

ELECTRON RECOMBINATION IN A MONATOMIC GAS

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The Fokker-Planck equation is used to compute the electron-ion recombination coefficient in three-body collisions between an electron, ion, and gas atom.

IT is the purpose of the present work to calculate the electron-ion recombination coefficient in a weakly ionized gas at high pressures and moderately high temperatures.

It is well-known that at low gas densities the chief contribution to the recombination process is due to radiative recombination. Three-body recombination processes become more important at higher pressures. If the gas is almost completely ionized then three-body collisions between charged particles* are important; if the gas is weakly ionized the important process is collisions between an ion, an electron and a neutral atom. We shall be interested specifically in a weakly ionized gas. An elementary calculation of the recombination coefficient α for this case has been given by Thomson (cf. [2]). We show below that the elementary calculation yields the proper form for α but gives an incorrect numerical coefficient.

A characteristic feature of collisions between electrons and atoms is the fact that the electron suffers a large change in momentum but a small change in energy (given approximately by m/M). This means that the time required for the electron energy distribution to reach equilibrium is much longer than is the case for the other variables. For this reason the electron energy distribution in the field depends only on the relative energy of the electron and the ion E and the recombination process can be regarded as a diffusion process in energy space in the direction of negative energies. This means that the Fokker-Planck equation can be used in place of the usual kinetic equation. In other words, the equation describing the electron distribution function in phase space $f(E)$ is a diffusion equation

$$\frac{\partial f}{\partial t} = -A(E) \frac{\partial}{\partial E} j(E). \quad (1)$$

*This case has been studied in detail by Belyaev and Budker[1].

The particle flux $j(E)$ must vanish for a Boltzmann distribution and is thus of the form [3]

$$j(E) = -B(E) (\partial f / \partial E + f / kT) \quad (2)$$

where k is the Boltzmann constant and T is the temperature.

We first clarify the meaning of the coefficients A and B . The coefficient A is determined immediately from the conservation of particles. The change per unit time of the number of particles with energies between E_1 and E_2 must be equal to the difference in the flow of particles to higher energies and to lower energies, i.e., $j(E_1) - j(E_2)$. In the phase space layer lying between energies E_1 and E_2 we integrate (1) with respect to

$$d\Gamma = d^3p d^3r = dp_x dp_y dp_z dx dy dz,$$

and find (N is the number of electrons in the layer):

$$\frac{dN}{dt} = \int dE \int \frac{d\Gamma}{dE} A(E) \frac{dj}{dE}.$$

As indicated above, the right side of this expression must be of the form $j(E_1) - j(E_2)$ so that

$$A(E) \int \frac{d\Gamma}{dE} = 1.$$

For an electron in the Coulomb field of a singly charged ion we have $E(\mathbf{p} \cdot \mathbf{r}) = p^2/2m - e^2/r$ (m is the mass and e is the charge of the electron) and

$$\int \frac{d\Gamma}{dE} = \int \delta \left(E - \frac{p^2}{2m} + \frac{e^2}{r} \right) d\Gamma. \quad (3)$$

In what follows it will be necessary to deal with negative values of E only. Computing the integral in (3) for the case $E < 0$ we have

$$A = \sqrt{2} |E|^{1/2} / 2\pi^3 e^6 m^{3/2}. \quad (4)$$

To find the recombination coefficient we need only find the stationary solution of (1). In this case

$$B(E) (\partial f / \partial E + f / kT) = -j = \text{const} \quad (5)$$

and

$$f = -je^{-E/kT} \int_C^E e^{\varepsilon/kT} \frac{d\varepsilon}{B(\varepsilon)}. \quad (6)$$

To determine the constants j and C we must know the boundary conditions. Since small (absolute) values of E correspond to large distances from the ion, in which case the electron distribution is a Boltzmann distribution, we have

$$f \rightarrow (2\pi kTm)^{-3/2} e^{-E/kT} \text{ for } E \rightarrow 0. \quad (7)$$

Negative values of E that are large in absolute value correspond to small distances from the ion. An electron that has come this close remains long enough to recombine by a radiative process and is eliminated from consideration. Thus, at some value of the energy E_0 that is large in modulus we can set the distribution function equal to zero:*

$$f = 0, \quad E = E_0, \quad |E_0| \gg kT \quad (8)$$

(the final result will be independent of the exact value of E_0). From (7) and (8) we have

$$C = E_0, \quad j = - \left[(2\pi m kT)^{3/2} \int_{E_0}^0 e^{E/kT} \frac{dE}{B(E)} \right]^{-1}. \quad (9)$$

The particle flux j , taken with opposite sign, is obviously equal to the number of electrons that reach a value E_0 per unit time, that is to say, the number of electrons that recombine. Hence, the recombination coefficient is given directly by j if the distribution function is normalized, as we have done, to one electron. Since $E_0 \gg kT$, we can replace E_0 by $-\infty$ in the integral in (9). We then obtain the final general expression for the recombination coefficient:

$$\alpha = \left[(2\pi m kT)^{3/2} \int_{-\infty}^0 e^{E/kT} \frac{dE}{B(E)} \right]^{-1}. \quad (10)$$

To determine the significance of the coefficient B , in (1) we substitute the distribution function f_0 that describes an electron of energy E

$$f_0 = A \delta(p^2/2m - e^2/r - E). \quad (11)$$

The function in (11) is obviously normalized to a single electron, i.e., $\int f_0 d\Gamma = 1$. We then multiply the left and right sides of (1) by $(\Delta E)^2 = [E(\mathbf{p}, \mathbf{r}) - E]^2$ and integrate over $d\Gamma$. As a result, on the left we obtain $\overline{\partial(\Delta E)^2/\partial t}$, i.e., the square of the change in energy of a particle per unit time averaged over the distribution (5); on the right, after a double integration by parts we obtain the quantity $2AB$. Thus,

*It is interesting to note that an equation of the form of (2), with the same boundary conditions, is encountered in the theory of boiling.^[4]

$$B = \frac{1}{2A} \frac{\overline{\partial(\Delta E)^2}}{\partial t} = \frac{\pi^3 \sqrt{2}}{2} \frac{e^6 m^{3/2}}{|E|^{5/2}} \frac{\overline{\partial(\Delta E)^2}}{\partial t}. \quad (12)$$

To compute $\overline{\partial(\Delta E)^2/\partial t}$ we must examine the actual details of a collision between an electron and an atom in the electric field of an ion. We assume that the gas is monatomic and limit ourselves to the case in which the collision between the atom and the electron is so fast that the electron may be assumed to be free during the collision. If this requirement is to be satisfied ω , the frequency of the perturbation produced by the atom at the point at which the electron is located, must be much higher than the frequency of rotation of the electron about the ion Ω . Since $\omega \sim V_a/a \sim a^{-1} \sqrt{kT/M}$ (V_a is the velocity of the atom, a is the atomic dimension, M is the mass of the atom) and $\Omega \sim mV^3/e^2 \sim (kT)^{3/2}/m^{1/2}e^2$ (V is the electron velocity), this requirement implies that

$$kT \ll \sqrt{m/M} e^2/a. \quad (13)$$

The condition in (13) sets the limits of applicability of the formulas obtained below.

We note further that it is valid to neglect collisions between ions and neutral atoms. The ion velocity is $\sqrt{M/m}$ times smaller than the electron velocity. For this reason ion-neutral collisions are much less frequent than electron-neutral collisions and have only a small effect on the energy. For this reason it may be assumed that the ion moves uniformly and that the entire analysis can be carried out in the coordinate system fixed in the ion.

Since the electron can be considered free if (13) holds, we can write $\overline{\partial(\Delta E)^2/\partial t}$ immediately as the average of the product of the square of the energy change per collision $(\Delta E)^2$ times the number of collisions per unit time:

$$\overline{\partial(\Delta E)^2/\partial t} = n \overline{V} \int (\Delta E)^2 d\sigma, \quad (14)$$

where n is the number of atoms per unit volume, V is the electron velocity and $d\sigma$ is the differential cross section for scattering of an electron by an atom. (We assume that V is much greater than the atom velocity V_a .)

The quantity $(\Delta E)^2$ can be computed easily if we assume that the magnitude of the relative velocity of the electron and atom is not changed in the collision:^[5]

$$(\mathbf{V} - \mathbf{V}_a)^2 = (\mathbf{V}' - \mathbf{V}_a')^2$$

(\mathbf{V}' and \mathbf{V}_a' are the velocities after the collision while \mathbf{V} and \mathbf{V}_a are the velocities before the collision; all velocities are measured in the coordi-

nate system in which the ion is at rest). Thus

$$V'^2 - V^2 \approx 2(V'V'_a - VV_a) \approx 2V_a(V' - V)$$

(we assume that the atom velocity is not changed greatly in the collision) and

$$(\Delta E)^2 = \frac{1}{4} m^2 (V'^2 - V^2)^2 \approx m^2 [(V_a V)^2 + (V_a V')^2 - 2(V_a V')(V_a V)]. \quad (15)$$

We must now average (15) over electron and atom velocities. For this purpose we express V_a in terms of the velocities of the atom and ion in a fixed coordinate system:

$$V_a = V_{a0} - V_{i0}.$$

Then, averaging (15) over V_{a0} and V_{i0} and assuming that $\overline{V_{i0}^2} = \overline{V_{a0}^2} = 3kT/M$, we have

$$(\Delta E)^2 = \frac{4}{3} m^2 [V^2 - (VV')^2] \overline{V_{a0}^2} = \frac{4kT}{M} (1 - \cos \vartheta) p^2$$

(p is the electron momentum and ϑ is the angle between V and V').

Introducing the effective transport cross section

$$\sigma^* = \int (1 - \cos \vartheta) d\sigma,$$

we have

$$\frac{\partial (\Delta E)^2}{\partial t} = \frac{4kTn}{Mm} \overline{\sigma^*(p) p^3}, \quad (16)$$

where the average is taken over the distribution in (5). We now must take account of the fact that if (13) is satisfied the electron wavelength is much greater than the atomic dimensions so that the electron scattering is independent of the electron momentum and is spherically symmetric. The cross section σ^* is then equal to the total cross section σ , being independent of momentum, and we need only average over the factor p^3 :

$$\begin{aligned} \overline{p^3} &= \int p^3 A \delta \left(\frac{p^2}{2m} - \frac{e^2}{r} - E \right) d^3 p d^3 r \\ &= 4\pi e^6 A \int p^3 \frac{d^3 p}{(p^2/2m + |E|)^4} = \frac{32\sqrt{2}}{3\pi} (m|E|)^{3/2}. \end{aligned} \quad (17)$$

Substituting (17) in (16) and (12) we have

$$B(E) = \frac{128\pi^2 e^6 m^2 \sigma kTn}{3 M |E|}. \quad (18)$$

Then, calculating the integral in (10) we find the recombination coefficient

$$\alpha = \frac{32\sqrt{2}\pi}{3} \frac{m^{1/2} e^6 \sigma n}{(kT)^{3/2} M}. \quad (19)$$

This quantity is approximately six times greater than that derived in [2] by an elementary calculation. We note that the condition given in (13) for the applicability of (19) is rather stringent. For example, the limit of applicability of this formula for helium is approximately 1500°K. At higher temperatures the transfer of energy by the electron to the atom in a collision is reduced sharply and this effect should reduce the recombination coefficient markedly as compared with the value given by (19). If our formula is to apply it is also necessary that the equilibrium over the electron coordinate system be established more rapidly than the energy equilibrium. This requirement implies that $kT \gg e^2 n \sigma \sqrt{m/M}$.

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