ATTENUATION OF ELECTROMAGNETIC WAVES IN AN INHOMOGENEOUS MEDIUM RELATED TO TRANSITION RADIATION

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Submitted to JETP editor April 26, 1962

J. Exptl. Theoret. Phys. (U.S.S.R.) **43**, 1419-1423 (October, 1962)

It is shown that electromagnetic waves can be absorbed in an inhomogeneous medium as a result of variation of the properties of the medium. The relation between this mechanism of wave absorption and transition radiation from charged particles of the medium is established by employing the Kirchhoff theorem.

It is well known at present that when electromagnetic waves propagate in a homogeneous plasma, there exists, along with the damping of the waves due to collisions of the plasma particles, also a damping connected with the absorption of energy by the plasma particles moving in the field of the normal electromagnetic wave. Thus, in an isotropic plasma, the longitudinal wave is damped as a result of Cerenkov absorption (Landau damping[1]). In a magnetoactive plasma, absorption of electromagnetic waves is possible also at Doppler frequencies, corresponding to radiation of normal waves in the plasma by electrons moving in a constant magnetic field (see, for example [2]). So far the analysis was essentially limited to absorption of electromagnetic waves and its connection, for example on the basis of the Kirchhoff formula, with radiation of particles with thermal velocities in a homogeneous medium. Yet, the medium dealt with is very frequently inhomogeneous, both under cosmic and laboratory conditions. On the other hand one should expect in the propagation of waves in an inhomogeneous medium an additional damping due to the changes in the properties of the medium.

To explain the singularities that occur here, we consider in the present article the propagation of a wave in a medium made up of an inhomogeneous plasma placed in an inhomogeneous dielectric. It turns out that the contribution to the absorption made by the electromagnetic wave because of the presence of the gradient of the dielectric constant can in many cases be decisive. This occurs, for example, in the case when no Cerenkov radiation is possible in the medium. It is shown further that the absorption in the inhomogeneous medium is connected in final analysis with the transition radiation of the thermal electrons of the plasma. Transition radiation is understood here in the broad sense, i.e., radiation from a charge moving with constant velocity in an arbitrary inhomogeneous medium. The particular case of such a phenomenon, namely the radiation from a charge passing through a boundary between two media, was first considered by Ginzburg and Frank[3].

1. We start from Maxwell's equations in an inhomogeneous medium[2]*

$$\mathrm{rot}\,\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{\varepsilon(\mathbf{r})}{c}\frac{\partial \mathbf{E}}{\partial t}, \qquad \mathrm{rot}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t},$$
$$\mathrm{div}\,\varepsilon(\mathbf{r})\,\mathbf{E} = 4\pi\rho, \qquad \mathrm{div}\,\mathbf{H} = 0, \tag{1}$$

where the current and charge density are determined with the aid of the kinetic equation for the homogeneous plasma[2]

$$\mathbf{j} = e\int \mathbf{v}\varphi\, d\mathbf{v}, \qquad \rho = e\int \varphi\, d\mathbf{v}, \quad \frac{\partial\varphi}{\partial t} + \mathbf{v}\nabla_{\mathbf{r}}\varphi + \frac{e\mathbf{E}}{m}\nabla_{\mathbf{v}} f_0 = 0$$
$$(f_0 = N v_T^{-3}\pi^{-3/2} e^{-v^2/v_T^2};\ \varphi \ll f_0). \tag{2}$$

Here $v_T^2 = 2T/m$ is the average thermal velocity of the plasma electrons and N the plasma electron density. The addition φ to the Maxwellian distribution function f_0 is assumed small; the motion of the ions and the collisions are neglected.

We assume that the plasma is placed in a planar stratified region, the properties of which vary along the coordinate axis Oz. We consider longitudinal (along Oz) propagation of a transverse electromagnetic wave in such a medium. If the electric field of the wave $\mathbf{E} \approx e^{-i\omega t}$ is oriented parallel to the Oy axis, we get from (1) and (2) for longitudinal propagation

$$\frac{d^2E_y}{dz^2} + \frac{\omega^2}{c^2}\varepsilon(z)E_y = -\frac{4\pi i\omega}{c^2} j_y = -\frac{4\pi i\omega}{c^2}\int v_y\varphi\, d\mathbf{v}. \tag{3}$$

*rot = curl.

Assuming that $\epsilon(z)$ is a slowly varying function, i.e., the condition

$$\lambda d\varepsilon/dz \ll \varepsilon \tag{4}$$

is satisfied (λ is the wavelength in the medium), we proceed to solve (3) in the following fashion. In a Maxwellian plasma ($v_T/c = \beta_T \ll 1$) the right half of (3) [see (2)] can be represented in the form of a sum

$$j_y = j_0 + j_s \qquad (\varphi = \varphi_0 + \varphi_s), \tag{5}$$

where [1])

$$-(4\pi i\omega/c^2)\, j_0 = (\omega_0^2/c^2)\, E_y \qquad (\omega_0^2 = 4\pi e^2 N/m),$$

and j_s is the term that accounts for spatial dispersion and vanishes as $v_T/c \to 0$ ($j_0 \gg j_s$). Taking this circumstance into account, we rewrite (3) in the form ($E_y \equiv E$)

$$\frac{d^2E}{dz^2} + \frac{\omega^2}{c^2}\left[\varepsilon(z) - \frac{\omega_0^2}{\omega^2}\right]E = -\frac{4\pi i\omega}{c^2}\int v_y \varphi_s\, d\mathbf{v} = -\frac{4\pi i\omega}{c^2} j_s. \tag{6}$$

The right half of (6) is now determined by the spatial dispersion and in the case considered of a nonrelativistic plasma it is a small quantity compared with the terms of the left half of (6).[2]) The solution of (6) without the right half under condition (4),[3]) i.e., in the geometrical-optics approximation, is known:

$$E^{(0)} \approx [\varepsilon'(z)]^{-1/4} \exp\left\{\pm \frac{\omega}{c} i \int^z \sqrt{\varepsilon'(z')}\, dz'\right\}, \tag{7}$$

where $\epsilon'(z) = \epsilon(z) - \omega_0^2/\omega^2$.

If we use, for example, the method of varying the constants to find the solution of the equation with the right half, we obtain from (6) with the aid of (7)

$$E = \frac{\alpha}{2i}\left\{\exp\left(i\int \gamma(z)\, dz\right)\int \delta(z) \exp\left(-i\int \gamma(z)\, dz\right) dz - \exp\left(-i\int \gamma(z)\, dz\right)\int \delta(z) \exp\left(i\int \gamma(z)\, dz\right) dz\right\},$$
$$\alpha = (\varepsilon'(z))^{1/4}; \quad \gamma = \omega\sqrt{\varepsilon'(z)}/c; \quad \delta = -(4\pi i\omega/c^2\alpha\gamma)\, j_s. \tag{8}$$

It is easy to see that (8) is a homogeneous integral equation [see (2)] with respect to E. To carry out further analysis of (8) it is necessary to specify the function $\epsilon(z)$. It is of interest to examine in detail the case when a wave propagates in a periodic medium, for which we put

$$\varepsilon' = \varepsilon_0 + \Delta \cos \xi z, \quad \Delta \ll \varepsilon_0, \quad c\sqrt{\varepsilon_0}\xi/\omega \ll 1, \tag{9}$$

where ϵ_0 and Δ are parameters that do not depend on the coordinates. Then

$$\gamma \approx \gamma_0 + \tilde{\Delta}\cos \xi z, \quad \tilde{\Delta} = \Delta\omega^2/2c^2\gamma_0, \quad \gamma_0 = \omega\sqrt{\varepsilon_0}/c, \quad \gamma_0^2 \gg \Delta\omega^2/2c^2 \tag{10}$$

and consequently $E^{(0)}$ [see (7)] has the form

$$E^{(0)} \approx \sum_{s=-\infty}^{\infty} J_s(\varkappa) \exp\left\{iz\left(\frac{\omega}{c}\sqrt{\varepsilon_0} + \xi s\right)\right\}, \tag{11}$$

where $\kappa = \omega\Delta/2c\sqrt{\epsilon_0}\,\xi$ and J_s is the Bessel function.

To obtain (11) it is necessary to use

$$e^{i\varkappa \sin \xi z} = \sum_{s=-\infty}^{\infty} J_s(\varkappa)\, e^{i\xi zs}.$$

If we substitute (11) and (8) and take the definitions (2) and (5) into account, Eq. (8) assumes the form

$$E = -\frac{2\pi e}{c\sqrt{\varepsilon_0}}\Big[\sum_{s,s'=-\infty}^{\infty} J_s(\varkappa) \exp\left\{iz\left(\frac{\omega}{c}\sqrt{\varepsilon_0} + s\xi\right)\right\}\int_{-\infty}^{\infty} d\mathbf{v} \times \int^z v_y\varphi_s(z',\mathbf{v})\, J_{s'}(\varkappa) \exp\left\{-iz'\left(\frac{\omega}{c}\sqrt{\varepsilon_0} + \xi s'\right)\right\} dz'$$
$$- \sum_{s,s'=-\infty}^{\infty} J_s(\varkappa) \exp\left\{-iz\left(\frac{\omega}{c}\sqrt{\varepsilon_0} + \xi s\right)\right\}\int_{-\infty}^{\infty} d\mathbf{v} \times \int^z v_y\varphi_s(z',\mathbf{v})\, J_{s'}(\varkappa) \exp\left\{iz'\left(\frac{\omega}{c}\sqrt{\varepsilon_0} + \xi s'\right)\right\} dz'\Big]. \tag{12}$$

Since the right half of (6) is small and vanishes as $\beta_T \to 0$, it is sensible to seek a solution of (12) in the same form as the solution of (6) without the right half, i.e.,

$$E = \sum_{s=-\infty}^{\infty} J_s(\varkappa) \exp\left\{iz\left(\frac{\omega}{c}\sqrt{\varepsilon^*} + \xi s\right)\right\}. \tag{13}$$

Here, generally speaking, ϵ^* is different from ϵ_0. For simplicity we shall neglect henceforth the correction to the real part of ϵ^* and assume [4]) that $\epsilon^* \approx \epsilon_0 + i\,\mathrm{Im}\,\epsilon^*$ ($\mathrm{Im}\,\epsilon^* \ll \epsilon_0$). We thus are considering a wave damped in space.

We could solve analogously the initial-value problem, i.e., seek an imaginary correction to the frequency ω, corresponding to the damping of the waves in time. Substituting (13) in (2) and (12),

[1])The term j_0 corresponds to complete neglect of spatial dispersion: $\varphi_0 = -(eEi/m\omega)\nabla_v f_0$.

[2])Of course, the function φ_s itself depends on E via Eq. (2).

[3])More accurately, it is assumed that all the conditions for the applicability of geometrical optics to the solution of the equation $d^2E/dz^2 + \omega^2c^{-2}\epsilon'(z)E = 0$ are fulfilled (see [2]).

[4])The inclusion of the correction to the real part of ϵ^* is of course trivial, but for a transverse wave ($E \perp k$) this correction is not essential in the case under consideration.

and using (5), we obtain, discarding the small terms for $\mathrm{Im}\,\frac{\omega}{c}\sqrt{\epsilon^*} \ll \xi$,

$$\sum_{s=-\infty}^{\infty} J_s(\varkappa)\, e^{iz\xi s} = -\frac{2\pi e^2}{mc\sqrt{\varepsilon_0}} \sum_{s,\, s'=-\infty}^{\infty} \frac{J_s(\varkappa)\, J_{s'}^{2'}(\varkappa)\, e^{iz\xi s}}{\omega(\sqrt{\varepsilon^*}-\sqrt{\varepsilon_0})/c} \int d\mathbf{v}\, v_y \nabla_{v_y} f_0 \times \left[\frac{1}{\omega} - \frac{1}{\omega - u(\omega\sqrt{\varepsilon_0}/c + \xi s')}\right], \tag{14}$$

where, as usual, the integration with respect to $dv_x dv_y = dv_\perp$ is carried out over real values, while integration with respect to $v_z = u$ is carried out in the complex plane along the path $-\infty < u < \infty$ with the pole $u_s = \omega/(\omega c^{-1}\sqrt{\epsilon_0} + \xi s)$ circuited from above. Thus, unlike the case of wave propagation in a homogeneous region, we obtain here an enumerable set of resonant points $(s = 0, \pm 1, \pm 2 \ldots)$.

The damping of the normal wave (13) is given by the imaginary part $\omega\sqrt{\epsilon^*}/c$. Equating the coefficients of equal powers of $e^{iz\xi s}$ in both halves of (14) we get

$$\alpha_\omega = 2\,\mathrm{Im}\,\frac{\omega}{c}\sqrt{\varepsilon^*} = \frac{\omega_0^2 \pi^{1/2}}{c\sqrt{\varepsilon_0}\, v_T} \sum_{s=s_m}^{\infty} \frac{J_s^2(\varkappa)\, e^{-\zeta u_s^2}}{\omega\sqrt{\varepsilon_0}/c + \xi s}, \quad \zeta = v_T^{-2}, \tag{15}$$

where s_m is determined from the requirement $u_s \le c$. As indicated above, $\mathrm{Im}\,\sqrt{\epsilon^*} \neq 0$ owing to the presence of poles in the integrand of (14) when $u = u_s$. The term $s = 0$ in (15) corresponds to absorption of waves of the Cerenkov type. It differs from zero only when $\epsilon_0 > 1$, whereas the next terms with $s > 0$ can, generally speaking, differ from zero also when $\epsilon_0 < 1$. In this case the absorption of the electromagnetic wave is due entirely to the change in the properties of the medium. The Cerenkov absorption predominates, for example, in the case when $\epsilon_0 > 1$ and $\kappa \ll 1$. We note that the quantity $\kappa = \Delta/2\epsilon_0\xi\lambda$ is a ratio of two small parameters [see (9)], so that generally speaking it can assume arbitrary values. The role of this mechanism of absorption of electromagnetic waves under astrophysical conditions, particularly in the earth's atmosphere, will be discussed in greater detail in another paper.

2. In order to indicate the connection between the resultant damping of an electromagnetic wave in the medium under consideration [see (15)] with radiation from the plasma electrons, we can use the Kirchhoff theorem [5)]:

$$\eta_\omega = \alpha_\omega I_\omega, \tag{16}$$

where η_ω is the radiating ability of the medium, α_ω the absorption ceofficient of the wave, and $I_\omega \approx (\omega^2 T/\pi^2 c^2)\epsilon$ is the equilibrium radiation intensity. We note that a connection of this form between the damping of the waves and the Cerenkov radiation in a homogeneous medium was discussed, for example, by Shafranov [5].

To calculate the radiating ability of the medium η_ω in the case (9) of interest to us, we use the previously obtained formula [6]. It was shown in [6] that the energy radiated by a charge moving in a periodic medium [see formula (9) in [6]] at an angle ϑ to the direction of change of the properties of the medium is [6)]

$$\frac{dW}{dt} = \frac{e^2 vi}{2\pi^2 c^2 \cos\vartheta} \sum_{s=-\infty}^{\infty} \int \frac{J_s^2(\varkappa)\, d\mathbf{k}_\perp\, d\omega\, (1 - 1/\varepsilon_0\beta^2)\,\omega}{[\delta/v\cos\vartheta - \xi s]^2 - \gamma_0^2} + \text{c.c.}, \tag{17}$$

where v is the particle velocity, $\beta = v/c$; $k_\perp$ is the transverse wave number, $k_\perp = \{k_x, k_y, 0\}$; $\kappa = \Delta\omega^2/2c^2\gamma_0$; $\gamma_0 = \sqrt{\omega^2\epsilon_0/c^2 - k_\perp^2}$, $\delta = \omega - vk_y \sin\vartheta$ $\gamma_0^2 \gg \Delta\omega^2/c^2$. For the energy radiated in a solid angle element $d\Omega$ by the wave propagating in a direction parallel to the z axis we have

$$\frac{dW}{dt} = \frac{e^2 vi}{2\pi^2 c^2 \cos\vartheta} \times \sum_{s=-\infty}^{\infty} \int \frac{\omega(1 - 1/\varepsilon_0\beta^2)\, J_s^2(\varkappa)\, \omega^2 c^{-2}\varepsilon_0\, d\omega}{(\omega/v\cos\vartheta - s\xi)^2 - \omega^2 c^{-2}\varepsilon_0}\, d\Omega + \text{c.c.} \tag{18}$$

If we average dW_ω/dt [see (18)] over the Maxwellian distribution, we obtain for the radiating ability of the medium

$$\eta_\omega = \frac{e^2 viN}{2\pi^2 c^2 v_T^3 \pi^{3/2}} \times \sum_{s=-\infty}^{\infty} \int \frac{J_s^2(\varkappa)\,\omega\,(1 - 1/\varepsilon_0\beta^2)\,\omega^2 c^{-2}\varepsilon_0 e^{-\zeta v^2}\, d\Omega}{(\omega/vx - s\xi)^2 - \omega^2 c^{-2}\varepsilon_0}\, v^2\, dx\, d\Phi\, d\vartheta, \tag{19}$$

where $\mathbf{v}$ is given in spherical coordinates: $\mathbf{v} = \{v, \vartheta, \Phi\}$, $x = \cos\vartheta$.

Of course, formula (19) gives the correct radiating ability of a medium consisting of a plasma placed in a dielectric only when the electron density is small, i.e., when the addition to the dielectric constant ϵ', due to the presence of plasma, can be neglected. This is obviously valid if the condition $\omega_0^2/\omega^2 \ll \epsilon_0$ is satisfied [5].

5)The applicability of the Kirchhoff theorem when the geometrical optics approximation is valid is not subject to any doubt (see [4]).

6)The corresponding formula (9) of [6] has a somewhat different form. The difference lies in the fact that the integration with respect to $dk_\perp$ has already been carried out in (9) of [6].

In (19) it is convenient to integrate first with respect to dx with the aid of the pole of the integrand

$$x = x_s = \frac{\omega/v}{\omega\sqrt{\varepsilon_0}/c + s\xi}, \quad |x_s| \leqslant 1, \tag{20}$$

after which the integration with respect to dv is elementary. As a result we have

$$\eta_\omega = \frac{\omega_0^2\omega^2\sqrt{\varepsilon_0}mv_T}{8\pi c^3\pi^{3/2}} \sum_{s=s_m}^{\infty} \frac{J_s^2(\varkappa)\exp\{-\zeta\omega^2/(\omega\sqrt{\varepsilon_0}/c+\xi s)^2\}}{\omega\sqrt{\varepsilon_0}/c + s\xi}, \tag{21}$$

where s_m is the number of the lowest harmonic, which is still radiated at a given frequency ω, i.e., expression (20) has real values for the radiation angle. Comparing now the value of η_ω with $\alpha_\omega I_\omega d\Omega/4\pi$ [see (15), (16), and (21)] we find that these quantities coincide.

We can thus conclude that the mechanism considered above for the damping of electromagnetic waves in an inhomogeneous medium is due to transition radiation. The plasma electrons, moving in the field of the normal wave, are accelerated by the latter, as a result of which the energy of the normal wave decreases.

The author is grateful to V. L. Ginzburg for several remarks.

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Translated by J. G. Adashko
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