \qquad (A.8)$$

If we replace the exponential in (A.8) by unity, and substitute formula (37) for the function $\varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23})$, then the integral obtained will diverge logarithmically at large k_{23}'. Consequently, in the integral (A.8) the large values of $k_{23}' \sim r_0^{-1}$ are significant, and the use of (37) is not valid. However, if we rewrite identically $\Sigma(\mathbf{R}', \mathbf{R}'')$ in the form

$$\Sigma(\mathbf{R}', \mathbf{R}'') = E\,\Sigma_0(\mathbf{R}', \mathbf{R}'') - \frac{1}{(2\pi)^3}\frac{2\mu_1}{4\pi}\int \varphi_{\mathbf{k}_{23}'}(\mathbf{r}_{23}')\varphi^*_{\mathbf{k}_{23}'}(\mathbf{r}_{23}'')$$
$$\times \exp\{-\gamma_0(k_{23}')\,|\rho_1'-\rho_1''|\}$$
$$\times \left[\gamma_E(k_{23}') - E\left(\frac{\partial\gamma_E(k_{23}')}{\partial E}\right)_{E=0} - \gamma_0(k_{23}')\right]d^3k_{23}', \qquad (A.9)$$

$$\Sigma_0(\mathbf{R}', \mathbf{R}'') = -\frac{1}{(2\pi)^3}\frac{2\mu_1}{4\pi}\int \varphi_{\mathbf{k}_{23}'}(\mathbf{r}_{23}')\varphi^*_{\mathbf{k}_{23}'}(\mathbf{r}_{23}'')$$
$$\times \exp\{-\gamma_0(k_{23}')\,|\rho_1'-\rho_1''|\}$$
$$\times \left(\frac{\partial\gamma_E(k_{23}')}{\partial E}\right)_{E=0} d^3k_{23}', \qquad (A.10)$$

then the $k_{23}' \sim a_{23}^{-1}$ are significant in the second term of (A.9) [$\Sigma_0(\mathbf{R}', \mathbf{R}'')$ is a certain function that does not depend on the energy]. Therefore the exponential in the second term of (A.9) can be replaced by unity and formula (32) can be used for the functions $\varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23})$, after which the second integral in (A.9) can be readily calculated. Upon substitution of the first term of (A.9) into formula (A.6) we obtain a certain indeterminate constant, multiplied by E, and upon substitution of the second term we obtain a certain known function of E, multiplied by $|a(0)|^2$. As a result of these calculations we obtain for $\tilde{f}(E)$ the expression

$$\tilde{f}(E) = f_1(E)\,a^*(0)/a(0), \qquad (A.11)$$

where $f_1(E)$ is determined by formula (36), and the constant ρ^{-2} is equal to

$$\frac{1}{\rho^2} = \frac{1}{\gamma\,|a(0)|^2}\int \Psi_0^*(0, 0; \mathbf{R}')V(\mathbf{R}')\,\Sigma_0(\mathbf{R}', \mathbf{R}'')V$$
$$\times(\mathbf{R}'')\,\Psi_0(0, 0; \mathbf{R}'')d^6R'd^6R''. \qquad (A.12)$$

In order to obtain for ρ^{-2} formula (30), it is necessary to replace in formula (A.10) the exponential under the integral sign by unity, use formula (28) for the functions $\varphi_{\mathbf{k}_{23}}(\mathbf{r}_{23})$, and calculate the resultant integral, which is cut off at $k'_{23} \sim r_0^{-1}$.

Let us explain now how to obtain formulas (40) and (41) of the text for $\widetilde{A}(k_{23}^2, E)$, valid when $a_{23} < 0$. In this case $a(0)$ is complex. The connection between $a(0)$ and $a^*(0)$ can be determined from the requirement that $C(E)$ [formula (29)] at $E = 0$ be equal to unity: $C(0) = 1$. The amplitude $a(0)$ must be expressed in terms of $a_d(0)$, using for this purpose formula (38) with $k_1 = 0$, $E = -\alpha^2/2\mu_{23}$. The function $\widetilde{A}(k_{23}^2, E)$ is determined from the obvious equality

$$\widetilde{A}\ (k_{23}^2, E) = A\ (k_{23}^2, E)/A\ (-\alpha^2,\ -\alpha^2/2\mu_{23}),$$

in which it is necessary to substitute in place of the amplitude $a(0)$ its expression in terms of $a_d(0)$.

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