DYNAMIC RESONANCES IN SOLUTIONS OF THE LOW EQUATION FOR SOME FIELD THEORY MODELS

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The possibility of existence of dynamic resonances in the solutions of the Low equation for some renormalized and non-renormalized field theory models is analyzed. It is shown that the constraint imposed on the coupling constant by the condition of analyticity of the amplitude precludes the appearance of dynamic resonances in the renormalized cases. The behavior of the S-phase shift in the nonrenormalized model of scalar charged mesons is similar to that of the P-phase shift in the Chew-Low model.

1. INTRODUCTION

MUCH attention is paid recently to the occurrence of resonances in the scattering of interacting particles. On the one hand, attempts are being made to attribute some of these resonances to the existence of new unstable particles which participate in the interaction [1,2]. (Resonances connected with unstable particles are called kinematic.) On the other hand, resonance solutions of the dispersion equations are investigated under the assumption that there are no unstable particles and that the resonances are due to the dynamics of the process (dynamic resonances). Attempts at a dynamic explanation of resonances have met so far with limited success, since no exact solutions have been obtained as yet for the dispersion equations. It is therefore of interest to consider simple field-theory models for which these equations can be solved exactly.

In the present article we consider the possible existence of dynamic resonances in the solutions of the Low equation for certain field-theory models, under the assumption that there are no contributions from the unstable particles. We show that resonant solutions are possible only for nonrenormalized models of field theory, while renormalized models have no such solutions. This conclusion agrees with the investigations of Shirkov, Efremov, and Chu [3] on resonance in the $\pi\pi$ system.

Resonance occurs even with the scattering S-phase in non-renormalized models. The energy behavior of the S-phase agrees qualitatively for a scalar charged model with derivative with the behavior obtained by Salzman [4] for the P-phase in the Chew-Low model.

2. RENORMALIZED MODELS OF FIELD THEORY

Castillejo, Dalitz, and Dyson [5] obtained a solution of the Low equation for the scattering amplitude $h_\alpha(\omega)$ of scalar charged mesons on a fixed nucleon (where $\alpha = 0$ for positive mesons and $\alpha = 1$ for negative ones). The solution has the form[1)]

$$h_\alpha(\omega) = \frac{\omega}{k} e^{i\delta_\alpha(\omega)} \sin\delta_\alpha(\omega)$$
$$= \left\{(-1)^\alpha \left(\frac{g^2}{2\pi}\right)^{-1} - \frac{1-\sqrt{1-\omega^2}}{\omega} - (-1)^\alpha R(\omega)\right\}^{-1}. \quad (1)$$

The function $R(\omega)$, as shown by Dyson [6] and by Klein [7], takes into account the contributions made by the unstable particles to the amplitude. If we require that the solution of (1) expanded in powers of g^2 coincide with perturbation theory, we must have $R(\omega) = 0$.

From the condition that the amplitude has in the unobservable region only a one-nucleon pole (for details see [5,8]), we obtain the limitation $g^2/2\pi < 1$ on the coupling constant. This limitation excludes the possibility of resonance in the solution of (1). Indeed, with $g^2/2\pi > 1$ we would have, on the one hand, an additional "nonphysical" pole in the unobservable region $\omega_0 < 1$ and, on the other hand, $\mathrm{Re}\, h_\alpha(\omega_{res}) = 0$ for $\alpha = 0$, where $\omega_{res} = g^2/2\pi$. This conclusion is illustrated in Fig. 1.

A similar situation obtains also in the other

1)We put everywhere $\hbar = c = \mu = 1$.

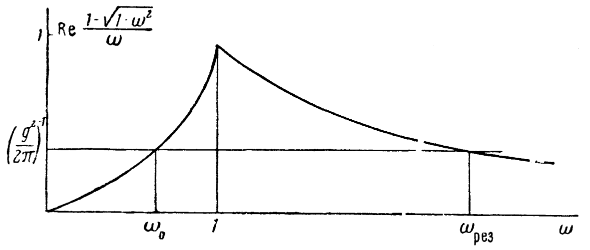

FIG. 1. The figure shows that the line $\psi(\omega) = (g^2/2\pi)^{-1}$ crosses the curve $\mathrm{Re}\,[(-\sqrt{1-\omega^2})/\omega]$ always at two points, $\omega_0 < 1$ and $\omega_{res} > 1$. Therefore the resonance at ω_{res} is always connected with the presence of a nonphysical pole at ω_0.

exactly-solvable renormalized model. The scalar mesons interact with the fixed nucleon, which can be in two states (proton, neutron) of different mass, $m_p = m_n + \Delta$, where we must put $\Delta < 1$ if we consider in the problem only stable particles. The Low equation and its solution for this model are written in the form

$$h_N(\omega) = -\frac{\delta_N g^2}{2(2\pi)^3}\left[\frac{\omega}{\Delta-\omega}+\frac{\omega}{\Delta+\omega}\right] + 4\pi\omega\int_1^\infty \frac{k'd\omega'}{\omega'^2}|h_N(\omega')|^2 \times\left[\frac{1}{\omega'-\omega-i\varepsilon}+\frac{1}{\omega'+\omega}\right],$$

$$h_N(\omega) = \frac{\omega\Delta}{2\pi^2(\omega^2-\Delta^2)} \times\left\{-\delta_N\left(\frac{g^2}{4\pi}\right)^{-1} + \frac{\Delta}{\sqrt{1-\Delta^2}}\,\frac{\sqrt{1-\omega^2}-\sqrt{1-\Delta^2}}{\sqrt{1-\omega^2}+\sqrt{1-\Delta^2}}\right\}^{-1}, \quad (2)$$

where

$$\delta_N = \begin{cases} 1 & \text{for } N = p \text{ (proton)} \\ -1 & \text{for } N = n \text{ (neutron)} \end{cases}$$

The R function is likewise discarded here.

Efimov and the author[8] have shown that

$$g^2/4\pi < \sqrt{1-\Delta^2}/\Delta. \quad (3)$$

If condition (3) is not satisfied, the amplitude will have an additional pole in the interval $0 \le \omega \le 1$. Condition (3) simultaneously with the nonphysical pole excludes the possibility of resonance in solution (2), for in this case we always have

$$\mathrm{Re}\left(\frac{\Delta}{\sqrt{1-\Delta^2}}\,\frac{\sqrt{1-\omega^2}-\sqrt{1-\Delta^2}}{\sqrt{1-\omega^2}+\sqrt{1-\Delta^2}}\right) < \left(\frac{g^2}{4\pi}\right)^{-1}.$$

Thus, in all the renormalized interaction cases discussed above, the resonance is excluded by the limitation on the coupling constant, which follows from the analytical properties of the amplitude.[2)]

[2)]Khalfin[10] has shown that the specified analytical properties of the amplitude lead to a limitation on the coupling constant in the case when the amplitude is restricted to a finite number of partial waves (in all the models considered here there is one S or P wave).

3. NONRENORMALIZED MODELS

Nonrenormalized models are obtained from those considered above by introducing in the interaction the time derivative of the meson operators. For example, for charged mesons we consider in place of

$$g\sum_{i=1}^{2}\tau_i\int d\mathbf{x}\varphi_i(\mathbf{x},t)\,\delta(\mathbf{x})$$

an interaction Hamiltonian in the form

$$f\sum_{i=1}^{2}\tau_i\int d\mathbf{x}\,\frac{\partial\varphi_i(\mathbf{x},t)}{\partial t}\,\rho(\mathbf{x}), \qquad \rho(\mathbf{x}) = \sum_k e^{i\mathbf{k}\mathbf{x}}v(\mathbf{k}),$$

where $\rho(\mathbf{x})$ is the form factor of the nucleon. (For further calculations we put $v(k) = L^2/(L^2+k^2)$, where L is the cutoff momentum.) In view of the presence of a derivative in the interaction, the Low equation for these models will contain under the integral sign an additional factor ω^2, as compared with the renormalized case. Besides, unlike (1), we have $h_\alpha(\omega) = \sin\delta_\alpha \exp[i\delta_\alpha]\,v^2(k)/k\omega$.

For charged mesons we now have in place of (1)

$$h_\alpha(\omega) = (-1)^\alpha\frac{f^2}{2\pi} + \frac{\omega}{\pi}\int_1^\infty d\omega' k' v^2(k')\left[\frac{|h_\alpha(\omega')|^2}{\omega'-\omega-i\varepsilon}+\frac{|h_{1-\alpha}(\omega')|^2}{\omega'+\omega}\right]. \quad (4)$$

The solution without the R function has the form

$$h_\alpha(\omega) = \left[(-1)^\alpha\left(\frac{f^2}{2\pi}\right)^{-1} - I(\omega)\right]^{-1}, \qquad I(\omega) = \frac{L^5\omega}{2(L+1)^2(L+\sqrt{1-\omega^2})^2}. \quad (5)$$

As in the first example [renormalized model (1)], we obtain from the properties of the amplitude for $0 \le \omega \le 1$ the inequality

$$f^2/2\pi \leqslant (2/L)(1+1/L)^2. \quad (6)$$

In the nonrenormalized model considered here, however, this limitation on f^2 does not exclude the possible existence of resonance in the solution (5).

Figure 2 shows the behavior of the function $\mathrm{Re}\, I(\omega)$. It is seen from this figure that if condition (6) is satisfied the equation $\mathrm{Re}\, h_\alpha^{-1}(\omega) = 0$ has for $\alpha = 0$ (positive mesons) two roots $1 \le \omega_{1res} \le L/\sqrt{6}$ and $L/\sqrt{6} \le \omega_{2res} \le \sqrt{L^2+1}$, i.e., there are two resonances. For sufficiently large L ($L \gg \sqrt{6}$), the second resonance at the point ω_{2res} lies in the region of large energies, where the static nucleon and the one-meson approximation cannot be employed. At the same time we can, by suitable choice of the coupling constant f^2, locate the first resonance in the low

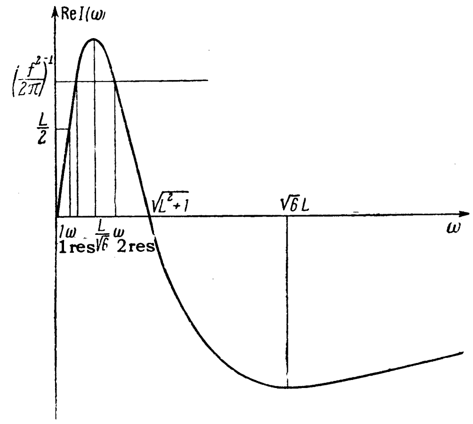

FIG. 2. Behavior of the function $\mathrm{Re}\,I(\omega)$. We see that the line $\psi_\alpha(\omega) = (-)^\alpha (f^2/2\pi)^{-1}$ can cross the curve $\mathrm{Re}\,I(\omega)$ both at the points $\omega_{1\,\mathrm{res}} > 1$ and $\omega_{2\,\mathrm{res}} > L/\sqrt{6}$ when $\alpha = 0$, and at the point $\omega_{3\,\mathrm{res}} > \sqrt{L^2+1}$ when $\alpha = 1$.

energy region $1 < \omega_{1\mathrm{res}} < 2$. From the form of the function $\mathrm{Re}\,I(\omega)$ shown in Fig. 2 it follows that resonance is possible also when $\alpha = 1$ (negative mesons). This resonance, however, lies at very large energies $\omega_{3\mathrm{res}} > \sqrt{L^2+1}$.

An interesting fact is the qualitative agreement between the curves*

$$(f^2/2\pi)\,\mathrm{Re}\,h_\alpha^{-1}(\omega) = (f^2/2\pi)\,k\omega v^2(k)\,\mathrm{ctg}\,\delta_\alpha \quad (\alpha = 0, 1),$$

obtained from formula (6), and the curves of $\mathrm{Re}\,g_{\alpha\alpha}(\omega)$ ($\alpha = 3, 1$), obtained by the Salzmans[4] for the P-phases in πN scattering from the solutions of the Chew-Low equation. (The discrepancy in the case of large energies is due to a different choice of the form factors.) In the solution of (6), we have $v(k) = L^2/(L^2 + k^2)$, whereas in [4] $v(k) = e^{-k/L}$. Thus, the presence of the S or P wave in the scattering is immaterial to the character of the solution of the Low equation. The decisive factor for the presence of resonant solutions is the nonrenormalizability of the interaction (the connection with the derivatives).

By way of another example of a nonrenormalized model we consider the Kemmer scalar symmetrical theory with derivative in the interaction

$$H_I = f \sum_{i=1}^{3} \tau_i \int dx\, \frac{\partial \varphi_i(x,t)}{\partial t}\, \rho(x).$$

Although it is impossible to obtain an exact solution of the Low equation in this case, for we have here a system of equations for two amplitudes $h_1(\omega)$ and $h_2(\omega)$ with total isotopic spinor $T = 1/2$ and $T = 3/2$, nevertheless we can obtain a limitation on the coupling constant f and show that it does not exclude the possibility of resonance in the amplitude $h_3(\omega)$.

*ctg = cot.

Let us examine the equation for $h_3(\omega)$

$$h_3(\omega) = \frac{f^2}{2\pi} + \frac{\omega}{\pi}\int_1^\infty d\omega' k' v^2(k') \times \left[\frac{|h_3(\omega')|^2}{\omega' - \omega - i\varepsilon} + \frac{1/3\,|h_3(\omega')|^2 + 2/3\,|h_1(\omega')|^2}{\omega' + \omega}\right]. \qquad (7)$$

It follows from (7) that $h_3(z)$ is an R function, since $\mathrm{Im}\,h_3(z) = \lambda(z)\,\mathrm{Im}\,z$ and $\lambda(z) > 0$. Using the Herglotz theorem [5] for the R function and taking into account the properties of $h_3(z)$ given by (7), we represent the inverse function $H_3(\omega) = -1/h_3(\omega)$ in the form

$$H_3(\omega) = R(\omega) - \left(\frac{f^2}{2\pi}\right)^{-1} + \frac{\omega}{\pi}\int_1^\infty d\omega' k' v^2(k') \times \left[\frac{1}{\omega' - \omega - i\varepsilon} + \frac{1}{\omega' + \omega}\left\{1 + 2\left|\frac{h_3 - h_1}{h_3 + 2h_1}\right|^2\right\}\right]. \qquad (8)$$

For agreement with perturbation theory we put $R(\omega) = 0$. Since $h_3(\omega)$ has no poles in the interval $0 \le \omega \le 1$, $H_3(\omega)$ does not vanish in this interval, and since $H_3(0) = -(f^2/2\pi)^{-1}$ and $dH_3(\omega)/d\omega > 0$ in the interval $0 \le \omega \le 1$, it follows that $H_3(1) \le 0$. This inequality leads to the following limitation on the coupling constant

$$\frac{f^2}{2\pi} \leqslant \left\{\frac{1}{\pi}\int_1^\infty d\omega' k' v^2(k')\left[\frac{1}{\omega' - 1} + \frac{1 + |x(\omega')|^2}{\omega' + 1}\right]\right\}^{-1}. \qquad (9)$$

If we discard under the integral sign in (9) the positive quantity

$$|x(\omega')|^2 = 2|(h_3(\omega') - h_1(\omega'))/(h_3(\omega') + 2h_1(\omega')|^2,$$

then the inequality becomes even stronger:

$$\frac{f^2}{2\pi} < \left\{\frac{1}{\pi}\int_1^\infty d\omega' k' v^2(k')\,\frac{2\omega'}{\omega'^2 - 1}\right\}^{-1} = \frac{2}{L}\left(1 + \frac{1}{L}\right)^2. \qquad (10)$$

The inequality (9) does not exclude the possibility of resonance in the amplitude $h_3(\omega)$. Indeed,

$$\mathrm{Re}\,h_3(\omega) = \frac{\left(\frac{f^2}{2\pi}\right)^{-1} - \frac{\omega}{\pi}\,\mathrm{P}\int_1^\infty d\omega' k' v^2(k')\left[\frac{1}{\omega' - \omega} + \frac{1 + |x(\omega')|^2}{\omega' + \omega}\right]}{\left|\left(\frac{f^2}{2\pi}\right)^{-1} - \frac{\omega}{\pi}\int_1^\infty d\omega' k' v^2(k')\left[\frac{1}{\omega' - \omega - i\varepsilon} + \frac{1 + |x(\omega')|^2}{\omega' + \omega}\right]\right|^2}. \qquad (11)$$

The vanishing of the numerator in (11) does not contradict (9) when $\omega = \omega_{\mathrm{res}}$, where $1 < \omega_{\mathrm{res}} < L$, since

$$\frac{\omega_{\mathrm{res}}}{\pi}\,\mathrm{P}\int_1^\infty d\omega' k' v^2(k')\left[\frac{1}{\omega' - \omega_{\mathrm{res}}} + \frac{1 + |x(\omega')|^2}{\omega' + \omega}\right] > \frac{1}{\pi}\int_1^\infty d\omega' k' v^2(k')\left[\frac{1}{\omega' - 1} + \frac{1 + |x(\omega')|^2}{\omega' + 1}\right]. \qquad (12)$$

In similar fashion we can obtain a limitation on f^2 in the Chew-Low model[1], too, by repeating the foregoing arguments for the amplitude h_3 (state $T = 3/2$ and $J = 3/2$). We obtain

$$\frac{f^2}{2\pi} \leqslant \frac{3}{4}\left\{\frac{1}{\pi}\int_1^\infty \frac{d\omega' k'^3 v^2(k')}{\omega'^2}\left[\frac{1}{\omega'-1}+\frac{1+|y(\omega')|^2}{\omega'+1}\right]\right\}^{-1}. \quad (13)$$

If we now discard the positive quantity

$$|y(\omega)|^2 = \frac{|h_2(\omega)-h_1(\omega)|^2 + 1/4\,|h_3(\omega)-h_1(\omega)|^2 + 1/4\,|h_3(\omega)-h_2(\omega)|^2}{|h_1(\omega)+h_2(\omega)+1/4\,h_3(\omega)|^2}$$

and put $L = 7$, we get $f^2/2\pi < 0.28$, which does not contradict the assumed value $f^2/2\pi = 0.08$.

From the examples considered above it follows that the occurrence of resonance in the solution of the Low equation is connected with the nonrenormalizability of the interaction. However, in the example given below, that of the nonrenormalized model of Bialynicki-Birul, the limitation on the coupling constant excludes, as in the renormalized models, the possibility of resonance. In this case we have in place of (2) and (3)

$$h_N(\omega) = -\frac{\delta_N f^2}{2(2\pi)^3}\left[\frac{\omega}{\Delta-\omega}+\frac{\omega}{\Delta+\omega}\right] + 4\pi\omega\int_1^\infty d\omega' k' v^2(k')\,|h_N(\omega')|^2\left[\frac{1}{\omega'-\omega-i\varepsilon}+\frac{1}{\omega'+\omega}\right] \quad (14)$$

with a solution

$$h_N(\omega) = \frac{\omega\Delta}{2\pi^2(\omega^2-\Delta^2)}\left\{-\delta_N\left(\frac{f^2}{2\pi}\right)^{-1} - \frac{\Delta^3}{\sqrt{1-\Delta^2}} + \frac{L\omega^2\Delta(2L+\sqrt{1-\Delta^2})}{(\sqrt{1-\Delta^2}+\sqrt{1-\omega^2})(L+\sqrt{1-\omega^2})^2}\right\}^{-1}, \quad (15)$$

where the coupling constant is limited by an inequality that follows from the requirement that $h_N(\omega)$ have no zeroes or poles outside the real axis

$$f^2/4\pi < \sqrt{1-\Delta^2}/\Delta\,(1+L\sqrt{1-\Delta^2}). \quad (16)$$

Inequality (16) can be readily obtained by the method first proposed by Gribov et al[12,13].

$$I(\omega^2) = \mathrm{Re}\,\frac{L\omega^2\Delta(2L+\sqrt{1-\Delta^2})}{(\sqrt{1-\Delta^2}+\sqrt{1-\omega^2})(L+\sqrt{1-\omega^2})^2}.$$

Figure 3 shows a plot of the function

$$\psi(\omega^2) = (f^2/2\pi)^{-1} + \Delta^3/\sqrt{1-\Delta^2}$$

It is seen from it that in order to exclude the additional pole in the interval $0 \le \omega \le 1$ it would be enough to stipulate that the line must not cross the curve $I(\omega_2)$ in this interval. From this we get a limitation for the constants f^2 when $L \gg 1$:

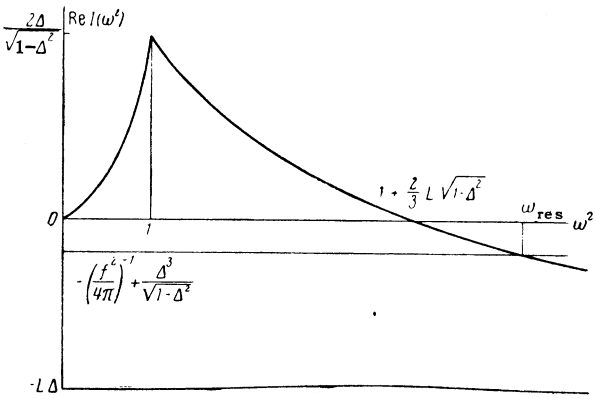

FIG. 3. It follows from inequality (17) that the line $\psi_N(\omega^2) = -\delta_N(f^2/2\pi)^{-1} + \Delta^3/\sqrt{1-\Delta^2}$ nowhere crosses the curve $I(\omega^2)$.

$$(f^2/4\pi)^{-1} + \Delta^3/\sqrt{1-\Delta^2} > 2\Delta/\sqrt{1-\Delta^2}. \quad (17)$$

But inequality (17) is clearly weaker than (16), since $L \gg \sqrt{1-\Delta^2}$. The satisfaction of the stronger inequality (17) is necessary not only to prevent the appearance of an additional pole in the interval $0 \le \omega \le 1$, but to exclude a possible pole of $h_N(\omega)$ on the imaginary axis, made possible by the fact that the function in the braces in (15) depends on ω^2. Resonance would be possible under inequality (17), but inequality (16) excludes this possibility.

4. CONCLUSION

From the examples considered it follows that in the case of nonrenormalized interactions, resonance exists for definite values of the coupling constant f^2 and of the cutoff parameter L. The possibility of resonance in the nonrenormalized scalar charged and symmetrical theories, considered in Sec. 3, is interesting because in this model there exists only an S-phase in the scattering amplitude, and therefore the resonance cannot be attributed to the centrifugal barrier.

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[1] M. Gell-Mann and F. Zachariasen, Preprint CTSL-26, (1961).

[2] F. Zachariasen, Preprint "A Self-consistent Calculation of the Mass and Width."

[3] Shirkov, Efremov, and Chu, Preprint, D-697 (1961).

[4] G. Salzman and F. Salzman, Phys. Rev. **108**, 1619 (1957).

[5] Castillejo, Dalitz, and Dyson, Phys. Rev. **101**, 453 (1956).
[6] F. Dyson, Phys. Rev. **106**, 157 (1957).
[7] F. Klein, Phys. Rev. **109**, 992 (1958).
[8] B. M. Barbashov and G. V. Efimov, Preprint, R-762 (1961).
[9] B. M. Barbashov and G. V. Efimov, JETP **40**, 848 (1961), Soviet Phys. JETP **13**, 595 (1961).
[10] L. A. Khalfin, JETP **41**, 1233 (1961), Soviet Phys. JETP **14**, 880 (1962).
[11] G. Chew and F. Low, Phys. Rev. **101**, 1571 (1956).
[12] Ansel'm, Gribov, Danilov, Dyatlov, and Shekhter, JETP **41**, 619 (1961), Soviet Phys. JETP **14**, 444 (1962).
[13] B. M. Barbashov, Preprint, Joint Institute for Nuclear Research, R-841, (1961).

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20