LOCATION OF SINGULARITIES OF FEYNMAN DIAGRAMS

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The denominator of the integrand in the parametric representation of the Feynman integrals is studied. The part of the resultant quadratic form depending on the external momenta is singled out. The results obtained are used to determine the location of singular curves. The equations for the asymptotes to the singular curves are written down for the case of the four-point function. It is shown that the second asymptote is not encountered in all the diagrams and that it is located at larger values of the finite invariant than the usual asymptote of the singular curve. The dependence of the singular curve on the internal masses in the diagram is studied.

1. INTRODUCTION

FOR the construction of the theory on the basis of the properties of unitarity and causality it is necessary to know the location of the singularities of quantum field-theoretical quantities.[1-4] To this end it is useful to study their location in perturbation theory. The general method for finding the singularities in a Feynman diagram was given by Landau.[5]

The denominator of the integral corresponding to the Feynman diagram may be written in the form

$$f = \sum_{ik} \alpha_{ik} (q_{ik}^2 - m_{ik}^2). \tag{1}$$

We make use of the following notation[6]: the subscripts i, k, etc, refer to the vertices in the diagrams, p_i represents the external 4-momentum entering the vertex i; the vertex l is called external if $p_l \neq 0$, and internal if $p_l \equiv 0$; $q_{ik} = -q_{ki}$ is the momentum of the virtual particle which corresponds in the diagram to the line ik; $\alpha_{ik} = \alpha_{ki}$, $m_{ik} = m_{ki}$ are respectively the Feynman parameter and the mass for that line. The summation in Eq. (1) is over all the lines in the diagram. The conservation laws at the diagram vertices will be written as

$$\sum_k q_{ik} = p_i. \tag{2}$$

In Eq. (2) the summation is over all vertices k that are connected by a line with the vertex i.

The problem is to represent f in the form

$$f = k_{ik,\, lm}\, q'_{ik} q'_{lm} + \varphi(\alpha, p, m), \tag{3}$$

where the q'_{ik} are obtained by translation from the independent vectors q_{ik} that remain after elimination of some of the internal momenta by means of Eq. (2). After integration over the internal momenta there remains a function of α, m, and the external momenta p, $\varphi(\alpha, m, p)$, whose properties must be investigated.

In order to find φ we observe that $f = \varphi$ for q' satisfying the extremum condition in q'_{ik}:

$$\partial f / \partial q'_{ik} = 0. \tag{4}$$

Since q'_{ik} was obtained from q_{ik} by translation it follows that Eq. (4) requires that f be an extremum as a function of the q_{ik} subject to the auxiliary conditions, Eq. (2). By making use of the method of Lagrange multipliers, which as a consequence of the vector nature of Eq. (2) are themselves 4-vectors, and using the notation $\beta_{ik} = 1/\alpha_{ik}$, we obtain

$$q_{ik} = \beta_{ik} (a_i - a_k). \tag{5}$$

Upon substitution of Eq. (5) into Eq. (2) we get

$$\sum_k \beta_{ik} (a_i - a_k) = p_i. \tag{6}$$

Since only differences of the vectors a_i appear in Eqs. (5) and (6) we may set at an arbitrarily chosen vertex of the diagram $a_l = 0$. The set of equations (6) will be written in the form (summation over repeated indices is understood)

$$B_{ik} a_k = p_i, \tag{7}$$

where $B_{ik} = -\beta_{ik}$ if $i \neq k$, and $B_{ii} = \sum_k \beta_{ik}$.

We note that

$$\varphi = \sum_{ik} \left[\beta_{ik} (a_i - a_k)^2 - \frac{m_{ik}^2}{\beta_{ik}} \right] = B_{ik} a_i a_k - \sum_{ik} \frac{m_{ik}^2}{\beta_{ik}}. \tag{8}$$

Therefore, making use of Eq. (7), we obtain

$$\varphi = p_i a_i - \sum_{ik} \frac{m_{ik}^2}{\beta_{ik}}. \tag{9}$$

It is clear that only a_i corresponding to external vertices enter, since for the internal vertices we have, by definition, $p_i \equiv 0$.

Let us consider an internal vertex, to be identified by the subscript 0. From Eq. (6) we have

$$a_0 = \sum_i \beta_{0i} a_i \Big/ \sum_k \beta_{0k}. \tag{10}$$

Let us substitute in the remaining Eqs. (6) for a_0 its value from Eq. (10). There will be changes only at the vertices that are connected by a line with the vertex 0. Namely

$$\sum_k \beta_{ik}(a_i - a_k) = \sum_{k\neq 0}\left(\beta_{ik} + \beta_{i0}\beta_{k0} \Big/ \sum_l \beta_{0l}\right)(a_i - a_k). \tag{11}$$

Let us introduce[6]

$$\widetilde{\beta}_{ik} = \beta_{ik} + \beta_{i0}\beta_{k0} \Big/ \sum_l \beta_{0l}. \tag{12}$$

The substitution of $\widetilde{\beta}_{ik}$ for β_{ik} transforms the system of equations (6) for the given diagram with n vertices into an analogous system for a diagram with $n - 1$ vertices:

$$\sum_{k\neq 0} \widetilde{\beta}_{ik}(a_i - a_k) = p_i. \tag{13}$$

If in the starting diagram two vertices i and k were connected by lines with the vertex 0 and were not connected with each other then, according to Eq. (12), a line joining them will appear with

$$\widetilde{\beta}_{ik} = \beta_{i0}\beta_{k0} \Big/ \sum_l \beta_{0l}. \tag{14}$$

Eliminating consecutively internal vertices we arrive at an irreducible diagram with vertices that are all external and, generally speaking, all connected with each other by lines. Thus for the three-point function we have the diagram of Fig. 1, for the four-point function the diagram of Fig. 2, for the five-point function the diagram of Fig. 3. The vectors a_i are now found by solving a set of the type (6) for the irreducible diagram, but the β for this diagram are given in a complicated way in terms of the β for the starting diagram with the help of Eq. (12). Writing the equation for the "irreducible diagram" in the form (7) we obtain for φ

$$\varphi = \varphi' - \sum_{ik} \frac{m_{ik}^2}{\beta_{ik}}; \quad \varphi' = \widetilde{B}_{ik}^{-1} p_i p_k. \tag{15}$$

Let us write out the part of φ depending on the external momenta for the case of the irreducible

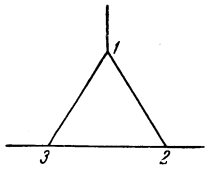

FIG. 1

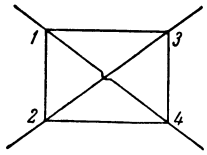

FIG. 2

three- and four-point functions (we assume that $p_i^2 = M_i^2$). For the three-point function (Fig. 1) we have

$$\Delta_3 \varphi_3' = M_1^2 \beta_{23} + M_2^2 \beta_{13} + M_3^2 \beta_{12}, \tag{16}$$

$$\Delta_3 = \beta_{12}\beta_{23} + \beta_{13}\beta_{23} + \beta_{12}\beta_{13}. \tag{17}$$

For the four-point function (Fig. 2) we have

$$\begin{aligned}\Delta_4 \varphi_4' = {} & \beta_{12}\beta_{34} s + \beta_{13}\beta_{24} t + \beta_{14}\beta_{23} u \\ & + M_1^2(\beta_{23}\beta_{34} + \beta_{24}\beta_{34} + \beta_{23}\beta_{24}) + M_2^2(\beta_{34}\beta_{14} + \beta_{34}\beta_{13} \\ & + \beta_{14}\beta_{13}) + M_3^2(\beta_{12}\beta_{24} + \beta_{12}\beta_{14} + \beta_{24}\beta_{14}) \\ & + M_4^2(\beta_{12}\beta_{23} + \beta_{12}\beta_{13} + \beta_{23}\beta_{13});\end{aligned} \tag{18}$$

$$\begin{aligned}\Delta_4 = {} & \beta_{12}\beta_{34}(\beta_{13} + \beta_{14} + \beta_{23} + \beta_{24}) + \beta_{12}(\beta_{13} + \beta_{23})(\beta_{14} + \beta_{24}) \\ & + \beta_{34}(\beta_{23} + \beta_{24})(\beta_{13} + \beta_{14}) + \beta_{13}\beta_{14}(\beta_{23} + \beta_{24}) \\ & + \beta_{23}\beta_{24}(\beta_{13} + \beta_{14}),\end{aligned} \tag{19}$$

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2,$$
$$u = (p_2 + p_3)^2, \quad s + t + u = \sum_{i=1}^{4} M_i^2. \tag{20}$$

2. LOCATION OF SINGULARITIES

It was shown by Landau[5] that the singularities in the diagram for real values of the invariants are determined by the equations

$$\partial\varphi / \partial\alpha_{ik} = q_{ik}^2 - m_{ik}^2 = 0, \quad \alpha_{ik} > 0. \tag{21}$$

Using Eq. (5) we find

$$\beta_{ik}^2 (a_i - a_k)^2 = m_{ik}^2. \tag{22}$$

Solving Eqs. (6) and (22) yields relations between the invariants for which the system of equations is soluble.

At the singular point all quantities in Eqs. (6) and (22) may be found as functions of the invari-

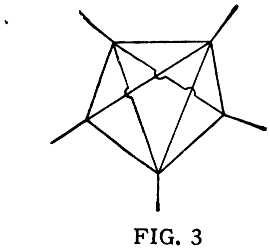

FIG. 3

ants. Let us therefore introduce $\tilde{m}_{ik}$ by means of the relation

$$\tilde{m}_{ik}^2 = -\tilde{\beta}_{ik}^2 \partial\varphi' / \partial\tilde{\beta}_{ik} = \tilde{\beta}_{ik}^2 (a_i - a_k)^2. \qquad (23)$$

Differentiating φ we obtain

$$\partial\varphi / \partial\alpha_{ik} = \tilde{m}_{lm}^2 \partial\tilde{\alpha}_{lm} / \partial\alpha_{ik} - m_{ik}^2, \qquad (24)$$

where $\tilde{\alpha}_{lm} = 1/\tilde{\beta}_{lm}$, and the summation is over all pairs of subscripts lm corresponding to lines in the diagram obtained from the starting diagram by elimination of internal vertices by means of formula (12).

As a result of consecutive eliminations of internal vertices we obtain a series of diagrams; let us denote by $\tilde{m}^{(n)}$ and $\tilde{\alpha}^{(n)} = 1/\tilde{\beta}^{(n)}$ the quantities referring to the diagram obtained from the starting diagram after elimination of n internal vertices. As was shown in Sec. 1 [see Eq. (15)], φ' is expressible in terms of the $\tilde{\alpha}^{(n)}$ corresponding to an arbitrary diagram in the series. Furthermore, $\tilde{\alpha}^{(n)}$ is easily expressed in terms of $\tilde{\alpha}^{(k)}$ for $n > k$.

Along the singular curve, $\partial\varphi/\partial\alpha_{ik} = 0$. Making use of Eq. (24) we obtain

$$(\tilde{m}_{lm}^{(n)})^2 \partial\tilde{\alpha}_{lm}^{(n)} / \partial\tilde{\alpha}_{is}^{(k)} = (\tilde{m}_{is}^{(k)})^2, \quad m > k. \qquad (25)$$

Equations (25) together with Eq. (12) are equivalent to Eq. (22). With their help all $\tilde{\alpha}^{(n)}$ and $\tilde{m}^{(n)}$ can be found as functions of the invariants. Consequently the locations of the singularities of all diagrams in the series coincide when the internal masses are certain functions of the invariants.

The number of lines changes with the elimination of internal vertices, so that the number of equations in (25) does not equal the number of unknowns. This imposes conditions on the internal masses in the diagram which must be satisfied if the system of equations which determine the location of the singularities is to have solutions for the diagram with the given structure. If the diagram has singularities then a solution for the Eqs. (25) exists and it is possible to establish certain relations for the $\tilde{m}^{(n)}$, with the help of which certain details of the behavior of the singular curves of the complex diagrams may be deduced from the singular curves of the irreducible diagram for which all vertices are external.[6]

Certain inequalities for the $m^{(n)}$ have been obtained previously.[6] We give here one more relation. The quantity

$$Z = \sum_{ik} \tilde{\alpha}_{ik}^{(n)} (\tilde{m}_{ik}^{(n)})^2 \qquad (26)$$

does not change when internal vertices are eliminated. To prove this we multiply Eq. (25) by $\tilde{\alpha}_{is}^{k}$ and sum over is. Taking into account the fact that $\tilde{\alpha}^{(n)}$ is a homogeneous function of first degree in $\tilde{\alpha}^{(k)}$ (for $n > k$) we obtain

$$\sum_{lm} \tilde{\alpha}_{lm}^{(n)} (\tilde{m}_{lm}^{(n)})^2 = \sum_{is} \tilde{\alpha}_{is}^{(k)} (\tilde{m}_{is}^{(k)})^2,$$

from which Eq. (26) follows. If the α_{ik} are so normalized that $\sum_{ik} \alpha_{ik} = 1$, as is conventional, then the value of Z lies between the square of the largest and the smallest of the masses of internal lines.

Let us pass to the consideration of the most complex of the irreducible diagrams with four external lines (Fig. 2). The masses of the internal lines will be assumed given. In this case (21) and (25) coincide and can be written in the form

$$\Delta_4^{-1} \Big\{ \beta_{34}s + M_3^2(\beta_{24} + \beta_{14}) + M_4^2(\beta_{13} + \beta_{23}) + \Delta_4 m_{12}^2/\beta_{12}^2 - [\beta_{34}(\beta_{13} + \beta_{14} + \beta_{23} + \beta_{24}) + (\beta_{13} + \beta_{23})(\beta_{14} + \beta_{24})] \sum_{ik} \frac{m_{ik}^2}{\beta_{ik}} \Big\} = 0, \qquad (27)$$

$$\Delta_4^{-1} \Big\{ \beta_{12}s + M_1^2(\beta_{24} + \beta_{23}) + M_2^2(\beta_{13} + \beta_{14}) + \Delta_4 m_{34}^2/\beta_{34}^2 - [\beta_{12}(\beta_{13} + \beta_{14} + \beta_{23} + \beta_{24}) + (\beta_{23} + \beta_{24})(\beta_{13} + \beta_{14})] \sum_{ik} \frac{m_{ik}^2}{\beta_{ik}} \Big\} = 0, \qquad (28)$$

$$\Delta_4^{-1} \Big\{ \beta_{24}t + M_2^2(\beta_{34} + \beta_{14}) + M_4^2(\beta_{12} + \beta_{23}) + \Delta_4 m_{13}^2/\beta_{13}^2 - [\beta_{12}\beta_{34} + \beta_{12}(\beta_{14} + \beta_{24}) + \beta_{34}(\beta_{23} + \beta_{24}) + \beta_{14}(\beta_{23} + \beta_{24}) + \beta_{23}\beta_{24}] \sum_{ik} \frac{m_{ik}^2}{\beta_{ik}} \Big\} = 0, \qquad (29)$$

$$\Delta_4^{-1} \Big\{ \beta_{13}t + M_1^2(\beta_{23} + \beta_{24}) + M_3^2(\beta_{12} + \beta_{14}) + \Delta_4 m_{24}^2/\beta_{24}^2 - [\beta_{12}\beta_{34} + \beta_{12}(\beta_{13} + \beta_{23}) + \beta_{34}(\beta_{13} + \beta_{14}) + \beta_{13}\beta_{14} + \beta_{23}(\beta_{13} + \beta_{14})] \sum_{ik} \frac{m_{ik}^2}{\beta_{ik}} \Big\} = 0, \qquad (30)$$

$$\Delta_4^{-1} \Big\{ \beta_{23}u + M_2^2(\beta_{13} + \beta_{34}) + M_3^2(\beta_{12} + \beta_{24}) + \Delta_4 m_{14}^2/\beta_{14}^2 - [\beta_{12}\beta_{34} + \beta_{12}(\beta_{13} + \beta_{23}) + \beta_{34}(\beta_{23} + \beta_{24}) + \beta_{13}(\beta_{23} + \beta_{24}) + \beta_{23}\beta_{24}] \sum_{ik} \frac{m_{ik}^2}{\beta_{ik}} \Big\} = 0, \qquad (31)$$

$$\Delta_4^{-1} \Big\{ \beta_{14}u + M_1^2(\beta_{34} + \beta_{24}) + M_4^2(\beta_{12} + \beta_{13}) + \Delta_4 m_{23}^2/\beta_{23}^2 - [\beta_{12}\beta_{34} + \beta_{12}(\beta_{14} + \beta_{24}) + \beta_{34}(\beta_{13} + \beta_{14}) + \beta_{13}\beta_{14} + \beta_{24}(\beta_{13} + \beta_{14})] \sum_{ik} \frac{m_{ik}^2}{\beta_{ik}} \Big\} = 0. \qquad (32)$$

At the same time $\varphi = 0$.

With the help of Eq. (20) we can eliminate u from (31) and (32):

$$\varphi = \Delta_4^{-1}(as + bt + c), \qquad (33)$$

$$a = \beta_{12}\beta_{34} - \beta_{14}\beta_{23}, \quad b = \beta_{13}\beta_{24} - \beta_{14}\beta_{23}. \tag{34}$$

As a consequence of (21) we have on the singular curve

$$\frac{\partial s}{\partial t} = -\frac{\partial \varphi}{\partial t} \Big/ \frac{\partial \varphi}{\partial s} = -\frac{\beta_{13}\beta_{24} - \beta_{14}\beta_{23}}{\beta_{12}\beta_{34} - \beta_{14}\beta_{23}}. \tag{35}$$

In (27) — (32) the coefficient of t is β_{13}/Δ_4, β_{14}/Δ_4, β_{23}/Δ_4, β_{24}/Δ_4. Let us suppose that s is bounded as $|t| \to \infty$. As is easy to verify, all the terms in (27) — (33) except for the term with t are bounded under these conditions, and consequently all terms are bounded. This means that for $|t| \to \infty$ and for s bounded we have

$$\partial s/\partial t \to 0, \quad (\beta_{13}\beta_{24} - \beta_{14}\beta_{23})/\Delta_4 \to 0$$

and the coefficients of t in Eqs. (27) — (32) tend to zero no slower than const/t. Since $\beta > 0$ and cannot vanish (we assume that $\sum_{ik}\alpha_{ik} = \sum_{ik} 1/\beta_{ik} = 1$), it follows that $\Delta_4 = \infty$, which requires some of the β to approach infinity.

The condition $\partial s/\partial t \to 0$ leads to $\beta_{12} \to \infty$, $\beta_{34} \to \infty$. We remark that if one expresses the $\tilde{\beta}$ appearing in Eq. (34) in terms of the β for the starting diagram, then for some diagrams $b = \tilde{\beta}_{13}\tilde{\beta}_{24} - \tilde{\beta}_{14}\tilde{\beta}_{23}$ cannot vanish. For example, for the "sealed envelope" diagram (Fig. 4)

$$b = \left(\beta_{13} + \frac{\beta_{15}\beta_{35}}{\beta_{15}+\beta_{35}+\beta_{25}+\beta_{45}}\right)\left(\beta_{24} + \frac{\beta_{25}\beta_{45}}{\beta_{15}+\beta_{35}+\beta_{25}+\beta_{45}}\right) - \frac{\beta_{15}\beta_{25}\beta_{35}\beta_{45}}{(\beta_{15}+\beta_{35}+\beta_{25}+\beta_{45})^2} = \beta_{13}\beta_{24} + \frac{\beta_{13}\beta_{25}\beta_{45} + \beta_{24}\beta_{15}\beta_{25}}{\beta_{15}+\beta_{35}+\beta_{25}+\beta_{45}} > 0. \tag{36}$$

An analogous result is also obtained for ladder diagrams. From among diagrams for which b can vanish the simplest is the "unsealed envelope" (Fig. 2). It will be shown below that this results in the appearance of an asymptote of the singular curve on which the finite invariant is not the square of the sum of the masses of the corresponding section of the diagram.

3. EQUATIONS FOR THE ASYMPTOTES

We shall obtain the equations for the asymptotes by considering Eqs. (27) — (32) in zeroth approximation in $1/\beta_{12}\beta_{34}$. Here it is convenient to make use of the normalization

$$\beta_{13} + \beta_{14} + \beta_{23} + \beta_{24} = 1. \tag{37}$$

We obtain (in what follows we consider asymptotic values of all quantities, denoted by the same symbols as the quantities themselves)

$$s = m_{13}^2/\beta_{13} + m_{14}^2/\beta_{14} + m_{23}^2/\beta_{23} + m_{24}^2/\beta_{24}, \tag{38}$$

$$\beta_{24}\varepsilon + m_{13}^2/\beta_{13}^2 - s = 0, \tag{39}$$

$$\beta_{13}\varepsilon + m_{24}^2/\beta_{24}^2 - s = 0, \tag{40}$$

$$\beta_{14}\varepsilon - m_{23}^2/\beta_{23}^2 + s = 0, \tag{41}$$

$$\beta_{23}\varepsilon - m_{14}^2/\beta_{14}^2 + s = 0; \tag{42}$$

$$\varepsilon = \lim_{t,\beta_{12},\beta_{34}\to\infty} (t/\beta_{12}\beta_{34}). \tag{43}$$

Multiplying Eq. (39) by β_{13}, Eq. (40) by β_{24}, Eq. (41) by β_{23} and Eq. (42) by β_{14} and subtracting from the sum of the first two equations the sum of the following two, we get, after making use of Eqs. (37) and (38)

$$(\beta_{13}\beta_{24} - \beta_{14}\beta_{23})\,\varepsilon = 0. \tag{44}$$

If $\epsilon = 0$, then we have the case of a usual asymptote

$$m_{13}/\beta_{13} = m_{14}/\beta_{14} = m_{23}/\beta_{23} = m_{24}/\beta_{24} = s^{1/2} \tag{45}$$

and from Eq. (38)

$$s = (m_{13} + m_{14} + m_{23} + m_{24})^2. \tag{46}$$

If instead $\epsilon \neq 0$, then

$$b = \beta_{13}\beta_{24} - \beta_{14}\beta_{23} = 0. \tag{47}$$

In that case Eqs. (38) — (42) determine the position of the second asymptote. A general solution of these equations was not found. In the case when the position of the second asymptote is near the position of the usual asymptote, the results were obtained in [6] by a somewhat different method.

For the case of a symmetric diagram the system of equations may be solved assuming β symmetry, i.e., if $m_{13} = m_{24} = m$, $m_{14} = m_{23} = 1$; $\beta_{13} = \beta_{24} = \beta$, $\beta_{14} = \beta_{23} = 1/2 - \beta$. Then Eqs. (44) and (37) yield $\beta = 1/4$ and

$$s = 8(1 + m^2), \tag{48}$$

$$\varepsilon = 8(1 - m^2); \tag{49}$$

Equation (48) has been obtained[7] by direct calculation of the location of the singularity of the symmetric diagram.

The expressions for s, Eq. (38), have under the condition (37) a minimum (for $\beta > 0$). As is not difficult to see the minimum condition coincides with Eq. (45). Consequently, the second asymptote lies always at larger values of the finite invariant than the usual asymptote.

Calculations analogous to the ones above may be carried out for the points on the singular curve with vertical and horizontal tangents.

4. LOCATION OF SINGULARITIES AS A FUNCTION OF THE INTERNAL MASSES OF THE DIAGRAM

In order to study the position of the singularities as a function of the internal masses we shall write φ in the form

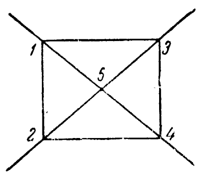

FIG. 4

$$\varphi = sa/\Delta_4 + tb/\Delta_4 - m^2/\beta + c. \qquad (50)$$

Here we have isolated the term due to one of the internal masses, which—as well as the β corresponding to it—is written without a subscript. The quantities a, b, c, Δ_4 are expressible in terms of $\tilde{\beta}$ and depend in a complicated way on the Feynman parameters of the diagram. Let us fix t and consider the location of the singularity in the variables s and m^2. Since along the singular curve $\partial\varphi/\partial\beta_{ik} = 0$, we find

$$\frac{\partial s}{\partial m^2} = -\frac{\partial\varphi/\partial m^2}{\partial\varphi/\partial s} = \frac{\Delta_4}{\beta a}. \qquad (51)$$

For $|t| \to \infty$ the quantity $a = \beta_{12}\beta_{34} - \beta_{14}\beta_{23} > 0$ ($\Delta_4 > 0$ always for $\beta > 0$).

Let us choose the invariants so that the singular curve lies asymptotically in the upper right hand quadrant. Then from the asymptote $|t| = \infty$ and to the point $a = 0$ the singular curve moves to the right, i.e., into the inner region of the curve, as the internal mass is increased. At the point $a = 0$, if a becomes negative $\partial s/\partial m^2$ changes sign; at that $\partial t/\partial s = -a/b$ also changes sign. Consequently also in this case the singular curve moves into the region it delineates. The singular curve has the tangent $t = \text{const}$ when $a = 0$ or $b = \infty$. In the latter case one of the Feynman parameters vanishes; this means that the curve under consideration terminates at a finite point in the plane of the invariants (the condition $\alpha > 0$ was used in the derivation of the formulas). In the case when there is another branch of the singular curve with $\alpha > 0$, which starts out from the point $b = \infty$, it is easily seen that the conclusion about the behavior of the singular curve as the internal mass is varied remains unchanged.

The requirement that we should have at some point simultaneously $a = 0$ and $b = 0$ gives, together with Eqs. (21), an overdetermined system of equations which, generally speaking, has no solutions. As was shown above the possibility of a or b vanishing is not present for all four-point functions and is connected with the appearance of the second asymptote.

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