# *CERTAIN RESONANCE SOLUTIONS OF LOW ENERGY PION-PION SCATTERING EQUATIONS*

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Submitted to JETP editor September 21, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **42**, 610-621 (February, 1962)

A method is proposed for the construction of resonant (in the limit $\lambda \to 0$) solutions of the pion-pion scattering equations, whose asymptotic behavior is that of a power law. Extension to the range of nonvanishing $\lambda$ is carried out by the N/D technique,[9] a second subtraction being performed for the D function. The three-parameter branch of solutions with one resonance at the same position in each wave was investigated by this method. A characteristic feature of the power-law resonant solutions is the sharpness of the p-wave resonance for reasonable values of the s-wave scattering lengths and a broad resonance in the $A_0$ wave. It is shown that the power-law branches are limiting cases of logarithmic branches, corresponding to the shifting of zeros (of the CDD type[3]) to infinity. It is shown that the experimental value of the p-wave resonance width can be a criterion of the role of high energy contributions to the low energy region.

## 1. INTRODUCTION

WE shall be concerned below with the study of a class of solutions of the equations for low-energy pion-pion scattering discussed previously.[1] Written without subtractions, these equations take the form

$$A_i(z) = \frac{1}{\pi}\int_1^\infty dz' \left\{ \frac{\operatorname{Im} A_i(z')}{z'-z} + \sum_k \frac{b_{ik} \operatorname{Im} A_k(z')}{z'+z} \right\} \quad (i = 0, 1, 2). \tag{1.1}$$

Here

$$A_0 = A_0^0, \quad A_1 = A_1^1, \quad A_2 = A_0^2,$$
$$z = 2\nu + 1 = 2q^2/\mu^2 + 1,$$

and the numerical matrix $b_{ik} = \delta_{ik} + l_i n_k$ is defined as follows:

$$l_0 = -1/3, \quad l_1 = -1/18, \quad l_2 = 1/6,$$
$$n_0 = 2, \quad n_1 = 9, \quad n_2 = -5. \tag{1.2}$$

The unitarity conditions for the partial waves are

$$\operatorname{Im} A_i(z + i0) = K(z)|A_i(z)|^2, \quad z \geqslant 1;$$
$$K(z) = \sqrt{(z-1)/(z+1)}. \tag{1.3}$$

The threshold condition for the p-wave gives

$$A_1(1) = 0. \tag{1.4}$$

The solutions of Eq. (1.1) satisfy the matrix condition of crossing symmetry

$$A_i(-z - i0) = \sum_k b_{ik} A_k(z + i0). \tag{1.5}$$

As was shown in [1], it is an algebraic consequence of Eq. (1.1) that the s-wave scattering lengths are positive

$$A_s(1) = a_s > 0 \quad (s = 0, 2) \tag{1.6}$$

It was also shown there that Eq. (1.1) admits of three kinds of asymptotic behavior: logarithmic decrease

$$\operatorname{Re} A_i(z) \approx d_i / \ln z,$$
$$d_0 = 2.13, \quad d_1 = -0.118, \quad d_2 = 0.640; \tag{1.7}$$

linear decrease

$$\operatorname{Re} A_i(z) \approx l_i c / z, \tag{1.8}$$

where c is an arbitrary parameter, and quadratic decrease

$$\operatorname{Re} A_i(z) \approx f_i / z^2, \tag{1.9}$$

with the coefficients $f_i$ satisfying the condition

$$\sum_k n_k f_k = 0,$$

from which it follows that there are two independent parameters among the $f_i$. It was also shown that a decrease according to a stronger power law is impossible.

We now concentrate on a study of the branch of solutions with the linear decrease, Eq. (1.8), as $\lambda \to 0$. Analogously to the procedure followed in the neutral model of pion-pion scattering, [2] we assume (Sec. 2) that we can neglect the contribution to the inverse amplitudes $(A_i)^{-1}$ from the integrals over the cuts for small $\lambda$ in the power law branches; as a consequence the real parts of the partial wave amplitudes are expressed in terms of partial fractions and the imaginary parts in terms of $\delta$ functions [see Eqs. (5.5), (5.6) of [2]].

In the succeeding section we obtain by means of algebraic considerations several very simple power-law branches with resonances in various partial wave amplitudes. In Secs. 4 and 5 it is demonstrated that an extension from the neighborhood of the point $\lambda = 0$ can be accomplished by means of the N/D technique, specially modified for solutions with a power-law decrease.

Section 6 is devoted to the establishing of a correspondence between solutions with a power law and solutions with a logarithmic asymptotic behavior. An example of a solution is constructed in which the transition from the power law to the logarithmic branch proceeds smoothly. It turns out that this transition is related to the shifting of the zeros of the logarithmic branch to infinity. This picture is analogous to the behavior of the zeros of the Castillejo, Dalitz, Dyson (CDD) type [3] in the neutral model.

## 2. METHOD OF δ-LIKE APPROXIMATIONS

We first define the parameter $\lambda$. In accordance with [1] we relate it to the values of the amplitudes at the symmetry point $z = 0$.

We obtain

$$A_s(0) = \gamma_s\lambda + 3A_1(0); \quad \gamma_0 = 5, \quad \gamma_2 = 2. \tag{2.1}$$

We wish to study the power law solutions, which vanish together with $\lambda$ and which can be expressed for $z > 1$ in the form

$$A_i(z) = \lambda / [\Phi_i(z, \lambda) + \lambda I_i(z, \lambda) - i\lambda K(z)], \tag{2.2}$$

where $\Phi_i$ are real bilinear functions of z with nonvanishing limiting values given by

$$\Phi_i(z, 0) = f_i(z), \tag{2.3}$$

and $I_i$ are integrals over the cuts, which increase as $|z| \to \infty$ no faster than logarithmically, with $I_i(z, 0)$ finite for finite z. Consequently in the limit of small $\lambda$ we have

$$\operatorname{Re} A_i(z) = \lambda / f_i(z), \tag{2.4}$$

$$\operatorname{Im} A_i(z) = \lambda^2 K(z) / [f_i^2(z) + \lambda^2 K^2(z)]. \tag{2.5}$$

We have kept in the numerator in Eq. (2.5) the term $\lambda^2K^2(z)$, since it turns out to be important in the integration of $\operatorname{Im} A_i(z)$ in the neighborhood of zeros of the functions $f_i(z)$. We seek resonant solutions, to which correspond simple zeros of the functions $f_i(z)$.

By making use of one of the representations of the $\delta$ function

$$\delta(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}, \tag{2.6}$$

we approximate the imaginary parts of Eq. (2.5) in the neighborhood of resonances by the expressions

$$\operatorname{Im} A_i(z) = \pi\,|\lambda\alpha_i|\,\delta(z - z_i), \tag{2.7}$$

where $\alpha_i = -1/f_i'(z_i)$. In order that Eqs. (2.7) and (2.4) be consistent with the analytic properties reflected by the integral equations it is necessary to require in addition that

$$\lambda\,\alpha_i > 0. \tag{2.8}$$

The condition (2.8) is equivalent to the requirement that the energy derivative of the phase shift be positive in the neighborhood of a resonance, which corresponds to an attraction.

The products $\lambda\alpha_i$ are connected to the widths $\Gamma_i$ of the resonances, defined by the formulas

$$A_k(\nu)\,|_{\nu\sim\nu_k} \approx \Gamma_k/[\nu_k - \nu - i\,\Gamma_k K(\nu)] \tag{2.9}$$

by the relations

$$\Gamma_i = \lambda\,\alpha_i/2. \tag{2.10}$$

At that the width $\Gamma_1$ of the p-wave differs from the width $\Gamma$ introduced by Frazer and Fulco [4] by the factor $\nu_1$:

$$\Gamma_1 = \nu_1\Gamma. \tag{2.11}$$

The construction of the functions $f_i$ with a prescribed number of zeros can be accomplished in various ways. In the most general approach the real parts of the partial wave amplitudes are constructed from the imaginary parts by means of integral equations. In the case under discussion it is more convenient to make use of the crossing symmetry conditions (1.5), written out for the real parts, Eq. (2.4).

Let us emphasize that the neglect of the terms $\lambda I_i(z, 0)$ in comparison with $f_i(z)$ is justified only in the case of a power law asymptotic behavior, when for large z the functions $f_i(z)$ increase at least linearly. For a logarithmic asymptotic behavior, when $f_i$ tends to a constant and $I_i$ to a logarithm, the term $\lambda I_i$ dominates at large z and cannot be neglected. For this reason the described procedure for obtaining solutions in the limit $\lambda \to 0$ is meant exclusively for the power-law branches and cannot be applied to the logarithmic solutions without appropriate modifications.

## 3. SOLUTIONS CONTAINING ONE RESONANCE IN EACH WAVE

We restrict ourselves to the discussion of the case when each partial wave goes through just one resonance. The location of the resonance in the wave $A_i$ will be denoted by $z_i$. In that case the most general form of $\mathrm{Re}\,A_i$, satisfying the conditions of crossing symmetry, is

$$\frac{\mathrm{Re}\,A_i}{\lambda} = \frac{\alpha_i}{z_i - z} + \frac{\alpha_i}{z_i + z} + l_i\left\{\frac{2\alpha_0}{z+z_0} + \frac{9\alpha_1}{z+z_1} - \frac{5\alpha_2}{z+z_2}\right\}. \quad (3.1)$$

Equation (3.1) contains seven parameters: $z_i$, $\alpha_i$, and $\lambda$. These parameters are subject to the conditions (2.8) and the requirement $z_i > 1$. The number of independent parameters turns out to be five, as a consequence of the threshold condition on the p-wave

$$\frac{A_1(1)}{\lambda} = \frac{2\alpha_1 z_1}{z_1^2 - 1} - \frac{1}{18}\left\{\frac{2\alpha_0}{z_0+1} + \frac{9\alpha_1}{z_1+1} - \frac{5\alpha_2}{z_2+1}\right\} = 0 \quad (3.2)$$

and the relations (2.1) at the symmetry point, which result in the formula:

$$2\alpha_0/z_0 - 9\alpha_1/z_1 + \alpha_2/z_2 = 6. \quad (3.3)$$

It is seen from Eq. (3.2), with Eq. (2.8) taken into account, that there are no solutions of the type under discussion with a resonance in just one wave (i.e., with two of the three coefficients $\alpha_i$ equal to zero). It also follows from Eq. (3.2) that we must have a resonance in the $A_0$ wave. Consequently, there exist two "two-resonance" branches: a) resonances in the $A_0$ and $A_1$ waves, b) resonances in the $A_0$ and $A_2$ waves.

Let us consider the branch a). This branch depends on three parameters: $\lambda$, $z_0$, and $z_1$. The widths of the resonances are expressed as follows:

$$\Gamma_0/\lambda = \alpha_0/2 = 3(1+z_0)(1+3z_1)/2T(z_0, z_1), \quad (3.4)$$

$$\Gamma_1/\lambda = \alpha_1/2 = (z_1^2 - 1)/3T(z_0, z_1), \quad (3.5)$$

where

$$z_0 z_1 T(z_0, z_1) = z_1 + 3z_1^2 + z_0(1 + z_1 + 2z_1^2).$$

It follows from these formulas that $\lambda$ may assume only positive values. The scattering lengths are of the form

$$a_0/\lambda = 2(3z_0 + 4z_1 + 5z_0 z_1)/(z_0 - 1)\,T(z_0, z_1), \quad (3.6)$$

$$a_2/\lambda = 4z_1/T(z_0, z_1). \quad (3.7)$$

The ratio of the resonance widths in this branch is given by

$$\Gamma_1/\Gamma_0 = 2(z_1^2 - 1)/9(1+z_0)(1+3z_1). \quad (3.8)$$

From this formula it is seen that for physically interesting solutions the resonance in the p-wave is always narrower than in the $A_0$ wave. Thus, for $\nu_1 < 6$ and $\nu_0 > 1$ we get $\Gamma_1 < \frac{7}{30}\Gamma_0$.

Another characteristic feature of this solution is the smallness of the width $\Gamma_1$ and the rather large value of the width $\Gamma_0$ for reasonable values of $\nu_1$ and $a_0$. Thus, for example, if we choose the position of the p-wave resonance to be at $\nu_1 = 3.5$ following Anderson et al,[5] and at $\nu_1 = 5.5$ following Stonehill et al,[6] then we get for a number of choices of $\nu_0$ the results for the scattering lengths and resonance widths shown in Table I.

**Table I.** Parameters of the two-resonance branch

| $\nu_1$ | $\nu_0$ | $a_0/\lambda$ | $a_2/\lambda$ | $\Gamma_0/\lambda$ | $\Gamma_1/\lambda$ | $\Gamma/\lambda$ |
|---|---|---|---|---|---|---|
| 3.5 | 3.5 | 5.31 | 1.58 | 16.7 | 1.04 | 0.30 |
| 3.5 | 10 | 5.11 | 1.83 | 45 | 1.14 | 0.33 |
| 5.5 | 9.5 | 5.01 | 1.78 | 43.2 | 1.77 | 0.32 |
| 5.5 | 19.5 | 5.08 | 1.85 | 87.5 | 1.84 | 0.34 |

The energy dependence of the scattering phase shifts is shown in Figs. 1—3 for the case $\nu_1 = 5.5$, $\nu_0 = 19.5$. We have plotted there the functions

$$\lambda K(z)\,\mathrm{ctg}\,\delta_i(z) = f_i(z) = \lambda/\mathrm{Re}\,A_i(z). \quad (3.9)^*$$

These results indicate that in order that $a_0$ be of the order of unity[7,8] it is necessary to take for $\lambda$ values of the order of 0.2. In that case, however, the width of the p-wave resonance is too small ($\Gamma$ is of the order of 0.06) to agree with the available data. At the same time the resonance in the $A_0$ wave will be rather broad ($\Gamma_0$ lies between 2 and 9). It should, of course, be kept in mind that for $\lambda \sim 0.2$ the formulas based on the $\delta$ approximation are no longer sufficiently exact (see Sec. 5) and it is necessary to perform exact numerical calculations to reach more reliable conclusions on the properties of the branch under discussion.

Let us consider the branch b). Since this branch has no p-wave resonance it is of no physical interest. For the moment we shall not consider the "three-resonance" solution in its most general form in order to avoid the complications involved in dealing with five parameters. We restrict ourselves to three-parameter solutions for which the positions of all three resonances coincide: $z_0 = z_1 = z_2 = z_r$.

For this family the wave $A_2$ will in general have a zero at the point $x_0$, which lies above the

*ctg = cot.

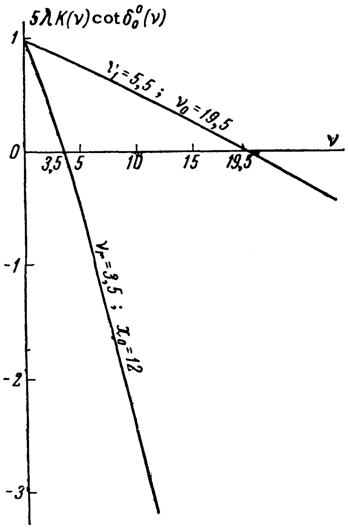

FIG. 1

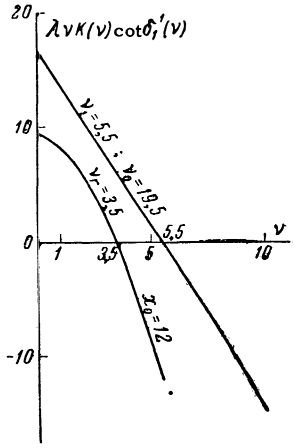

FIG. 2

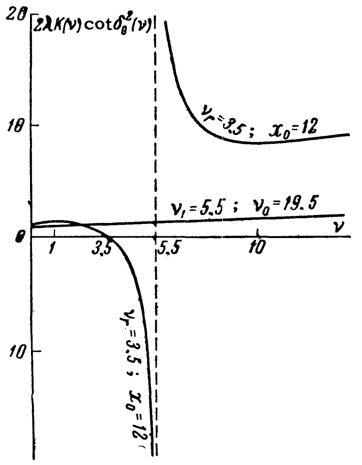

FIG. 3

resonance. As independent parameters we choose $\lambda$, $z_r$, and $x_0$. Calculations lead to the following expressions for the resonance widths:

$$\Gamma_0 = \lambda z_r (3 + 5x_0 + 4z_r)/4(1 + x_0), \quad (3.10)$$

$$\Gamma_1 = \lambda z_r (z_r - 1)/6(1 + x_0), \quad (3.11)$$

$$\Gamma_2 = \lambda z_r (x_0 - z_r)/2(1 + x_0) \quad (3.12)$$

and scattering lengths

$$a_0 = \lambda \frac{z_r^2}{z_r^2 - 1} \frac{7 + 5x_0}{1 + x_0}, \qquad a_2 = 2\lambda \frac{z_r^2}{z_r^2 - 1} \frac{x_0 - 1}{x_0 + 1}. \quad (3.13)$$

Numerical results for a number of versions are given in Table II.

The dependence of the scattering phase shifts on energy is shown in Figs. 1—3 for $\nu_r = 3.5$, $x_0 = 12$. It is seen that the main peculiarities of the branch a) are also characteristic of the "three-parameter three-resonance branch." For this branch the resonance in the $A_2$ wave turns out to be narrower than the resonance in the $A_0$ wave (for $x_0$ not too close to $z_r$) and wider than the resonance in the p-wave.

**Table II.** Parameters of the three-resonance branch with coincident resonances

| $\nu_r$ | $x_0$ | $a_0/\lambda$ | $a_2/\lambda$ | $\Gamma_0/\lambda$ | $\Gamma_1/\lambda$ | $\Gamma/\lambda$ | $\Gamma_2/\lambda$ |
|---|---|---|---|---|---|---|---|
| 3.5 | 12.0 | 5.23 | 1.72 | 14.6 | 0.72 | 0.20 | 1.23 |
| 3.5 | 24.0 | 5.16 | 1.87 | 12.4 | 0.37 | 0.11 | 2.56 |
| 5.5 | 24 | 5.12 | 1.85 | 20.5 | 0.88 | 0.16 | 2.88 |
| 5.5 | 48 | 5.08 | 1.93 | 17.8 | 0.45 | 0.08 | 4.40 |

## 4. EXTENSION FROM THE REGION OF SMALL $\lambda$

An extension from the region of small $\lambda$ can be accomplished for the $\delta$-like solutions by means of the N/D technique.[9] However the original version of this technique, utilized by Chew and Mandelstam, must be modified somewhat because of the presence of resonances. As mentioned in [2], the spectral representation for the functions $D_i$ is determined accurate to within a polynomial, whose degree is determined by the asymptotic values of the scattering phase shifts at high energies. In the case under consideration, when each wave contains one resonance, this degree is equal to unity and in the formulas for $D_i$ it is necessary to subtract the linear term. If the subtraction is performed at the symmetry point we get

$$A_i(z) = N_i(z)/D_i(z), \quad (4.1)$$

$$N_i(z) = A_i(0) + \frac{z}{\pi} \int_{-\infty}^{-1} \frac{\operatorname{Im} A_i(z') D_i(z')}{z'(z' - z)} dz', \quad (4.2)$$

$$D_i(z) = 1 - zg_i - \frac{z}{\pi} \int_1^{\infty} \frac{K(z') N_i(z')}{z'(z' - z)} dz'. \quad (4.3)$$

The threshold condition for the p wave gives

$$A_1(0) = -\frac{1}{\pi} \int_{-\infty}^{-1} \frac{\operatorname{Im} A_1(z') D_1(z')}{z'(z' - 1)} dz'. \quad (4.4)$$

The parameter $\lambda$ is introduced by means of Eq. (2.1) at the symmetry point. The coefficients $g_i$ should satisfy the asymptotic conditions of crossing symmetry, Eq. (1.8):

$$g_i = -N_i(\infty)/\lambda l_i c. \quad (4.5)$$

After substitution of Eq. (4.2) into Eq. (4.3) we obtain for $D_i$ the integral equation

$$d_i(z) = 1 + zg_i + zA_i(0) K(z, 0) + \frac{z}{\pi} \int_1^{\infty} \frac{dz'}{z'} K(z, z') \varphi_i(z') d_i(z'), \quad (4.6)$$

in which we have introduced the following abbreviations

$$d_i(z) = D_i(-z),$$

$$\varphi_i(z) = \operatorname{Im} A_i(-z) = -\sum_k b_{ik} \operatorname{Im} A_k(z), \quad (4.7)$$

$$K(x, y) = \frac{1}{\pi} \int_1^\infty \frac{K(z')\, dz'}{(z'+x)(z'+y)}. \quad (4.8)$$

Equation (4.6) represents a Fredholm equation for the function $d_i$ if the $\varphi_i$ are assumed known. It is convenient to make use of this fact in a numerical solution of the problem by iteration. Such a solution is now being carried out on the electronic computer of the Institute of Mathematics of the Siberian Division of the U.S.S.R. Academy of Sciences.

Since in the zeroth approximation the $\varphi_i$ are expressed in terms of $\delta$ functions, the Fredholm equation (4.6) becomes an algebraic equation for the first iteration, and the iteration itself reduces to algebraic manipulations. We shall make use of this circumstance to obtain first order corrections to the $\delta$-like approximations of the preceding section.

## 5. INCLUSION OF TERMS OF ORDER $\lambda^2$ IN THE BRANCH WITH COINCIDENT RESONANCES

The two-parameter branch a) represents a special case of the three-parameter branch, in which the positions of the zero and the resonance in the amplitude $A_2$ coincide. Since the position of the zero for fixed values of the limiting coefficient c and the position of the resonances $z_r$ are functions of $\lambda$, the character of the branch a) changes upon leaving the point $\lambda = 0$, with the result that a zero and a resonance appear in the second wave. For this reason the application of the N/D technique to the branch a) is more complicated than to the three-parameter branch with coincident resonances; we therefore restrict the discussion for the time being to the latter branch.

We note first that the $\delta$-like approximation to the three-parameter branch with coincident resonances [Eqs. (3.1), (3.10)—(3.12)] may be expressed in the N/D form with

$$D_i(z) = 1 - z/z_r, \quad (5.1)$$

$$N_i(z) = \lambda \frac{z_r - z}{z_r} \left\{ \frac{\alpha_i}{z_r - z} + \frac{\beta_i}{z_r + z} \right\}, \qquad \beta_i = \alpha_i + l_i c, \quad (5.2)$$

$$c = 2\alpha_0 + 9\alpha_1 - 5\alpha_2 = 12 z_r^2 / (1 + x_0), \quad (5.3)$$

$$\varphi_i(z) = -\pi\lambda\beta_i\delta(z - z_r). \quad (5.4)$$

We may say that the $\delta$-like approximation yields terms of first order in $\lambda$ in the solutions of the N/D equations. In order to obtain terms of order $\lambda^2$ it is necessary to evaluate more precisely the integrals in Eqs. (4.2), (4.4), and (4.6), containing the functions $\varphi_i$. To this end we use instead of Eq. (2.6) the following relation:

$$\lim_{\lambda\to 0} \frac{1}{\pi} \int_B^A \frac{\varphi(x)\, dx}{(x - x_r)^2 + \lambda^2} - \frac{\varphi(x_r)}{\lambda} = \frac{1}{\pi} \mathfrak{S}_B^A \frac{\varphi(x)\, dx}{(x - x_r)^2}$$

$$\equiv \frac{1}{\pi} \{\Psi(A) - \Psi(B)\}, \quad (5.5)$$

where

$$d\,\Psi(x) = \varphi(x)\, dx / (x - x_r)^2.$$

Generally speaking, the solution that we are trying to determine depends on five parameters. This number is reduced by two by making all resonances coincide at the point $z_r$. As independent parameters we choose $\lambda$, $z_r$, and c. The parameter $x_0$ in Eq. (5.4) now no longer corresponds precisely to the position of the zero in the $A_2$ wave, which is displaced by a small quantity of the order of $\lambda$ [see Eq. (5.11) below].

The fact that it is possible to arbitrarily assign the positions of the resonances even when higher order terms in $\lambda$ are taken into account has important significance. It means that already the $\delta$-like approximation of Sec. 3 reflects the main features of the exact solutions belonging to the class under discussion.

As can be shown, under these conditions the solutions of the N/D equations of Sec. 4 accurate to terms of order $\lambda^2$ are of the form*

$$D_i(z) = 1 - \frac{z}{z_r} + \lambda \frac{z}{z_r} \{B_i(z) - B_i(z_r)\}, \quad (5.6)$$

$$N_i(z) = N_i^0(z) + \frac{\lambda^2}{z_r(z_r + z)} \{A_i + z\Phi_i(\infty)$$

$$- (z + z_r)\Phi_i(z) - z l_i c B_i(z_r)\}, \quad (5.7)$$

where we used the notation

$$A_s = 3A_1 \qquad (s = 0,2),$$

$$A_1 = l_1 c B_1(z_r) - \Phi_1(\infty) + (1 + z_r)\Phi_1(1),$$

$$B_i(z) = 2\beta_i K(-z, z_r) - (\alpha_i + \beta_i) K(-z, 0),$$

$$\Phi_i(z) = \sum_k b_{ik} I_k(z), \quad I_k(z) = \frac{z z_r}{\pi\lambda^2} \mathfrak{S}_1^\infty \left[\frac{N_i^0(z')}{D_i^0(z')}\right]^2 \frac{d_i(z')\, dz'}{z'(z' + z)}, \quad (5.8)$$

with the matrix $b_{ik}$ and the function K(x, y) defined by Eqs. (1.2) and (4.8). The superscript zero in Eq. (5.8) indicates the functions N and D of the previous approximation, Eqs. (5.1) and (5.2).

The corrections to the resonance widths

*This solution can also be obtained in another way without using the N/D technique. In that method a kind of perturbation theory is used, wherein the unitarity and crossing symmetry conditions must be satisfied in each order of $\lambda$.

$$\lambda^2 \delta\Gamma_i = \Gamma_i - \lambda a_i / 2 \quad (5.9)$$

are

$$4\delta\Gamma_i = A_i / z_r + \Phi_i(\infty) - 2\Phi_i(z_r) - l_i c B_i(z_r) + a_i z_r B_i'(z_r), \quad (5.10)$$

where

$$B_i'(z_r) = \frac{d}{dz} B_i(z)\big|_{z=z_r}.$$

The position of the zero in the second wave is shifted by

$$x_0^\lambda = x_0 + \lambda\Delta,$$

$$\Delta = \frac{1+x_0}{2z_r^2}\{A_2 + x_0\Phi_2(\infty) - (z_r + x_0)\Phi_2(x_0) - x_0 l_2 c B_2(z_r)\}. \quad (5.11)$$

The shifted zero always lies above the resonance: $x_0^\lambda > z_r$. This reflects the fact that the $A_2$ wave is a generalized R-function.[1]

As was noted above, a peculiar feature of the branch under discussion is the small width of the p resonance for reasonable values of the scattering length $a_0$. This feature persists when terms of order $\lambda^2$ are taken into account. Thus, for example, for one of the cases given in Table II, Eqs. (5.6), (5.7), and (5.10) give for $\nu_r = 3.5$, $x_0 = 12$:

$$a_0 = 5.23\lambda + 12\lambda^2, \qquad \Gamma_1 = 0.72\lambda + 0.87\lambda^2.$$

This shows that the linear approximation in $\lambda$, Eqs. (3.4)—(3.6), is reliable in the region $\lambda < 0.2$.

Equations (5.6) and (5.7) may be used as the starting point when solving the equations of Sec. 4 by numerical iterations.

## 6. RELATION BETWEEN THE LOGARITHMIC AND POWER LAW BRANCHES

Let us establish now the connection between the logarithmic and power law branches. In the neutral model[2] the power law branch was a special case of the logarithmic branch with an R-term, corresponding to the passing of the CDD zero to infinity. We shall now show that a similar correspondence can be established between the resonant power law branches discussed here, and the logarithmic branch studied previously.[1]

In analogy to the neutral model we shall assume that the logarithmic solution with R-terms may be described by expressions (2.2), with the partial-fractional functions $\Phi_i(z, \lambda)$ tending to constant limits as $z \to \infty$. Considering only small $\lambda$, let us investigate Eqs. (2.4) and (2.5). We bear in mind that for small $\lambda$ these expressions are a good approximation to the solution in the region where $\lambda \ln \nu \ll 1$ and the integral terms $\lambda I_i$ are negligibly small.

For simplicity we consider only the case when all three waves have a resonance at the same fixed point $z_r$. We express the functions $f_i$ in the form

$$f_i(z) = (z_r^2 - z^2) / z_r^2 Q_i(z), \quad (6.1)$$

where $Q_i$ is a second degree polynomial:

$$Q_i(z) = b_i z^2 + c_i z + d_i, \qquad c_1 = -d_1 - b_1. \quad (6.2)$$

The conditions of crossing symmetry and the conditions at the symmetry point leave only three of the coefficients $b_i$, $c_i$, and $d_i$ independent. The limiting transition to the power law branch may be accomplished by having simultaneously all $b_i$ coefficients vanish. Let us perform this transition by assuming that all the $b_i$ are proportional to some small parameter $\epsilon$. We take into consideration that these coefficients are related by the condition of crossing symmetry

$$2b_0 + 9b_1 - 5b_2 = 0, \quad (6.3)$$

and also that $b_2 < 0$, in view of the fact that $A_2(z)$ is a generalized R-function.[1] These considerations limit the possible signs of $b_0$ and $b_1$ to the three combinations:

a) $b_1 > 0, \quad b_0 < 0, \quad b_2 < 0;$
b) $b_1 < 0, \quad b_0 > 0, \quad b_2 < 0;$
c) $b_1 < 0, \quad b_0 < 0, \quad b_2 < 0.$

In all three cases the $A_2$ wave has a resonance at $z_r$ and a zero at $x_0$, whose position is independent of $\epsilon$ and may be taken as fixed. In the waves $A_k$ $(k = 0, 1)$ one has for negative values of the corresponding $b_k$ beside the resonance at $z_r$ also a zero, whose position goes to infinity like $1/\epsilon$ as $\epsilon \to 0$. In addition, in the waves $A_0$ and $A_2$, depending on the sign of the logarithmic terms, we have in the corresponding cases high-energy resonances $z^{(\epsilon)} \sim e^{1/\lambda\epsilon}$. The behavior of the scattering phase shifts is shown schematically for all three cases in Fig. 4.

It is therefore clear that in the limit $\epsilon \to 0$ we obtain the three-parameter branch with coincident resonances discussed previously. This then establishes that the power law branch is a special case of the logarithmic branch, just like in the neutral model.

It is also clear that for $\epsilon$ sufficiently small the discussed logarithmic branches will be practically indistinguishable from the limiting power law branch in the region of not too large $\nu$. In other words, for each of the power law solutions discussed in the previous sections there exists a

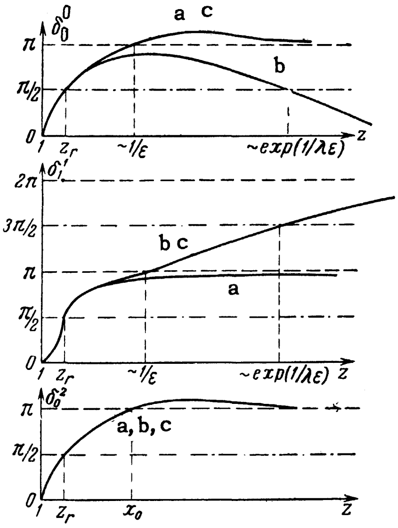


FIG. 4

solution nearly identical to it in the energy region of interest but behaving logarithmically at infinity. Therefore there is no reason for rejecting [10] these solutions as being in disagreement with experiment in the region of high energies.

These solutions appear to us of great importance, since, as was shown, they provide us at the moment with the only possibility for quantitative analysis of many-parameter solutions, which fact is connected with their rapid decrease in the high energy region.

## 7. DISCUSSION OF RESULTS

Let us investigate our solutions from the point of view of their agreement with experiment.

The first group of experimental results refers to the determination of the difference of the scattering lengths $a_0 - a_2$. These refer to the $K \to 3\pi$ decay [11] and to the analysis of the reaction $\pi^- + p \to \pi^+ + \pi^- + n$. [12] Khuri and Treiman obtain a negative difference [11] $a_0 - a_2 = -0.7$. At the same time Batusov et al [12] find this difference to be positive: $a_0 - a_2 = 0.35 \pm 0.30$.

As was shown, our power law solutions have the property

$$2a_0 - 5a_2 > 0 \qquad (7.1)$$

for positive $a_0$ and $a_2$. They are therefore in disagreement with the Khuri and Treiman result and could correspond to the result of Batusov et al. At that the parameter $\lambda$ may not exceed the value 0.2.

An estimate of $a_0$ based on an analysis of pd collisions was obtained by Truong [7]: $0.5 < a_0 < 1.5$. This estimate corresponds to our solutions for $0.1 < \lambda < 0.3$. Truong's result is also not in disagreement with the conclusions of Ishida et al [8] and Efremov, Meshcheryakov and one of the authors [13] on $\pi N$ scattering.*

A second group of results refers to the determination of the parameters of the p resonance.

In the work of Anderson et al [5] and Erwin et al [15] on the analysis of the processes $\pi + p \to 2\pi +$ nucleon the position of this resonance was determined as $\nu_r = 3.5$ and 5.5 respectively. These numbers are not in disagreement with other papers on theoretical interpretation of the data on nucleon structure. [16,17]

The width of the resonance is determined by Anderson et al [5] as $\Gamma = 0.3$. From the point of view of our power law solutions this is much too wide. To obtain such a value it would be necessary to set $\lambda \sim 1$, i.e., we would be forced into contradiction with the data on scattering lengths. The width determined by Stonehill et al [6] seems preferrable. Their data place the resonance at $\nu_r \approx 5.5$ with a width $\Gamma \sim 0.15$. However even this width is rather large for our solutions.

Taking into account the fact that for $\lambda$ of the order of 0.2 the formulas obtained in Sec. 3 must be made more precise by numerical calculations on electronic computers, one may reach the preliminary conclusion that the power-law branches are not in contradiction with experimental data. To get a more definitive conclusion on the correspondence of these solutions to experiment one must compare the results of numerical calculations with more reliable experimental data. A significant property of our solutions is the broad resonance in the $A_0$ wave.

Let us consider the mechanism responsible for the p resonance in our solutions. Let us remark first that the equation for the p wave has no solutions compatible with the threshold condition if the contribution of the s wave is ignored. Thus, in our scheme the "bootstrap" mechanism [18] for producing the p resonance is impossible. The $A_2$ wave enters the equation for $A_1$ with a positive sign, and the $A_0$ wave with a negative sign. Therefore the $A_0$ wave helps and the $A_2$ wave hinders the p resonance. The larger the $A_0$ wave the broader the p resonance. The larger $A_2$, the narrower the p resonance. Consequently in our solution the p resonance is completely determined by the $A_0$ wave. As was seen, the resonance in $A_0$ is much broader than the resonance in $A_1$. This is easily understood since the integral over $\mathrm{Im}\, A_0$ enters the equation for $A_1$ with

*We take this opportunity to remark that [13] contains a somewhat unsuccessful approximation (in this connection see [14]) so that the conclusions obtained there require revision.

the small coefficient $1/9$. Therefore even if the $A_0$ wave were close to saturation in a large interval the resonance in $A_1$ would remain narrow.

In order to obtain an estimate of the possible width of the resonance let us consider the threshold condition for the p wave, in which the term with $\operatorname{Im} A_2$ is ignored, and the term with $\operatorname{Im} A_0$ is replaced by its maximum value $K^{-1}(z)$ in the interval from $\nu = 0$ to $\nu = \Lambda$ and by zero in the high energy region $\nu > \Lambda$. Making use for $\operatorname{Im} A_1$ of the $\delta$-like approximation

$$\operatorname{Im} A_1 = 2\pi \Gamma_1 \delta (z - z_1) = \pi \Gamma_1 \delta (\nu - \nu_1),$$

we find for $\Lambda \gg 1$

$$\frac{\Gamma_1}{\nu_1} + \frac{\Gamma_1}{2(\nu_1 + 1)} - \frac{\ln 4\Lambda}{9\pi} = 0.$$

Setting here $\nu_1 = 5.5$ and placing the limit of the low energy region at $\Lambda = 10$ we find $\Gamma_1 = 0.50$, which corresponds to an energy total width of the dipion of 50 MeV.

Under these assumptions the parameter $\lambda$, calculated from Eq. (2.7) of [1], turns out to be equal to 0.27, which corresponds approximately to the connection between $\Gamma_1$ and $\lambda$ as given by Eq. (3.5).

Let us emphasize that the estimate here given is based on the assumption that the high energy contributions may be ignored in the unsubtracted equations. Therefore a reliable experimental indication that the total width of the dipion is more than 50 MeV would mean that the high energy contributions cannot be ignored in the equations without subtractions. In that case the high energy contributions should be described by the first subtraction parameter $\lambda$ by ascribing to it values larger than 0.3. At that the role of high energy processes may turn out to be small in equations with one subtraction.

To clarify the role of the high energy contributions in the low energy region it is important to have more precise experimental data on the width of the p resonance as well as to obtain information on the energy dependence of the phase shift $\delta_0^0$.

The authors are grateful to Chou Hung-Yuan, A. V. Efremov, and I. F. Ginzburg for useful discussions.


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Translated by A. M. Bincer
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