ON SPIN-WAVE STATISTICS

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We discuss a method for taking account in spin wave theory of the fact that the spin of an atom is finite; this method consists in expressing the spin operators in terms of fermion operators.

As a model in the Weiss region one considers usually (in the Heitler-London approximation) a crystal in which the interaction between the electrons is described by Dirac exchange operators. One can introduce in that case quasi-particles that pertain only to the spin variable systems. In this connection it is of interest to elucidate how the fact that the atomic spin is finite manifests itself in the statistical properties of the quasi-particles.

Such a quasi-particle, which is a "localized" spin deviation, corresponds to a deviation by unity from the maximum projection of the atomic spin S_i. Because the atomic spin is finite, the localized quasi-particle is not a normal boson, since its state cannot be occupied more than $2S_i$-fold. We need thus use for the operator of the atomic spin S_i not the Holstein-Primakoff[1] representation, but those of Izyumov[2] and of Frank[3]:

$$S_i^+ = (2S_i - b_i^+ b_i)^{1/2} b_i, \qquad S_i^- = b_i^+ (2S_i - b_i^+ b_i)^{1/2},$$
$$S_i^z = S_i - b_i^+ b_i = S_i - n_i;$$
$$[b_i, b_j^+]_- = \delta_{ij}\{1 - (2S_i + 1)\delta_{n_i, 2S_i}\}, \quad [b_i, b_j]_- = 0. \tag{1}$$

From these commutation relations it follows that b_i, b_i^+, and $b_i^+ b_i$ are, respectively, the operators of the annihilation, the creation, and the number of particles. The eigenvalues of the operator $b_i^+ b_i$ run, in accordance with the statistical properties, through the integer values from 0 to $2S_i$.

In the $S_i = 1/2$ case (we shall in that case denote the quasi-particle operators by c_i instead of by b_i) the commutation relations have the simpler form

$$[c_i, c_i^+]_- = 1 - 2c_i^+ c_i.$$

We can consider them to be a combination of the usual fermion and boson commutation relations,

$$[c_i, c_i^+]_+ = 1, \qquad c_i^2 = c_i^{+2} = 0,$$
$$[c_i, c_j^+]_- = [c_i, c_j]_- = 0, \qquad i \neq j. \tag{2}$$

The introduction of non-localized (moving) quasi-particles which are connected with non-localized spin deviations (which may be thought of as being energy levels of the spin system) runs into difficulties because of the boundedness of $b_i^+ b_i$. In order that the non-localized quasi-particles show a well-defined statistical behavior, it is necessary that the operators corresponding to them satisfy fermion or boson commutation relations, or else (1). In the $S_i = 1/2$ case it is possible to reduce the operators c_i to the usual fermion operators a_i and to perform a Fourier transformation. This is attained through the transformation[4-6]

$$a_i = U_1 U_2 \ldots U_{i-1} c_i, \qquad U_i = 1 - 2c_i^+ c_i. \tag{3}$$

The transformation (3) does, of course, not change the dynamical properties of the system, since these are determined by the Hamiltonian and the commutation relations only. The statistical properties of localized particles affect only the form of the interaction between the non-localized particles. The former and the latter are completely different particles.

We can proceed similarly also in the more general case of half-odd-integral spin $S_i = (2^r - 1)/2$. The spin system can be described as a gas of several kinds of interacting fermions. In particular, for $r = 2$ we obtained[7] a transformation which enabled us to consider the system as consisting of interacting fermions. This transformation is based upon the possibility of representing the operator $\mathbf{S}_i$ of spin $3/2$ as the sum of two spin-$1/2$ operators $\mathbf{S}_{\nu i}$:

$$S_i^z = S_{1i}^z + 2S_{2i}^z, \qquad S_i^{\pm} = \sqrt{3} S_{1i}^{\pm} + 2S_{2i}^{\pm} S_{1i}^{\mp};$$
$$S_{2i}^{\pm} = \tfrac{1}{2} 3^{-1/2} (S_i^{\pm})^2, \qquad S_{1i}^{\pm} = 3^{-1/2} [(S_i^{\pm})^3 (S_i^{\mp})^2 + (S_i^{\mp})^2 (S_i^{\pm})^3].$$

Moreover, the operators $\mathbf{S}_{\nu i}$ can be expressed through (3) in terms of the usual fermion operators.

The possibility of introducing fermion type spin waves was disputed by Vonsovskiĭ and Svirskiĭ[8]. They pointed to the fact that the square of the operator $c_{\mathbf{k}}$, defined by

$$c_{\mathbf{k}} = N^{-1/2}\sum_j e^{-i\mathbf{k}\mathbf{r}_j} c_j \, ,$$

does not vanish. This does, however, not mean at all that, as was asserted in [8], the system has states in which there are more than one spin wave with quasi-momentum **k**. To the contrary, it is clear from the commutation rules (2) that $c_{\mathbf{k}}$ is not an annihilation operator for any kind of quasi-particle, just as $c_{\mathbf{k}}^{+}c_{\mathbf{k}}$ is not a particle number operator. This is so because states corresponding to the operators $c_{\mathbf{k}}$ are not orthogonal to one another, for otherwise the number of states obtained in this way would by far exceed the number of possible states of the whole system. It is therefore impossible to consider non-localized spin deviations in the form where they are described by the $c_{\mathbf{k}}^{+}$ operators as quasi-particles. It is impossible to use a Fourier transformation of the c_i operators to introduce any kind of quasi-particles, in particular boson-type spin waves.

Vonsovskiĭ and Svirskiĭ did not notice that only the transformations (3) lead to the possibility of introducing fermions and to the appearance of the Jordan-Wigner sign function. Contrary to their statement, the fact that one can make such an introduction bears no relation whatever to the statistical properties of the electrons. Vonsovskiĭ and Svirskiĭ's objections are unfounded as can be seen most distinctly from the example of a linear chain with nearest-neighbor interactions only. Nambu has already shown[9] that one can in that case replace all commutators in (2) by anticommutators without changing the energy spectrum of the system. The same result has also been obtained by us before.[4,5]

The advantage of the fermion formulation of the spin wave theory consists in the fact that it enables us to exclude fictitious states and to introduce non-localized quasi-particles. It is as yet not clear whether it is possible to overcome the difficulties which prevent the introduction of fermion operators in the case of two- or three-dimensional lattices. These difficulties are connected with the necessity of a linear ordering of the operators in (3). However, as one would expect, the fermion formulation has completely justified itself for the linear chain. For instance, this method gives for $J > 0$ a finite "spontaneous magnetization" although the temperature dependence of the latter shows up the absence of a phase transition point which is in fact connected with the absence of a ferromagnetic region. The fermion formulation enabled us, moreover, to determine rather accurately the ground state of the antiferromagnetic linear chain.[10]

Although it is, of course, not necessary to introduce fermion quasi-particle operators for the model considered here, it enables us to describe correctly the states of the system, using these fermion operators. This general statement does, of course, not solve the problem of the advisability of such a description.

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