A DIAGRAM TECHNIQUE FOR EVALUATING TRANSPORT COEFFICIENTS IN STATISTICAL PHYSICS AT FINITE TEMPERATURES

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We use a direct analytical continuation of the diagrams for the temperature-dependent Green's functions in the Matsubara method [1-3] from discrete points on the imaginary axis onto the whole plane of the complex frequency ω to construct a diagram technique which involves quantities depending on the real frequency. This method enables us to evaluate the transport coefficients of a system at finite temperatures

1. INTRODUCTION

RECENTLY quantum field theory methods have been applied intensively in statistical physics at non-zero temperatures. One of these methods, that of Matsubara, [1] uses the so-called temperature-dependent Green's functions which depend on a fictitious "imaginary" time. A special advantage of this method is that it allows the use of the Fourier transform with respect to the imaginary time. [2,3] The diagram technique which arises in this way differs from the usual one in field theory at T = 0 in that the integration over the frequency ω is replaced by a summation over discrete "imaginary" frequencies $i\omega_n$.

An application of Matsubara's method involving an expansion in imaginary frequencies has made it possible to solve many statistical problems. However, its application to a study of transport phenomena runs into serious difficulties. The difficulty is the following: transport properties of a system are naturally, described by the usual field-theory Green's functions (we shall give several examples) which depend on the time t.

If we know the temperature-dependent Green's functions of the Matsubara technique, we can in principle obtain also the usual Green's functions by an analytical continuation of the former from the discrete points on the imaginary axis into the whole complex ω-plane. [2] Although such an analytical continuation can in various concrete cases be done without any particular difficulty, there exists in principle no definite algorism.

In the following we shall describe a diagram technique which uses directly quantities that depend on the real frequency ω. It is obtained by a direct analytical continuation of the Matsubara diagrams from the points on the imaginary axis into the upper (or lower) half-plane of the complex variable ω. This technique is a generalization of the technique suggested by Luttinger [4] for the case T = 0. The technique proposed here is, in my opinion, appreciably simpler and more convenient than the one of Konstantinov and Perel' [5] which uses complicated contours in the complex time plane.

2. ANALYTICAL CONTINUATION

We start with the analytical continuation of the diagrams for the single-particle temperature-dependent Green's function $\mathfrak{G}_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2; \tau_1 - \tau_2)$.* The diagram technique to evaluate its Fourier transforms $\mathfrak{G}(\mathbf{p}, \omega_n)$ differs from the usual quantum field theory diagram technique by the replacement of the integration over the real frequencies ω by a summation over discrete "imaginary" frequencies† $i\omega_n = i\pi nT$ (where it is well known [2] that the Green's function for fermions contains only the "odd" frequencies $\omega_n = \pi(2n+1)T$, and the Green's function for bosons only the "even" ones $\omega_n = 2\pi nT$). The analytical continuation of $\mathfrak{G}(\omega_n)$ from the discrete points on the upper imaginary semi-axis is a function which is analytical in the upper half-plane of the complex variable ω and is the same as the so-called retarded Green function $G^R(\omega)$. We similarly obtain from the continuation from the discrete points $i\omega_n$ on the lower imaginary semi-axis the ad-

*We shall assume in the following that the system is homogeneous and nonferromagnetic. In that case $\mathfrak{G}_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2; \tau_1 - \tau_2) = \delta_{\alpha\beta}\mathfrak{G}(\mathbf{r}_1, \mathbf{r}_2; \tau_1 - \tau_2)$.

†We use units such that $\hbar = c = 1$. The temperature is measured in energy units.

vanced function $G^A(\omega)$ which is analytical in the lower half-plane ω.

The functions $G^R(\mathbf{p}, \omega)$ and $G^A(\mathbf{p}, \omega)$ have a common Lehmann expansion [2]

$$\int_{-\infty}^{\infty} \frac{\rho(\mathbf{p}, \eta)}{\eta - \omega} d\eta \qquad (1)$$

with a real function ρ, where G^R is that branch of the function (1) which is analytical in the upper half-plane while G^A is the branch analytical in the lower half-plane. The function $\rho(\mathbf{p}, \omega)$ is for real ω the same, apart from a factor, as the imaginary part of G^R (or G^A)

$$\rho(\omega) = \pi^{-1} \operatorname{Im} G^R(\omega) = -\pi^{-1} \operatorname{Im} G^A(\omega). \qquad (2)$$

The temperature-dependent Green's function $\mathfrak{G}(\omega_n)$ is the same as G^R on the upper imaginary semi-axis and the same as G^A on the lower semi-axis:

$$\mathfrak{G}(\omega_n > 0) = G^R(i\omega_n),$$
$$\mathfrak{G}(\omega_n < 0) = G^A(i\omega_n). \qquad (3)$$

The functions G^R and G^A are connected with the usual Green's function by Landau's well-known relations [6]

$$\operatorname{Re} G(\omega) = \operatorname{Re} G^R(\omega) = \operatorname{Re} G^A(\omega),$$
$$\operatorname{Im} G(\omega) = \operatorname{th}\frac{\omega}{2T} \operatorname{Im} G^R(\omega) = -\operatorname{th}\frac{\omega}{2T} \operatorname{Im} G^A(\omega) \qquad (4a)^*$$

for fermions, and

$$\operatorname{Re} G(\omega) = \operatorname{Re} G^R(\omega) = \operatorname{Re} G^A(\omega),$$
$$\operatorname{Im} G(\omega) = \operatorname{cth}\frac{\omega}{2T} \operatorname{Im} G^R(\omega) = -\operatorname{cth}\frac{\omega}{2T} \operatorname{Im} G^A(\omega) \qquad (4b)$$

for bosons.

From (1) and (3) follows an equation for $\mathfrak{G}(\mathbf{p}, \omega)$ in terms of $\rho(\mathbf{p}, \omega)$, which is important for what follows. Performing the inverse Fourier transformation in (1) and using (3) we get for fermions

$$\mathfrak{G}(\mathbf{p}, \tau) = \begin{cases} \int_{-\infty}^{\infty} (1 - n(\eta)) e^{-\eta\tau} \rho(\mathbf{p}, \eta) d\eta, & \tau > 0 \\ -\int_{-\infty}^{\infty} n(\eta) e^{-\eta\tau} \rho(\mathbf{p}, \eta) d\eta, & \tau < 0 \end{cases}, \qquad (5)$$

where $n(\eta)$ is the Fermi function

$$n(\eta) = [e^{\eta/T} + 1]^{-1}.$$

Similarly for bosons

$$\mathfrak{G}(\mathbf{p}, \tau) = \begin{cases} \int_{-\infty}^{\infty} (1 + n(\eta)) e^{-\eta\tau} \rho(\mathbf{p}, \eta) d\eta, & \tau > 0, \\ \int_{-\infty}^{\infty} n(\eta) e^{-\eta\tau} \rho(\mathbf{p}, \eta) d\eta, & \tau < 0 \end{cases}, \qquad (6)$$

$$n(\eta) = [e^{\eta/T} - 1]^{-1}.$$

*th = tanh, cth = coth.

We turn now directly to the analytical continuation of diagrams. It is most convenient to do this without performing a Fourier transformation with respect to the time τ in the internal lines. Moreover, we shall at once analytically continue the diagrams which contain already the total Green's functions $\mathfrak{G}$ (depicted by heavy lines), instead of the free-particle Green's function $\mathfrak{G}^{(0)}$ (light lines), i.e., compact, irreducible diagrams which do not contain internal self-energy parts.

For the case of a two-particle interaction the simplest of such diagrams is given by Fig. 1.

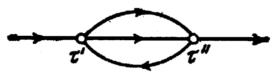

FIG. 1

After performing the Fourier transformation with respect to the spatial coordinates the expression corresponding to this diagram is (apart from a factor) of the form*

$$\delta\mathfrak{G}(\mathbf{p}, \tau_1 - \tau_2) = \int \mathfrak{G}^{(0)}(\mathbf{p}, \tau_1 - \tau') \Gamma^{(0)}(\mathbf{p}, \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}; \mathbf{p}_1, \mathbf{p}_2)$$
$$\times \mathfrak{G}(\mathbf{p}_1, \tau' - \tau'') \mathfrak{G}(\mathbf{p}_2, \tau' - \tau'')$$
$$\times \mathfrak{G}(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}, \tau'' - \tau') \Gamma^{(0)}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}, \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p})$$
$$\times \mathfrak{G}(\mathbf{p}, \tau'' - \tau_2) d\mathbf{p}_1 d\mathbf{p}_2 d\tau' d\tau''. \qquad (7)$$

$\Gamma^{(0)}$ is the function describing the interaction; it is independent of the time.

This expression is, as is the total Green's function itself, a function of the difference of the coordinates $\tau' - \tau''$ (see [2]), and

$$\delta\mathfrak{G}(\tau < 0) = -\delta\mathfrak{G}(\tau + 1/T).$$

Its Fourier transform $\delta\mathfrak{G}(\omega_n)$ is thus connected with a double Fourier transform through the relation

$$\delta\mathfrak{G}(\omega_{n1}, \omega_{n2}) = \frac{1}{4} \int_{-1/T}^{1/T} \int_{-1/T}^{1/T} d\tau' d\tau'' e^{i\omega_{n1}\tau_1 - i\omega_{n2}\tau_2} \delta\mathfrak{G}(\tau_1 - \tau_2)$$
$$= \frac{1}{T} \delta_{\omega_{n1}, \omega_{n2}} \delta\mathfrak{G}(\omega_{n1}).$$

We evaluate the double Fourier transform of (7) and substitute instead of the Green's functions corresponding to external lines their Fourier series expansions

$$\mathfrak{G}(\tau) = T \sum_h e^{-i\omega_n\tau} \mathfrak{G}(\omega_n),$$

and instead of the functions corresponding to internal lines their Lehmann expansions (5). Equa-

*We shall not write down the spin indices of the $\mathfrak{G}$ functions. Moreover, we restrict ourselves solely to the fermion case; the transition to bosons is completely obvious.

tion (7) is written in the form (we shall henceforth write out only integrals with respect to the time)

$$\delta\mathfrak{G}(\omega_n,\omega_n')\sim-\int d\eta_1 d\eta_2 d\eta_3 \mathfrak{G}^{(0)}(\mathbf{p},\omega_n)\,\mathfrak{G}(\mathbf{p},\omega_n')\,\rho(\mathbf{p}_1,\eta_1)$$
$$\times\rho(\mathbf{p}_2,\eta_2)\,\rho(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p},\eta_3)\,\{(1-n(\eta_1))(1-n(\eta_2))$$
$$\times n(\eta_3)\int\limits_0^{1/T}d\tau'\int\limits_0^{\tau'}d\tau'' e^{i\omega_n\tau'-i\omega_n'\tau''}e^{(-\eta_1-\eta_2+\eta_3)(\tau'-\tau'')}$$
$$-n(\eta_1)\,n(\eta_2)$$
$$\times(1-n(\eta_3))\int\limits_0^{1/T}d\tau''\int\limits_0^{\tau''}d\tau' e^{i\omega_n\tau'-i\omega_n'\tau''}e^{(-\eta_1-\eta_2+\eta_3)(\tau'-\tau'')}\}.$$

When we evaluate the integrals over τ' and τ'' we obtain two kinds of expressions, one proportional to $\delta_{\omega_n,\omega_n'}$ (it is obtained when we substitute the upper limit), and the other not. One verifies easily by using the explicit expression for the Fermi function that the two terms of the second kind cancel one another. This cancellation follows, incidentally, from general considerations. Indeed, since $\mathfrak{G}$ (and with it also any Feynman diagram) is a function of the difference between the coordinates only, its double Fourier transform is non-vanishing only if $\omega_n \neq \omega_n'$. The structure of the terms which do not contain $\delta_{\omega_n,\omega_n'}$ is, however, such that if they cancel when $\omega_n \neq \omega_n'$ (as should be the case), the same happens also when $\omega_n = \omega_n'$. We get thus finally ($\delta\mathfrak{G} = \delta\mathfrak{G}_a + \delta\mathfrak{G}_b$)

$$\delta\mathfrak{G}_a(\omega_n)\sim-\mathfrak{G}^{(0)}(\omega_n)\,\mathfrak{G}(\omega_n)\int d\eta_1 d\eta_2 d\eta_3\rho(\eta_1)\,\rho(\eta_2)\,\rho(\eta_3)$$
$$\times\frac{(1-n(\eta_1))(1-n(\eta_2))\,n(\eta_3)}{\eta_1+\eta_2-\eta_3-i\omega_n},\qquad(8a)$$

$$\delta\mathfrak{G}_b(\omega_n)\sim\mathfrak{G}^{(0)}(\omega_n)\,\mathfrak{G}(\omega_n)\int d\eta_1 d\eta_2 d\eta_3\rho(\eta_1)\,\rho(\eta_2)\,\rho(\eta_3)$$
$$\times\frac{n(\eta_1)\,n(\eta_2)\,(1-n(\eta_3))}{-\eta_1-\eta_2+\eta_3+i\omega_n}.\qquad(8b)$$

One obtains the analytical continuation of expressions (8a) and (8b) into the upper ω half-plane simply by replacing $i\omega_n$ by ω and using Eq. (3). From the form of (8a) and (8b) it follows that the analytically continued expressions are analytical in the upper halfplane and are thus corrections to the retarded function G^R which is expressed in terms of $\rho(\omega)$, i.e., in terms of Im G^R. We introduce the mass operator for the retarded function

$$G^R(\omega)=G_0^R(\omega)+G_0^R(\omega)\,\Sigma^R(\omega)\,G^R(\omega)$$

and obtain finally (Im $\omega > 0$)

$$\delta\Sigma_a^R(\mathbf{p},\omega)\sim-\int d\eta_1 d\eta_2 d\eta_3\rho(\mathbf{p}_1,\eta_1)\,\rho(\mathbf{p}_2,\eta_2)\,\rho$$
$$\times(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p},\eta_3)\,\frac{(1-n(\eta_1))(1-n(\eta_2))\,n(\eta_3)}{\eta_1+\eta_2-\eta_3-\omega},\qquad(9a)$$

$$\delta\Sigma_b^R(\mathbf{p},\omega)\sim\int d\eta_1 d\eta_2 d\eta_3\rho(\mathbf{p}_1,\eta_1)\,\rho(\mathbf{p}_2,\eta_2)\,\rho(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p},\eta_3)$$
$$\times\frac{n(\eta_1)\,n(\eta_2)\,(1-n(\eta_3))}{-\eta_1-\eta_2+\eta_3+\omega}.\qquad(9b)$$

We need not write down again the expressions for the corresponding corrections to G^A, since G^A for Im $\omega < 0$ is connected with G^R for Im $\omega > 0$ by the relation

$$G^A(\omega)=G^{R*}(\omega^*),$$

which follows immediately from (1).

We can compare Eqs. (9a) and (9b) with the well-defined diagrams obtained by a simple modification of the usual Feynman diagram. We arrange the time coordinates corresponding to the vertices of the usual Feynman diagram of Fig. 1 in such order that they decrease from top to bottom. We get then two diagrams corresponding to two possible cases: $\tau' > \tau''$ and $\tau' < \tau''$ (Fig. 2).

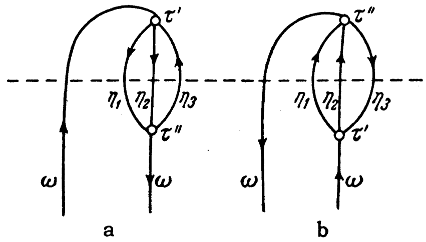

FIG. 2

The external lines must always be drawn vertically as shown in Fig. 2. We now draw a dotted horizontal line (called in the following a section) separating one vertex from the other.

We assign now to each internal line a function $\rho(\mathbf{p}_i, \eta_i)$ such that each internal line will carry a momentum $\mathbf{p}_i$ and a "frequency" η_i; we shall also assume that the external lines carry a frequency and momentum ω and $\mathbf{p}$. We integrate over the momenta and "frequencies" of the internal lines in such a way that in each vertex the usual momentum conservation laws $\Sigma\mathbf{p}_i = 0$ are satisfied; the integrations over the frequencies are, however, independent. Moreover, we assign to each internal line directed from top to bottom a factor $1-n(\eta_i)$ and to each line directed from bottom to top a factor $-n(\eta_i)$. Finally, we assign to a section a denominator equal to the sum of the frequencies which are intersected by the horizontal line, where the frequency carries a minus sign if the line is directed upward and a plus-sign if the line is directed downward.

We must, of course, assign to each vertex the interaction operator which depends only on the momenta, and multiply each of the two diagrams by the factor corresponding to the initial Feynman diagram (Fig. 1).

One verifies easily that the diagram of Fig. 2a corresponds exactly to Eq. (9a), and the diagram of Fig. 2b to Eq. (9b).

By similar considerations, taking the above-mentioned cancellation of the terms which do not contain $\delta_{\omega_n, \omega_n'}$ into account, we can verify that the diagram rules written down just now remain the same also for more complicated Feynman diagrams. As before, we must draw all possible diagrams obtained from a given Feynman diagram for different relative values of the times corresponding to the vertices, and draw in each diagram all possible horizontal sections separating one vertex from another. The rules of assigning well-defined expressions to the elements of the diagrams remain the same as before; one need only take into account that in integrating over the singularities of the denominators which correspond to the sections which intersect both external lines or do not intersect one of them (these latter denominators do not contain external frequencies) one must take principal value integrals. This is connected with the fact that in the expression for the sum of all diagrams of a given type there is no such singularity (the quantity occurring in the numerator vanishes at the same time as the denominator).

Let us consider, for instance, the correction to Σ^R given by the Feynman diagram of Fig. 3. The six diagrams given in Fig. 4 correspond to

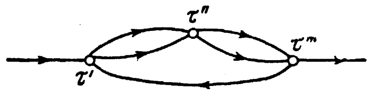

FIG. 3

this diagram. Using the rules stated a moment ago we can easily write down the expressions corresponding to Figs. 4a, 4c, and 4e (we again leave out the integrals over the momenta):

$$\delta\Sigma_a^R(\omega) \sim -\int d\eta_1 \ldots d\eta_5 \rho(\eta_1) \ldots \rho(\eta_5)(1-n(\eta_1)) \times (1-n(\eta_2))(1-n(\eta_3))(1-n(\eta_4))\, n(\eta_5) \times \frac{1}{(\eta_1+\eta_2-\eta_5-\omega)(\eta_3+\eta_4-\eta_5-\omega)},$$

$$\delta\Sigma_b^R(\omega) \sim -\int d\eta_1 \ldots d\eta_5 \rho(\eta_1) \ldots \rho(\eta_5)(1-n(\eta_1)) \times (1-n(\eta_2))\, n(\eta_3)\, n(\eta_4)\, n(\eta_5) \times \frac{1}{(\eta_1+\eta_2-\eta_5-\omega)(\eta_1+\eta_2-\eta_3-\eta_4)},$$

$$\delta\Sigma_c^R(\omega) \sim \int d\eta_1 \ldots d\eta_5 \rho(\eta_1) \ldots \rho(\eta_5)\, n(\eta_1)\, n(\eta_2) \times (1-n(\eta_3))(1-n(\eta_4))(1-n(\eta_5)) \times \frac{1}{(-\eta_1-\eta_2+\eta_3+\eta_4)(-\eta_1-\eta_2+\eta_5+\omega)}.$$

The sum of all possible diagrams for $\Sigma^R(\omega)$ expresses this quantity, and at the same time also $G^R(\omega)$ in terms of $\rho(\omega)$, i.e., in terms of Im $G^R(\omega)$. The equation

$$G_R^{-1}(\omega) = G_{R0}^{-1}(\omega) - \Sigma^R(\omega)$$

is thus the analogy of the Dyson equation where

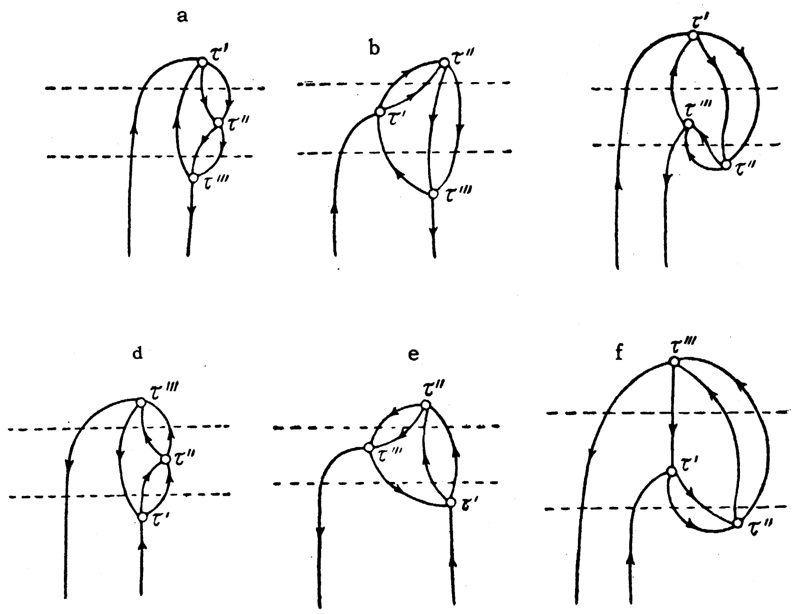

FIG. 4

the unknown function will be here not $G^R(\omega)$ itself, but its imaginary part.

3. MANY-PARTICLE GREEN'S FUNCTIONS

We now turn to the problem of a diagram technique suitable for many-particle Green's functions in terms of which the different transport coefficients of the system can be expressed. Generally speaking, many-particle temperature-dependent Green's functions depend on several discrete "frequencies" ω_n and in the general case it is a hopeless task to find an analytical continuation for all of them. Fortunately, however, it is sufficient for a calculation of all the important transport quantities to continue analytically the many-particle functions only with respect to one frequency. For instance, the electrical conductivity, the dielectric constant, and the viscosity are expressed in terms of the retarded two-particle function

$$K^R(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4; t_1 - t_2) = \mathrm{Sp}\left\{\exp\left(\frac{\Omega - \hat{H} + \mu\hat{N}}{T}\right)[\psi^+(\mathbf{r}_1, t_1)\psi(\mathbf{r}_2, t_1) \times \psi^+(\mathbf{r}_3, t_2)\psi(\mathbf{r}_4, t_2) - \psi^+(\mathbf{r}_3, t_2)\psi(\mathbf{r}_4, t_2)\psi^+(\mathbf{r}_1, t_1)\psi(\mathbf{r}_2, t_1)]\right\} \quad t_1 > t_2;$$
$$K^R(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4; t_1 - t_2) = 0, \qquad t_1 < t_2, \tag{10}$$

where $\psi(\mathbf{r}, t)$ and $\psi^+(\mathbf{r}, t)$ are the usual Heisenberg operators. In particular, using the well-known expression for the current operator (see, for instance, [7]) one verifies easily that in the general case where there is both temporal and spatial dispersion the electrical conductivity $\sigma(\mathbf{k}, \omega)$ is proportional to the Fourier transform of

$$Q(\mathbf{r} - \mathbf{r}'; t_1 - t_2) = \left(\frac{\partial}{\partial x_{i1}} - \frac{\partial}{\partial x_{i2}}\right)\left(\frac{\partial}{\partial x_{i3}} - \frac{\partial}{\partial x_{i4}}\right) \times K^R(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3, \mathbf{r}_4; t_1 - t_2)\Big|_{\substack{\mathbf{r}_1=\mathbf{r}_2=\mathbf{r} \\ \mathbf{r}_3=\mathbf{r}_4=\mathbf{r}'}}.$$

Repeating the arguments of [2] discussing the analytical properties of the single-particle Green's functions, and replacing the matrix elements of $\psi(\mathbf{r}_1, t_1)$ and $\psi^+(\mathbf{r}_2, t_2)$ which occur there by those of $\psi^+(\mathbf{r}_1, t_1)\psi(\mathbf{r}_2, t_1)$ and $\psi^+(\mathbf{r}_3, t_2)\psi(\mathbf{r}_4, t_2)$ respectively, one can show easily that the Fourier transform of (10) with respect to the time, $K^R(\omega)$, is an analytical function of ω which is regular in the upper half-plane, and which is the analytical continuation from the discrete points on the upper imaginary axis of the Fourier transform corresponding to the temperature-dependent Green's function $\mathfrak{K}(\omega_n)$ ($\omega_n = 2\pi nT$):

$$\mathfrak{K}(\tau_1 - \tau_2) = \mathrm{Sp}\left\{\exp\left(\frac{\Omega - \hat{H} + \mu\hat{N}}{T}\right)T_\tau[\bar{\psi}(\mathbf{r}_1, \tau_1)\psi(\mathbf{r}_2, \tau_1) \times \bar{\psi}(\mathbf{r}_3, \tau_2)\psi(\mathbf{r}_4, \tau_2)]\right\}, \tag{11}$$

where

$$\psi(\mathbf{r}, \tau) = e^{\tau(\hat{H} - \mu\hat{N})}\psi(\mathbf{r})e^{-\tau(\hat{H} - \mu\hat{N})},$$
$$\bar{\psi}(\mathbf{r}, \tau) = e^{\tau(\hat{H} - \mu\hat{N})}\psi^+(\mathbf{r})e^{-\tau(\hat{H} - \mu\hat{N})}.$$

The analytical continuation of the diagrams for $\mathfrak{K}(\omega_n)$ proceeds exactly as for the single-particle Green's function. The first Feynman diagram for $\mathfrak{K}$ is given in Fig. 5a. To emphasize the fact that the ends of the lines in Fig. 5a on the right (and also on the left) correspond to the same "time" τ we shall draw the diagrams for $\mathfrak{K}$ as diagrams for some single-particle function to which we shall assign a double line (see Fig. 5b). We emphasize, however, that in the new "vertex" which appears in this way and which is formed by one double and two single lines there is no integration over the times or coordinates.

One verifies now easily that the evaluation of the contribution to $K^R(\omega)$ from the Feynman diagram of Fig. 5a can be performed using the rules described in the previous section, if we apply them to diagram 5b for the "single-particle" function. Diagram 5b will then, of course, depend on all four external momenta (or spatial coordinates) of diagram 5a. The definition of the momentum dependence of the diagram remains, of course, the same as in the Feynman method.

The contribution to $K^R(\omega)$ given by the Feynman diagram 5b is, as in the previous section, determined by the two diagrams of Figs. 6a and 6b

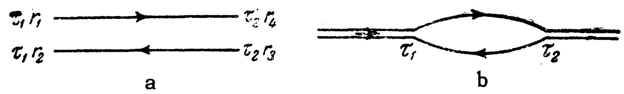

FIG. 5

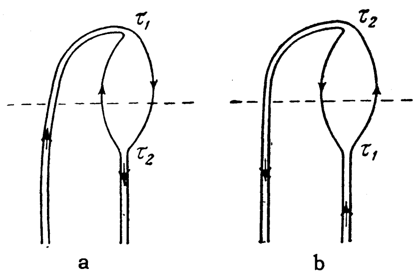

FIG. 6

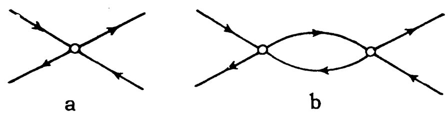

FIG. 7

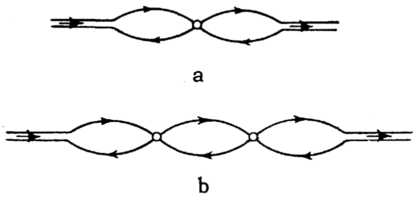

FIG. 8

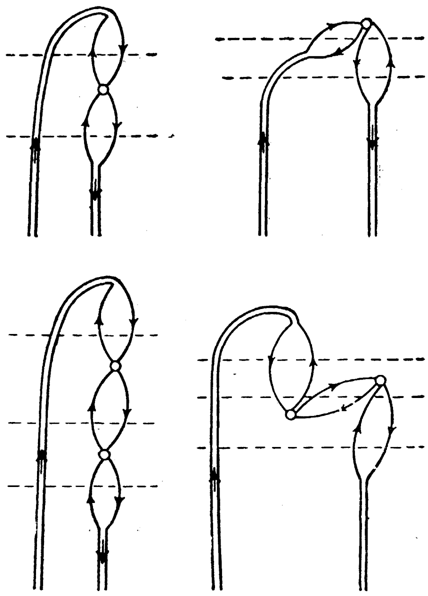

FIG. 9

and can easily be evaluated. Figures 7 to 9 give examples of more complicated diagrams.

One can perform an analytical continuation with respect to one of the discrete frequencies for diagrams of three-particle and similar temperature-dependent Green's functions in a completely similar manner.

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Translated by D. ter Haar
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