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# *PROPAGATION OF A NONEQUILIBRIUM HEAT WAVE WITH ACCOUNT OF THE FINITE VELOCITY OF LIGHT*

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Quasi-stationary propagation of heat in the form of radiation through an opaque cold gas is investigated. The gas within the heated region is initially not in equilibrium with the radiation and is transparent to it. Of decisive importance is the thin intermediate layer between the transparent hot gas and the totally opaque cold gas. The balance between the radiation and absorption in this layer can be described in the diffusion approximation by taking into account the fact that the velocity of light c is finite. The displacement velocity v of the boundary of the hot region is determined for the case when the nonequilibrium radiation energy density $U_1$ in the transparent region is much greater than the radiation energy density at its boundary. It is demonstrated that irrespective of the value of $U_1$ the propagation velocity v is always smaller than $c/\sqrt{3}$.

## 1. INTRODUCTION

It is well known that at a sufficiently high temperature any substance, particularly a gas, is strongly or completely ionized and becomes highly transparent to radiation (the radiation free path exceeds the linear dimensions of the heated region). In such a substance the thermal equilibrium between the radiation and the substance is established relatively slowly, and if changes do occur in the substance with finite speed, the radiation may not have time to "follow" the changes and go into equilibrium.

An example of such a process was considered by us earlier in [1], where we studied the expansion of a heated region in gas. We solved the problem of the velocity of quasi-stationary propagation in an opaque cold gas boundary of a heated region, where the gas inside the region was not in equilibrium at the initial instant of time with the radiation and was transparent to the latter. The decisive factor was the thin transition layer between the transparent heated gas and the completely opaque cold gas. The balance between the radiation and absorption in this layer did indeed determine the rate of propagation of heat by radiation.

In [1] we found the dependence of this velocity on the ratio of the nonequilibrium radiation energy density $U_1$ in the transparent region to the energy density in the gas $\epsilon(T_0)$ on the transparency boundary (the transparency boundary is arbitrarily defined by means of the temperature $T_0$ at which the radiation free path is on the order of the dimensions of the entire heated region). Under certain conditions ($U_1 \gg U_{eq0}$, where $U_{eq0} = aT_0^4$ is the equilibrium radiation energy density on the transparency boundary), the speed v of the boundary of the heated region through the cold gas was determined by the formula

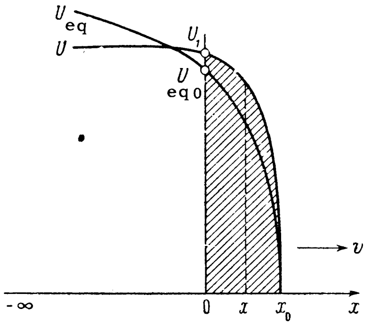
$$v = cU_1/\sqrt{3}\,\varepsilon(T_0),$$

where c is the speed of light [see formula (16) in [1] ]. Since $U_1$ and $\epsilon(T_0)$ are physically unrelated, this formula can lead to an impossible result $v > c$ at sufficiently large $U_1$.

To avoid this, we determine in the present article the velocity v with allowance for the fact that the velocity of light is finite.

## 2. INTEGRAL EQUATIONS FOR RADIATION TRANSPORT WITH ACCOUNT OF THE FINITE VELOCITY OF LIGHT

We align the origin $x = 0$ at the instant of time $t = 0$ with the transparency boundary and direct the x axis toward the cold gas. Then the transition layer, which can be regarded as plane in view of its small thickness, will occupy the space between the point $x = 0$ (where it borders on the transparent region, which in turn extends by definition to $x = -\infty$) and the point $x = x_0$ (where it borders on the cold gas, see the figure).




The state of radiation in the transition layer is described by the differential equation for radiation transport, which for our case has the form [2]

$$\frac{1}{c}\frac{\partial I}{\partial t}+\mu\frac{\partial I}{\partial x}+kI=\frac{kc}{4\pi}U_{eq}. \tag{1}$$

Here $I(x,t,\mu)$ is the radiation intensity integrated over the frequencies, $\mu$ is the cosine of the angle between the direction of beam propagation and the x axis, $k(x,t)$ is the radiation absorption coefficient averaged over the frequencies, and $U_{eq} = aT^4$ is the equilibrium radiation energy density, where $T = T(x,t)$ and $a = 7.55 \times 10^{-15}$ erg-cm$^{-3}$ deg$^{-4}$.

It is convenient to consider separately the intensity $I(x,t,\mu)$ for $1 \geq \mu \geq 0$ and $-1 \leq \mu \leq 0$, that is, for "forward" and "backward" radiation. We stipulate that the functions $I(\mu \geq 0)$ and $I(\mu \leq 0)$ satisfy the following boundary conditions

$$I(\mu \geqslant 0,\ x=0)=\frac{c}{4\pi}U_1(t),\quad I(\mu \leqslant 0,\ x=x_0)=0. \tag{2}$$

The first of conditions (2) describes the transfer of radiant energy from the transparent region to the transition region on the transparency boundary. It takes into account the nonequilibrium state of the radiation within the transparent region, where the radiation energy density has the same order of magnitude as $U_1$ (as was shown in [1], the radiation energy density at the transparency boundary is practically the same as inside the transparency region, that is, its approximate value is $U_1 = aT^4R/l(T)$, where R is the dimension of the entire heated region of the gas, $l(T)$ is the radiation free path averaged in a certain manner over the frequencies and characterizing the emissivity of the gas heated to a temperature T; it is also shown in [1] that $U_1 \geq U_{eq0}$).

The second condition in (2) denotes that no radiation enters the opaque transition layer at its boundary with the cold gas.

It can be readily shown that

$$1 \geqslant \mu \geqslant 0:$$

$$I(x,t,\mu \geqslant 0)=\frac{c}{4\pi}U_1\left(t-\frac{x}{\mu c}\right)E_\mu(0,x)$$
$$+\frac{c}{4\pi}\int_0^x k(x',t')\,U_p(x',t')\,E_\mu(x',x)\frac{dx'}{\mu},$$

$$-1 \leqslant \mu \leqslant 0:$$

$$I(x,t,\mu \leqslant 0)=-\frac{c}{4\pi}\int_x^{x_0} k(x',t')\,U_{eq}(x',t')\,E_\mu(x',x)\frac{dx'}{\mu}, \tag{3}$$

are solutions of Eq. (1) satisfying the boundary conditions (2). Here

$$E_\mu(x',x)=\exp\left\{-\int_{x'}^{x} k(x'',t'')\frac{dx''}{\mu}\right\},$$

$$t'=t-\frac{x-x'}{\mu c},\quad t''=t-\frac{x-x''}{\mu c}.$$

By definition, the radiation energy density $U(x,t)$ and the radiation flux $S(x,t)$ are connected with the intensity $I(x,t,\mu)$ by the relations

$$U(x,t)=\frac{2\pi}{c}\int_{-1}^{1} I(x,t,\mu)\,d\mu,\quad S(x,t)=2\pi\int_{-1}^{1} I(x,t,\mu)\,\mu d\mu. \tag{4}$$

Substituting (3) in (4) we arrive at integral equations for the radiation transfer, which relate $U(x,t)$ and $S(x,t)$ with the equilibrium radiation energy density $U_{eq} = aT^4$ and in which the retardation is taken into account automatically

$$U(x,t)=\frac{1}{2}\int_0^1 d\mu\, U_1\left(t-\frac{x}{\mu c}\right)E_\mu(0,x)$$
$$+\frac{1}{2}\int_0^1\frac{d\mu}{\mu}\int_0^x k(x',t')\,U_{eq}(x',t')\,E_\mu(x',x)\frac{dx'}{\mu}$$
$$+\frac{1}{2}\int_0^{-1}\frac{d\mu}{\mu}\int_x^{x_0} k(x',t')\,U_{eq}(x',t')\,E_\mu(x',x)\frac{dx'}{\mu}$$
$$\equiv U_0+U_++U_-, \tag{5}$$

$$S(x,t)=\frac{c}{2}\int_0^1 \mu d\mu\, U_1\left(t-\frac{x}{\mu c}\right)E_\mu(0,x)$$
$$+\frac{c}{2}\int_0^1 d\mu\int_0^x k(x',t')\,U_{eq}(x',t')\,E_\mu(x',x)\frac{dx'}{\mu}$$
$$+\frac{c}{2}\int_0^{-1} d\mu\int_x^{x_0} k(x',t')\,U_{eq}(x',t')\,E_\mu(x',x)\frac{dx'}{\mu}$$
$$\equiv S_0+S_++S_-. \tag{6}$$

The symbols $U_0$, $S_0$, $U_\pm$, and $S_\pm$ are defined in the equations; $U_0$ and $S_0$ are respectively the energy density and the radiation flux arriving at the point x from the transparency boundary; $U_+$, $S_+$, and $U_-$, $S_-$ are the energy density and the radiation flux arriving at the point x from the opaque transition layer, from the right and from the left of the point under consideration, respectively. It is recognized in all these quantities that the radiation experiences absorption along the path to the point x.

### 3. DIFFUSION APPROXIMATION WITH ACCOUNT OF FINITE SPEED OF LIGHT

Let us determine the solution of the radiation transport equations corresponding to the quasi-stationary mode wherein a plane heat wave propagates through the cold gas with constant velocity v, assuming all the functions in (5) and (6) to depend on the coordinate x and on the time t only in the combination $x - vt$ [by the same token, t under the integral sign must be taken with allowance for the retardation, as in (5) and (6)].

We solve the problem in the diffusion approximation; in other words we assume that over one radiation free path $l = 1/k$ the temperature of the substance does not change very much. In this case $U_{eq}$ can be expanded in (5) and (6) in powers of $(x - x')$ and only the first three terms of this series need be considered.

It is also convenient to introduce the "retarded" optical thickness $\zeta$ defined by the formulas

$$(d\zeta)_{x=\mathrm{const}} = -dx'k\left[x' - v\left(t - \frac{x-x'}{\mu c}\right)\right],$$

$$\zeta = \int_{x'}^{x} k(x'', t'')\,dx'' = \begin{cases} \infty & \text{for } x' = 0 \\ -\infty & \text{for } x' = x_0 \end{cases}. \tag{7}$$

Then the expanded expressions for $U(x - vt)$ and $S(x - vt)$ assume for a point x located not too close to the transparency boundary the form

$$U(x - vt) \approx U_{+} + U_{-}$$
$$= \frac{1}{2}\left[\int_0^1 \frac{d\mu}{\mu}\int_0^\infty [U_{eq}]\,e^{-\zeta/\mu}\,d\zeta + \int_0^{-1}\frac{d\mu}{\mu}\int_{-\infty}^{0}[U_{eq}]\,e^{-\zeta/\mu}\,d\zeta\right], \tag{8}$$

$$S(x - vt) \approx S_{+} + S_{-}$$
$$= \frac{c}{2}\left[\int_0^1 d\mu\int_0^\infty [U_{eq}]\,e^{-\zeta/\mu}\,d\zeta + \int_0^{-1} d\mu\int_{-\infty}^{0}[U_{eq}]\,e^{-\zeta/\mu}\,d\zeta\right]; \tag{9}$$

here

$$[U_{eq}] \equiv U_{eq}(x - vt)$$
$$+ (x' - x)\left(1 - \frac{\beta}{\mu}\right)U'_{eq} + \frac{(x'-x)^2}{2}\left(1 - \frac{\beta}{\mu}\right)^2 U''_{eq},\quad \beta \equiv \frac{v}{c}, \tag{10}$$

where $U_{eq}$ has as its argument $x - vt$, and $U'_{eq}$ and $U''_{eq}$ denote respectively the first and second derivatives of the function $U_{eq}(x - vt)$ with respect to the same argument. In (8) and (9) we left out $U_0$ and $S_0$, since they make exponentially small contributions to U and S at distances not very close to the transparency boundary $(x = 0)$.

To carry out the integration with respect to $\zeta$ and $\mu$ it is necessary to express $x' - x$ in terms of $\zeta$ and $\mu$. For this purpose we make use of the definition of the "retarded" optical thickness (7):

$$x' - x = -\int_0^\zeta l(\zeta)\,d\zeta \approx -\zeta l(x - vt)$$
$$-\frac{\zeta^2}{2}\left(\frac{dl}{d\zeta}\right)_{x-vt} = -\zeta l + \frac{\zeta^2}{2}\left(1 - \frac{\beta}{\mu}\right)ll'. \tag{11}$$

In (11), the argument of $l$ is $x - vt$ and $l'$ is the derivative of $l$ with respect to this argument.

Substituting (11) in (10) we obtain

$$[U_{eq}] = U_{eq}(\zeta, \mu) = U_{eq} - \zeta\left(1 - \frac{\beta}{\mu}\right)U'_{eq}l$$
$$+ \frac{\zeta^2}{2}\left(1 - \frac{\beta}{\mu}\right)(U''_{eq}l^2 + U'_{eq}l'l). \tag{12}$$

If we now again introduce the optical thickness $\tau$, the differential of which is $d\tau = d(x - vt)/l(T)$, with $\tau = 0$ when $x = 0$ and $\tau = \infty$ when $x = x_0$, we obtain in place of (12)

$$U_{eq}(\zeta, \mu) = U_{eq} - \zeta\left(1 - \frac{\beta}{\mu}\right)\frac{dU_{eq}}{d\tau} + \frac{\zeta^2}{2}\left(1 - \frac{\beta}{\mu}\right)^2\frac{d^2U_{eq}}{d\tau^2}. \tag{12'}$$

In the integrals $U_-$ and $S_-$ it is convenient to make the change of variables $\zeta \to -\zeta$ and $\mu \to -\mu$; then the expressions $U(x - vt)$ and $S(x - vt)$ assume, not too close to the transparency boundary, the symmetrical form

$$U(x - vt) \approx \frac{1}{2}\int_0^1 \frac{d\mu}{\mu}\int_0^\infty [U_{eq}(\zeta, \mu) + U_{eq}(-\zeta, -\mu)]\,e^{-\zeta/\mu}\,d\zeta, \tag{13}$$

$$S(x - vt) \approx \frac{c}{2}\int_0^1 d\mu\int_0^\infty [U_{eq}(\zeta, \mu) - U_{eq}(-\zeta, -\mu)]\,e^{-\zeta/\mu}\,d\zeta. \tag{14}$$

If we substitute (12′) in these equations and carry out integration with respect to $\zeta$ and $\mu$, we obtain expressions for U and S in the diffusion approximation with account of the finiteness of the speed of light

$$U \approx U_{eq} + \beta dU_{eq}/d\tau + \tfrac{1}{3}d^2U_{eq}/d\tau^2, \tag{15}$$

$$S \approx -\tfrac{1}{3}cdU_{eq}/d\tau - \tfrac{2}{3}\beta cd^2U_{eq}/d\tau^2. \tag{16}$$

We can eliminate the derivatives of $U_{eq}$ from (16) and (15), leaving the terms linear in $\beta$ and neglecting the derivatives of order higher than the second. We then obtain in their place two other equations, of which one relates the diffusion flux S with the radiation density U (with allowance for the finite speed of light),

$$-\beta dS/d\tau + S = -\tfrac{1}{3}cdU/d\tau, \tag{17}$$

and the other is the equation for the balance of radiant energy

$$-\beta c dU/d\tau + dS/d\tau = c\,(U_{eq} - U) \tag{18}$$

(we note that we used in [1] Eq. (17) without the first term in the left side.)

Equations (17) and (18) can also be obtained directly from the differential equation for the radiation transfer (1), if the latter is solved in the diffusion approximation [linear dependence of $I(\mu)$ on $\mu$]. For this purpose it is necessary to multiply successively (1) by to $2\pi\mu$ and $2\pi$ and integrate with respect to $\mu$ from −1 to +1, taking the definitions (4) into consideration.

We prefer to use the integral equations (5) and (6), since we simultaneously obtain from (6) an expression for the diffusion flux S on the transparency boundary. In deriving this expression we can no longer neglect $S_0$. By the same method used to derive (16) we obtain from (6), by putting in it $x = 0$,

$$S\,(\tau = 0) = \frac{c}{4}\left(U_1 - U_{eq0}\right) - c\left(\frac{1}{6} + \frac{1}{4}\beta\right)\left(\frac{dU_{eq}}{d\tau}\right)_0 - c\left(\frac{1}{8} + \frac{1}{3}\beta\right)\left(\frac{d^2U_{eq}}{d\tau^2}\right)_0 \tag{19}$$

(the subscript 0 denotes that the derivatives are taken at $\tau = 0$). The derivatives of $U_{eq}$ must be eliminated from (19) with the aid of (15) and (16); we then obtain in the desired approximation an expression for the diffusion flux on the transparency boundary

$$S\,(\tau = 0) + \frac{c}{4}(U_1 - U_{eq0}) + \frac{1}{2}S\,(0) - \frac{\beta c}{4}\left(\frac{dU}{d\tau}\right)_0 + \frac{3}{8}\left(\frac{dS}{d\tau}\right)_0. \tag{20}$$

Thus, the quantities T, U, and S on the boundaries of the transition layer should satisfy the following conditions:

on the transparency boundary

$$\tau = 0\begin{cases} T = T_0 \\ U = U_1 \\ S = S\,(\tau = 0) \end{cases} \quad (\text{i.e., } \varepsilon = \varepsilon\,(T_0) \text{ and } U_{eq} = U_{eq0} = aT_0^4); \tag{21}$$

on the boundary with the cold gas

$$\tau \to \infty, \qquad T = U = S = 0. \tag{22}$$

From the equation for the energy balance $\partial(\epsilon + U)/\partial t + \partial S/\partial x = 0$ and from condition (22) it follows that

$$S = v\,(\varepsilon + U), \tag{23}$$

where $\epsilon(T)$ is the energy density of the gas matter at the temperature T. On the transparency boundary (23) assumes the form

$$S\,(\tau = 0) = v\,(\varepsilon\,(T_0) + U_1). \tag{24}$$

## 4. SPEED OF PROPAGATION OF QUASI-STATIONARY HEAT WAVE

The purpose of the present paper was to ascertain how the finite speed of light influences the speed of propagation v of a heat wave in cold gas. As was indicated in Sec. 1, it is shown in [1] that $v > c$ is possible in the case when $U_1 \gg U_{eq0}$ on the transparency boundary. It is therefore of interest to investigate precisely this case.

We neglect $U_{eq}$ in (18) and eliminate S from (17) and (18); we then obtain an equation containing U only:

$$\left(\frac{1}{3} - \beta^2\right)\frac{d^2U}{d\tau^2} + 2\beta\frac{dU}{d\tau} - U = 0, \qquad \beta = \frac{v}{c}. \tag{25}$$

Exactly the same equation is obtained also for S, if we eliminate U from (17) and (18). We seek a solution of these equations in the form $C\exp(-\tau\lambda/\beta)$, where $\lambda$ must be real and positive for U and S to satisfy condition (22) as $\tau \to \infty$. An analysis of the characteristic equation for $\lambda$

$$\left(\frac{1}{3\beta^2} - 1\right)\lambda^2 - 2\lambda - 1 = 0 \tag{26}$$

shows that it can have the required positive solutions only if the coefficient of $\lambda^2$ is positive, that is, when

$$\beta < \frac{1}{\sqrt{3}}. \tag{27}$$

When this coefficient is equal to zero or is negative, Eq. (26) has no real positive roots.

The inequality (27) denotes that in the approximation considered here the plane quasi-stationary heat wave cannot propagate with arbitrarily large speed, for this speed v is always less than $c/\sqrt{3}$.

One of the two roots of (26) is negative and must be discarded; the positive root is

$$\lambda_1 = \sqrt{3}\,\beta/(1 - \sqrt{3}\,\beta). \tag{28}$$

The solutions of (26) for U and of the analogous equation for S, satisfying condition (22), are thus

$$U = C_1 e^{-\tau\lambda_1/\beta}, \qquad S = C_2 e^{-\tau\lambda_1/\beta}, \tag{29}$$

where the integration constants $C_1$ and $C_2$ must be determined from conditions (21) on the transparency boundary. It is obvious that $C_1 = U_1$. Using (24), we can also obtain $C_2$.

It follows from (18) and (23) that when $U \gg U_{eq}$ we have

$$\beta d\varepsilon/d\tau = -U.$$

Integrating this equation subject to condition (21), we obtain

$$U_1/\varepsilon(T_0) = \lambda_1 = \sqrt{3}\beta/(1-\sqrt{3}\beta). \qquad (30)$$

Equation (30) defines the ratio $U_1/\epsilon(T_0)$ as a function of $\beta = v/c$. It shows that when $\beta < 1/\sqrt{3}$ this ratio can assume an arbitrary finite value. Thus, no matter what the value of $U_1$ may be, the propagation speed v of the quasi-stationary heat wave is always smaller than $c/\sqrt{3}$.

When $U_1/\epsilon(T_0) \ll 1$, (that is, $\beta \ll 1/\sqrt{3}$), relation (30) goes over into the corresponding result of [1] for the case when $U_1 \gg U_{eq0}$.

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