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ON THE SOLUTION OF EQUATIONS OF THE CHEW-MANDELSTAM TYPE

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It is shown that the Chew-Mandelstam equation for the interaction amplitude of neutral mesons $A(\nu)$ ($\nu = q^2\mu^2$, q is the momentum in the c.m.s.) has a solution which does not oscillate at infinity only in cases that have no physical meaning, i.e., when the coupling constant is negative. This result can be obtained from some very general properties of analytic functions and may be of methodological interest irrespective of the specific problem under consideration.[1)]

1. In the Chew-Mandelstam theory, the interaction amplitude $A(\nu)$ satisfies the equation

$$A(\nu) = A(0) + \frac{1}{\pi}\int_0^\infty \sqrt{\frac{\nu'}{1+\nu'}}\left(\frac{1}{\nu'-\nu} - \frac{1}{\nu'}\right)|A(\nu')|^2 d\nu' + \frac{2}{\pi}\int_{-\infty}^{-1}\left(\frac{1}{\nu'-\nu} - \frac{1}{\nu'}\right)\frac{d\nu'}{\nu'}\int_0^{-\nu'-1}\sqrt{\frac{\nu''}{1+\nu''}}|A(\nu'')|^2 d\nu''. \quad (1)$$

The function $A(\nu)$ is considered in the complex ν plane with two cuts $(-\infty; -1)$ and $(0; +\infty)$. The function $A(\nu)$ is assumed to be bounded as $\nu \to \infty$.[2)]

It follows from (1) that for real ν we have

$$\operatorname{Re} A(\nu) = A(0) + \frac{1}{\pi}P\int_0^\infty \sqrt{\frac{\nu'}{1+\nu'}}\left(\frac{1}{\nu'-\nu} - \frac{1}{\nu'}\right)|A(\nu')|^2 d\nu' + \frac{2}{\pi}P\int_{-\infty}^{-1}\left(\frac{1}{\nu'-\nu} - \frac{1}{\nu'}\right)\frac{d\nu'}{\nu'}\int_0^{-\nu'-1}\sqrt{\frac{\nu''}{1+\nu''}}|A(\nu'')|^2 d\nu''. \quad (2)$$

Im $A(\nu)$ vanishes on the interval $(-1, 0)$, and satisfies on the upper edges of the cuts the relations

$$\operatorname{Im} A(\nu) = \sqrt{\frac{\nu}{1+\nu}}|A(\nu)|^2, \qquad 0 \leqslant \nu < +\infty; \quad (3)$$

$$\operatorname{Im} A(\nu) = \frac{2}{\nu}\int_0^{-\nu-1}\sqrt{\frac{\nu'}{1+\nu'}}|A(\nu')|^2 d\nu', \quad -\infty < \nu < -1. \quad (4)$$

It follows from (3) and (4) that on the upper edge of the right-hand cut Im $A(\nu) > 0$, and on the upper edge of the left cut Im $A(\nu) < 0$. At the points $\nu = 0$ and $\nu = -1$ the function $A(\nu)$ has root-type singularities.

If $A(\nu)$ does not oscillate at infinity, then $A(\nu)$ tends to limits along the upper edges of the left-hand and right-hand cuts as $\nu \to -\infty$ and $\nu \to +\infty$. These limits are equal to each other, for otherwise, in accord with the Lindelöf theorem (see, for example, [3]), the function $A(\nu)$ could not remain bounded[3)]. Thus

$$\lim_{\nu\to+\infty} A(\nu) = \lim_{\nu\to-\infty} A(\nu) = C. \quad (5)$$

It follows from (3) and (4) that this common limit is equal to zero, otherwise Im $A(\nu)$ would tend to different limits $|C|^2$ and $-2|C|^2$ as $\nu \to +\infty$ and $\nu \to -\infty$, which is impossible by virtue of (5).

2. We consider an arbitrary function $F(\nu)$, analytic in a plane with cut $(-\infty, -1)$ and $(0, +\infty)$, and having the following properties, which follow from Eq. (2) for the function $A(\nu)$:

Im $F(\nu) < 0$ on the upper edge of the left cut, Im $F(\nu) = 0$ on the segment $(-1; 0)$, and Im $F(\nu) > 0$ on the upper edge of the right-hand cut. From the fact that the function $F(\nu)$ is real on the segment $(-1; 0)$ it follows that the function $F(\nu)$ assumes conjugate values on opposite points of the cuts. We assume further that as $\nu \to \infty$ the function $F(\nu)$ tends uniformly to a limit $F(\infty)$. This limit is obviously real.

Let us show that the foregoing properties of the function $F(\nu)$ imply the inequalities

$$F(-1) > F(\infty), \qquad F(0) > F(\infty). \quad (6)$$

We assume that this statement is incorrect. Let, for example, $F(0) > F(\infty)$ and $F(-1) < F(\infty)$. Let us consider the mapping of the ν plane with the cuts on the F plane (see Fig. 1). From the

[1)]The incompatibility of the Chew-Mandelstam equations for S and P amplitudes of the interaction of charged pions was demonstrated by Lovelace.[2]

[2)]A logarithmic growth can also be assumed.

[3)]It can be shown that in this case the function $A(\nu)$ increases more rapidly than any polynomial.

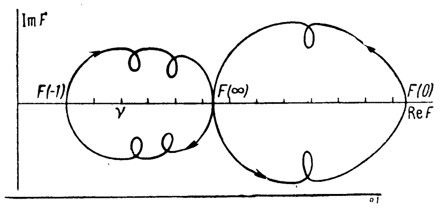

FIG. 1

inequality Im F(ν) > 0 along the upper edge of the right-hand cut it follows that the mapping of the lower edge lies completely in the lower half plane and joins the points F(∞) and F(0). The mapping of the upper plane is symmetrically situated. The inequalities F(0) > F(∞) and F(− 1) < F(∞) determine the circuiting directions indicated by the arrows in Fig. 1. Let γ be an arbitrary real point located inside the left half of the curve. It is obvious that the increment arg [F(ν) − γ] along the entire curve is equal to − 2. In accordance with a well-known theorem, this means that the number of poles of the function F(ν) − γ in the plane with cuts is larger by one than the number of zeroes of this function, i.e., the function F(ν) has at least one pole, which contradicts the assumption.

We can prove analogously the impossibility of the combinations F(− 1) < F(∞) and F(0) < F(∞) or F(− 1) > F(∞) and F(0) < F(∞).

The case where F(− 1) > F(∞) and F(0) > F(∞) is shown schematically in Fig. 2. If each

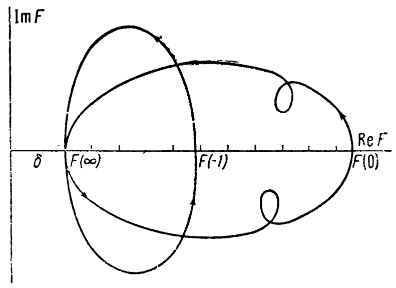

FIG. 2

closed loop in this figure is differently cross hatched, then the function F(ν) − a has as many roots as the times the point a is cross hatched. If δ is a real number, δ < F(∞), then the increment arg [F − δ] along the curve is zero, i.e., the function F(ν) does not assume real values δ < F(∞) in a plane with cuts. Going to the limit δ → F(∞), we find that the function F(ν) does not assume the value F(∞) at any finite point.

We have shown above that A(ν) = 0, and therefore the inequalities (6) go over in this case into the inequalities

$$A(0) > 0, \qquad A(-1) > 0. \qquad (7)$$

Thus, Eq. (1) has no solution for A(0) ≤ 0. Finally, if Eq. (1) has a solution for A(0) > 0, then the function A(ν) does not assume a value ≤ 0.

3. The case A(0) > 0, when the equation does have a solution, has apparently no physical meaning whatever. Indeed, a solution of the exact system of equations which follow from analyticity and unitarity (for the neutral meson interaction model) is equivalent to the solution for the case when the renormalized interaction energy of the mesons has the form

$$H' = \frac{\lambda}{4!}\int \varphi^4 dv, \qquad (8)$$

where the value of λ is close to the value of − A(0) [it can be shown rigorously that the sign of λ is always opposite to that of A(0)]. If the amplitude φ of the meson field (φ real) is large, then the field can be regarded as a classical object. If λ is negative (i.e., A(0) > 0), then the interaction energy H′ can be made as large as desired and negative. The free field energy H_0 is proportional only to φ^2. Therefore, in view of the growth of the meson field, its total energy $H_0 + H'$ can be reduced as much as desired when λ is negative. This means that the amplitude φ of such a meson field will increase as much as desired and that there exists no stationary solution. On the other hand if λ > 0, no such catastrophe occurs.[4)]

4. We obtained by interaction an approximate solution of Eqs. (2)—(4) [we introduced here a variable κ = ν/(1 + ν)]. Figure 3 shows the mapping of the plane $\nu = \nu_1 + i\nu_2$ with cuts (− ∞; − 1) and (0; ∞) on the plane A = Re A + i Im A for the case A(0) = 0.3. In this mapping, the equation A = A(ν) has one root in the region shaded once and two roots in the region shaded twice. The mapping of the cut (0; + ∞) is the curve I; the mapping of the cut (− ∞; − 1) is the dashed curve II. The segment (0; 0.3) of the Re A axis is obtained by mapping the segment (− 1; 0) of the ν_1 axis and a certain curve Γ, the crossing of which is accompanied by a reversal of the sign of the imaginary part (see Fig. 4).

The part of the curve Γ lying in the upper half-plane ν is mapped into the segment [0; α_2] of the Re A axis; the mapping of the portion of Γ located in the lower half-plane is [α_2; 0]. The segment [− 1; 0] of the ν_1 axis is mapped in the following fashion: [− 1; ν] goes over into [α_1; α_2], and

[4)]This item was written by K. A. Ter Martirosyan.

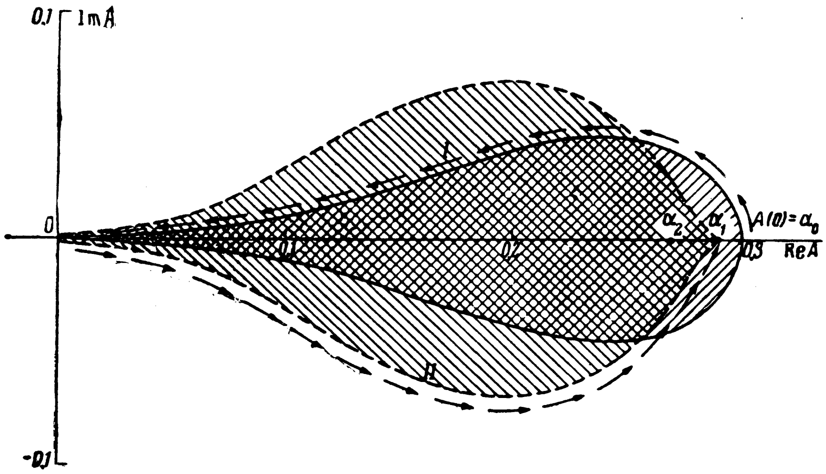

FIG. 3

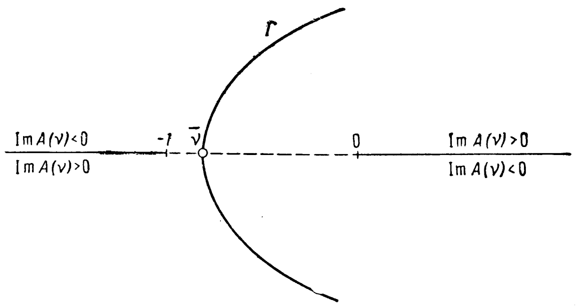

FIG. 4

$[\bar{\nu}; 0]$ goes over into $[\alpha_2; \alpha_0]$. In the calculation carried out here $\alpha_0 = A(0) = 0.3$ $\alpha_2 = 0.27$, and $\alpha_1 = 0.29$.

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[1] G. Chew, S. Mandelstam, UCRL-8728, April 1959.

[2] C. Lovelace, Nuovo cimento **21**, 305 (1961).

[3] R. Nevanlinna, Odnoznachnye analiticheskie funktsii (Single-Valued Analytic Functions), OGIZ, 1941, p. 68.

Translated by J. G. Adashko
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