CONTRIBUTION TO THE THEORY OF THE $\pi + N \to N + \pi + \pi$ *AND* $\gamma + N \to N + \pi + \pi$ *REACTIONS NEAR THRESHOLD*

V. V. ANISOVICH, A. A. ANSEL'M, and V. N. GRIBOV

Leningrad Physico-Technical Institute

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The amplitudes and cross sections for the reactions indicated in the title are considered with an accuracy to terms quadratic in the threshold momenta. The dependence of the cross sections on the kinetic energy of the produced particles is expressed in explicit form. From an analysis of the experimental data on the reactions it is possible in principle to determine the two $\pi\pi$-interaction amplitudes, a_0 and a_2, at zero energy and in states with isotopic spins $T = 0$ and $T = 2$, unlike the analysis accurate to terms linear in the momenta,[1,2] which yields only the charge-exchange amplitude $a_2 \to a_0$. The first part of the paper contains a convenient dispersion method for calculating the diagrams that are important in the problem under consideration.

It was shown earlier[1,2] that experiments in which γ quanta or negative pions react with nucleons to yield low-energy pions make it possible to determine the π-meson charge exchange amplitude at zero energy. This can be done by merely separating the terms linear in the relative momenta of the particles from the energy dependence of the cross section. The first experimental data have by now been obtained (see [3,4]). The quadratic terms are only partially determined by the pair interactions, but, as shown in [5], the quadratic terms responsible for the angular correlations of the produced particles are completely determined by the pair scattering amplitudes.

If the energy distribution of the particles in the final state is of interest, then, as noted recently,[6] there are terms of two types. These are, on the one hand, complicated functions of the ratios $k_{il}\sqrt{2\mu_{il}E}$ (k_{il} — relative momentum of the i-th and l-th particle, μ_{il} — their reduced mass, E — kinetic energy of the three particles), which yield a non-trivial energy distribution and are proportional to the pair interaction amplitudes at zero energy. The other terms are of the form $C_{il}k_{il}^2$, where the constants C_{il} are not connected with the amplitudes, and can be experimentally separated from the terms of the first kind.

In the present investigation we derived the appropriate formulas for the reactions listed in the title. Compared to the case of $K_{\pi 3}$ decay, considered in [6], the point of interest in reactions where two pions are formed is obviously the dependence of the total and differential cross sections on the total energy of the produced particles. We shall show in this paper that the differential cross sections of the reactions contain terms of the form $k_{il}^2 \ln(\mu/E)$ and $E \ln(\mu/E)$ ($1/\mu$ — effective interaction radius), making the total reaction cross sections dependent on $E \ln(\mu/E)$. These terms are proportional to the pair amplitudes and are greater than the other quadratic terms in the direct vicinity of the threshold [$\ln(\mu/E) \gg 1$]. [In the case of $K_{\pi 3}$ decay, where E is fixed ($E = m_K - 3m_\pi$), these terms are included in the aforementioned $C_{il}k_{il}^2$.] To determine the $\pi\pi$-interaction amplitudes it is sufficient in principle to study the dependence of total cross sections of the reactions $\pi + N \to N + \pi + \pi$ and $\gamma + N \to N + \pi + \pi$ near threshold on the energy of the incoming particles.

An evaluation of the quadratic terms makes it possible to determine not only the pion charge exchange amplitude, as in an experimental data reduction accurate to terms linear in k_{il}, but the two pion scattering amplitudes at zero energy in states with isotopic spin $T = 0$ and $T = 2$. In addition, formulas more accurate than those obtained in [1,2] permit the use of experimental data that are not so close to the reaction thresholds.

In the first section we consider the amplitude of the conversion of two neutral spinless particles into three low-energy particles. The dispersion method will be used to calculate the contribution of certain diagrams that are significant near threshold.[6] In the second section we employ the result obtained in an analysis of the reactions $\pi + N \to N + \pi + \pi$ and $\gamma + N \to N + \pi + \pi$.

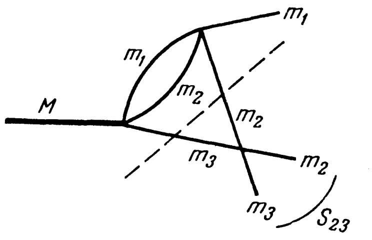

FIG. 1.

1. AMPLITUDE OF CREATION OF THREE LOW-ENERGY NEUTRAL PARTICLES

In the analysis of the conversion of two particles into three near threshold, diagrams of a certain type are significant, one of which is shown in Fig. 1. Each vertex corresponds not to the interaction constant but to the scattering or conversion amplitude at zero energy. For neutral particles there are altogether six diagrams of this type, in which the particles commute cyclically. A detailed motivation for the choice of such diagrams is given in [6]. We merely recall here that in these diagrams the small three-particle phase volume E^2 is partially compensated for by the pole character of the particle interaction amplitude in the final state. The value of the diagram near threshold is therefore $\sim E \ln E$.

The diagram shown in Fig. 1 is essentially triangular and depends on two invariants, the square of the c.m.s. energy of the incoming particles M^2 (the incoming particles enter the diagram at one point and are designated by the heavy line in the figure) and the square of the energy of particles 2 and 3 in their c.m.s., S_{23}. Let us calculate the contribution of this diagram, using the dispersion relations in S_{23} and M^2.

We examine the analytic properties of this diagram as functions S_{23} for different M^2. If M^2 is sufficiently small there exists, as for any triangular diagram, a dispersion relation in S_{23} with a lower integration limit $S_{23} = (m_2 + m_3)^2$

$$A(S_{23}, M^2) = A((m_2+m_3)^2, M^2) + \frac{S_{23}-(m_2+m_3)^2}{\pi} \int\limits_{(m_2+m_3)^2}^{\infty} \frac{A_1(S', M^2)\, dS'}{[S'-(m_2+m_3)^2][S'-S]}. \tag{1}$$

As M^2 is increased, anomalous singularities usually appear, modifying the form of the dispersion equation (1). This is caused by the fact that the singularity of the absorption part A_1 touches the contour of integration in S' and captures it.[7] At the decay value $M^2 = (m_1 + m_2 + m_3)^2$ the contour of integration goes over into the complex plane.[8] In the case of interest to us the situation is somewhat different. Since real scattering takes place at the vertex (m_1m_2, m_1, m_2) (the corresponding cosine on the Landau diagram is equal to unity), the diagram under consideration does not have a Karplus singularity. The absorption part $A_1(S_{23}, M^2)$ has along with the trivial singularity $S_{23} = (m_2 + m_3)^2$ also, as will be shown later on, a singularity $S_{23} = (M - m_1)^2$, which falls on the contour of integration in (1) when $M^2 = (m_1 + m_2 + m_3)^2$, so that when $M^2 > (m_1 + m_2 + m_3)^2$ the dispersion relation (1) retains its form, but $A_1(S_{23}, M^2)$ becomes complex in the interval $(m_2 + m_3)^2 < S_{23} < (M - m_1)^2$.

It is interesting to note that, in contradiction to Cutkosky,[9] who states that all the singularities of the absorption parts and of the amplitudes themselves occur upon vanishing of the denominators corresponding to different lines in the diagrams, the singularity $A_1(S_{23}, M^2)$ does not exhibit this behavior when $S_{23} = (M - m_1)^2$. This singularity is connected with the boundary of the physical region — the creation of the first particle with zero momentum. However, the entire amplitude $A(S_{23}, M^2)$, as will be shown later on, has no singularity at this point. On the other hand, the point

$$S_{23} = (m_2+m_3)^2 + m_2(m_1+m_2)^{-1}[M^2-(m_1+m_2+m_3)^2],$$

which corresponds to the vanishing of all denominators of the diagram, is not singular for the absorption part in this case. This is also seen from a more detailed investigation of A_1.

We first calculate $A_1(S_{23}, M^2)$ with M^2 smaller than $(m_1 + m_2 + m_3)^2$. We shall later continue it to the region $M^2 > (m_1 + m_2 + m_3)^2$ of interest to us. At small M^2, corresponding to the physical region of the transformation $m_2, m_3 \to M, m_1$, the value A_1 is determined from the unitarity condition in the channel S_{23}:

$$A_1(S_{23}, M^2) = \frac{k_{23}}{\sqrt{S_{23}}} a_{23}(m_2+m_3) \int\limits_{-1}^{+1} \frac{dz}{2} B(t), \tag{2}$$

$$k_{23} = \tfrac{1}{2} S_{23}^{-1/2} \{[S_{23}-(m_2+m_3)^2][S_{23}-(m_2-m_3)^2]\}^{1/2}.$$

Here a_{23} — scattering length of particles 2 and 3. The normalization of the unitarity condition corresponds to the ordinary determination of the invariant amplitudes, the product $a_{23}(m_2 + m_3)$ being the invariant amplitude of the scattering of particles 2 and 3 at zero energy.

The amplitude $B(t)$ of the conversion of the particle M into m_1, m_2, and m_3 is shown in Fig. 1 to the left of the dashed line. Since particles m_1 and m_2 approach the vertex (m_1m_2, m_1, m_2) at a single point, $B(t)$ depends only on t — the square of the energy of the particles m_1 and m_2 in their c.m.s. Integration with respect to z denotes averaging over the angle between the momenta of

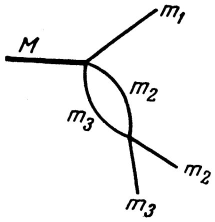

FIG. 2.

the particles m_1 and m_2 in the c.m.s. of particles 2 and 3. The quantities t and z are obviously related as

$$t = m_1^2 + m_2^2 - 2\sqrt{(q_1^2 + m_1^2)(q_2^2 + m_2^2)} + 2q_1q_2z,$$
$$q_1 = \tfrac{1}{2}S_{23}^{-1/2}\{[S_{23} - (M + m_1)^2][S_{23} - (M - m_1)^2]\}^{1/2}, \quad (3)$$
$$q_2 = \tfrac{1}{2}S_{23}^{-1/2}\{[S_{23} - (m_2 + m_3)^2][S_{23} - (m_2 - m_3)^2]\}^{1/2},$$

where q_1 and q_2 are the momenta of particles 1 and 2 in the c.m.s. of particles 2 and 3.

For B (t) we can write a dispersion relation in the variable t:

$$B(t) = \frac{t - (m_1 + m_2)^2}{\pi}\int\limits_{(m_1+m_2)^2}^{\infty}\frac{B_1(t')\,dt'}{[t' - (m_1 + m_2)^2][t' - t]}. \quad (4)$$

$B[(m_1 + m_2)^2]$ is set equal to zero, since the constant terms in B (t) make the same contribution as the diagrams of the type shown in Fig. 2, which are taken into account separately.[6] The absorption part of B_1 is equal to

$$B_1(t) = \lambda a_{12}(m_1 + m_2)\,t^{-1/2}k_{12}, \quad (5)$$
$$k_{12} = \tfrac{1}{2}t^{-1/2}\{[t - (m_1 + m_2)^2][t - (m_1 - m_2)^2]\}^{1/2},$$

where k_{12} — momentum of relative motion of particles 1 and 2 in their c.m.s.

Expression (2) for $A_1(S_{23}, M^2)$ is an analytic function of M^2 up to $M^2 = (\sqrt{S_{23}} + m_1)^2$. This can be readily verified by recognizing that B (t) has a singularity only at $t = (m_1 + m_2)^2$. Indeed, if we change over in the integral (2) to the variable t, we obtain an integral of an analytic function with a cut along the real axis from $t = (m_1 + m_2)^2$ to $t = \infty$. The integration is between the points t^- and t^+ [$z = \pm 1$ in formula (3)], located on the real axis outside the cut when $M^2 < (S_{23} - m_1)^2$ (the contour in Fig. 3). When $M^2 = (\sqrt{S_{23}} - m_1)^2$ the quantities $t^\pm$ are complex conjugate, but this point is obviously not a singularity. The singularity appears at $M^2 = (\sqrt{S_{23}} + m_1)^2$, when $t^\pm$ lie on the real axis, so that the integration contour includes the cut. To obtain a correct analytic continuation of $A(S_{23}, M^2)$ when M^2 exceeds $(m_1 + m_2 + m_3)^2$, it is necessary to make the substitution $M^2 \to M^2 + i\epsilon$ ($\epsilon > 0$). Then the value of A_1 when $M^2 > (S_{23} + m_1)^2$ is determined by the integral along the contour 3 in Fig. 3. Further increase in M^2 causes t^- to circuit the point $t = (m_1 + m_2)^2$ from the left and the contour of integration assumes the form 4 of Fig. 3. The value of M^2 corresponding to $t^- = 0$ is not a singularity of $A_1(S_{23}, M^2)$. This is equivalent to the earlier statement that the vanishing of all the denominators of the diagram (a condition to which this point precisely corresponds) produces no singularities in the absorption part.

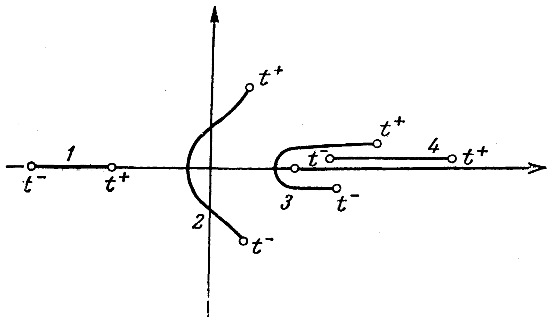

FIG. 3.

Following this analytic continuation of A_1, it is obvious that if we are interested in $A_1(S_{23}, M^2)$ for the nonrelativistic region

$$S_{23} - (m_2 + m_3)^2 \ll (m_2 + m_3)^2,$$

then, as can be seen from (3), the variable t in the integral for A_1 varies near $t = (m_1 + m_2)^2$. We therefore obtain from (4) and (5)

$$B(t) = i\lambda k_{12}a_{12}, \qquad t = (m_1 + m_2)^2 + (m_1 + m_2)\,k_{12}^2/\mu_{12},$$
$$\mu_{12} = m_1m_2/(m_1 + m_2), \quad (6)$$

if we omit the terms of order k_{12}^2. Substituting (6) in (2) and integrating in the manner indicated above, we obtain the value of A_1 accurate to terms quadratic in the momenta

$$A_1(k_{23}^2, \varkappa^2) = \lambda a_{12}a_{23}\frac{k_{23}}{\sqrt{k_{23}^2 - \varkappa^2}}\left(\varkappa^2 + \frac{1 - 4\beta_2}{3\beta_2}k_{23}^2\right)\sqrt{\frac{\beta_2\mu_{12}}{\mu_{23}}}, \quad (7)$$

$$S_{23} = (m_2 + m_3)^2 + (m_2 + m_3)\,k_{23}^2/\mu_{23},$$
$$\varkappa^2/2\mu_{23} = E = M - m_1 - m_2 - m_3,$$
$$\beta_2 = m_2(m_1 + m_2 + m_3)/(m_1 + m_2)(m_2 + m_3). \quad (8)$$

It now remains to calculate the dispersion integral (1) with the absorption part (7). It is convenient first to rewrite (1) in the form of an integral with respect to $k_{23}'^2$:

$$A(k_{23}^2, \varkappa^2) = A(0, \varkappa^2) + \frac{k_{23}^2}{\pi}\int\limits_0^{\infty}\frac{A_1(k_{23}'^2, \varkappa^2)\,dk_{23}'^2}{k_{23}'^2(k_{23}'^2 - k_{23}^2)}. \quad (9)$$

At large values of $k_{23}'^2$ the function $A_1(k_{23}'^2, \kappa^2)$ contains a term of order $k_{23}'^2$ and the integral (9) diverges logarithmically. Naturally, this divergence is due to the expansion of the absorption part of A_1, whereas the exact expression would cut off the integral at a value of k_{23}' on the order of the particle mass. It is meaningless to define the character of this cutoff in greater detail by starting from our specific diagram, since the cutoff can also be, for example, the result of the decrease in the exact amplitudes, which we have replaced by constants, at the vertices of the diagram. This can be formulated in somewhat different fashion as follows. The difference between the two values of A obtained from (9) at cutoffs that differ by a numerical factor is of the form Ck_{23}^2, where C is a certain constant. Terms of this type are contained in many diagrams and can in any case be expressed in terms of the pair scattering amplitudes.[6] In view of the foregoing, we shall cut off the integral (9) at $k_{23}' \sim \mu$, where $1/\mu$ is a quantity of the order of the interaction radius. The resultant logarithmic term of the form $k_{23}^2 \ln(\mu^2/\kappa^2)$ differs from terms such as Ck_{23}^2 only in the dependence of κ^2 on the total energy, and therefore these terms have been omitted in the analysis of the $K_{\pi 3}$ decay.[6] They do make, however, the greatest contribution, compared with other quadratic terms, at very small energies.

The integration of (9) is elementary and yields

$$A(k_{23}^2, \varkappa^2) = A(0, \varkappa^2) + \lambda a_{12}a_{23}\sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}}$$
$$\times\left[-\frac{\varkappa k_{23}}{\mu_{23}}\, 2\arccos\frac{k_{23}}{\varkappa}\left(1-\frac{k_{23}^2}{\varkappa^2}\right)^{-1/2}\left(\beta_2+\frac{k_{23}^2}{\varkappa^2}\,\frac{1-4\beta_2}{3}\right)\right.$$
$$\left.+\frac{1-4\beta_2}{3}\,\frac{k_{23}^2}{\mu_{23}}\,\frac{1}{\pi}\ln\frac{\mu^2}{\varkappa^2}\right]. \qquad (10)$$

Apart from the last term, which was referred to earlier, this expression coincides with the expression in [6].

The quantity $A(0, \kappa^2)$ contains a term of the order of $a_{12}a_{23}\kappa^2 \ln(\mu^2/\kappa^2)$. This term, naturally, should be taken into account in the same approximation as the last term in (10). It is necessary to calculate thus the diagram shown in Fig. 1, with $S_{23} = (m_2 + m_3)^2\,(k_{23} = 0)$ as the function of κ^2, accurate to terms $\sim\kappa^2 \ln(\mu^2/\kappa^2)$. The function $A(0, \kappa^2)$ obeys the dispersion equation

$$A(\varkappa^2) = \frac{\varkappa^2}{\pi}\int_0^\infty \frac{A_1(\varkappa'^2)\,d\varkappa'^2}{\varkappa'^2(\varkappa^2-\varkappa'^2)}, \qquad A(\varkappa^2) \equiv A(0, \varkappa^2). \qquad (11)$$

The subtraction term in this equation is set equal to zero, since it contributes only to the amplitude of the process at zero energy.

The absorption part of A_1 can be obtained from the three-particle unitarity condition

$$A_1(\varkappa^2) = -\frac{\lambda a_{12}a_{23}(m_1+m_2)(m_2+m_3)}{\pi^3}\int\frac{d^4q_1d^4q_2d^4q_3}{q_2'^2-m_2^2}$$
$$\times\,\delta^4(q_1+q_2+q_3-P)\,\delta(q_1^2-m_1^2)\,\delta(q_2^2-m_2^2)$$
$$\times\,\delta(q_3^2-m_3^2). \qquad (12)$$

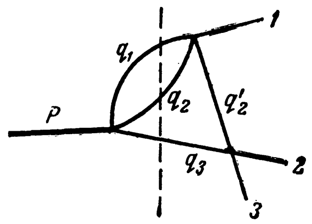

FIG. 4.

Here q_1, q_2, and q_3 — four-momenta of the particles in the intermediate state and q_2' — momentum of particle 2 in the pole amplitude of conversion of the three particles into three. Figure 4 shows the same diagram as in Fig. 1, illustrating the notation.

To calculate the integrals (12) it is convenient to use a well known device, namely introduce the additional integrations

$$\delta^4(q_1+q_2-q_{12})\,\delta(q_{12}^2-m_{12}^2)\,d^4q_{12}dm_{12}^2.$$

Then the integrations

$$\delta(q_1^2-m_1^2)\,\delta(q_2^2-m_2^2)\,\delta^4(q_1+q_2-q_{12})\,d^4q_1d^4q_2$$

yield the two-particle phase volume of particles 1 and 2, while the remaining integrations represent integration over the phase space of two particles with masses m_{12} and m_3 and momenta q_{12} and q_3 and integration over the mass m_{12}^2. After long but rather elementary algebraic calculations, we arrive at an expression for the absorption part of $A_1(\kappa^2)$:

$$A_1(\varkappa^2) = \lambda a_{12}a_{23}\sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}}\,\frac{\varkappa^2}{\mu_{23}}\,\frac{1}{6}(1+2\beta_2). \qquad (13)$$

Substituting (13) in (11) and integrating up to $\kappa' \sim \mu$, we obtain

$$A(\varkappa^2) = \lambda a_{12}a_{23}\sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}}\,\frac{1}{6}(1+2\beta_2)\frac{\varkappa^2}{\mu_{23}}\,\frac{1}{\pi}\ln\frac{\mu^2}{\varkappa^2}. \qquad (14)$$

Formulas (10) and (14) yield the final expression for the diagram under consideration. The total amplitude of the process $M \to m_1, m_2, m_3$, accurate to quadratic terms, is obtained by summing six diagrams such as Fig. 1 and three diagrams such as Fig. 2, and is equal to

$$M = M_0\,\{1 + ik_{12}a_{12} + ik_{13}a_{13} + ik_{23}a_{23} + a_{12}a_{13}\,[J_1(k_{12})$$
$$+ J_1(k_{13}) + \mathcal{K}_1(k_{12}) + \mathcal{K}_1(k_{13})] + a_{12}a_{23}\,[J_2(k_{12})$$
$$+ J_2(k_{23}) + \mathcal{K}_2(k_{12}) + \mathcal{K}_2(k_{23})] + a_{13}a_{23}\,[J_3(k_{13})$$
$$+ J_3(k_{23}) + \mathcal{K}_3(k_{13}) + \mathcal{K}_3(k_{23})] + C_1k_{12}^2 + C_2k_{13}^2 + C_3k_{23}^2\};$$

$$J_\alpha(k_{il}) = I_\alpha(x_{il}), \quad \mathcal{K}_\alpha(k_{il}) = K_\alpha(x_{il}), \quad x_{il} = k_{il}/\sqrt{2\mu_{il}E},$$

$$E = M - m_1 - m_2 - m_3,$$

$$\beta_1 = m_1(m_1+m_2+m_3)/(m_1+m_2)(m_1+m_3),$$

$$I_\alpha(x) = -2E\sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}}\,\frac{2x\arccos x}{\pi\sqrt{1-x^2}}\left[\beta_\alpha + x^2\frac{1-4\beta_\alpha}{3}\right],$$

$$K_\alpha(x) = -2E\sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}}\,\frac{1}{\pi}\ln\frac{\mu}{E}\left[-\frac{1}{6}(1+2\beta_\alpha)\right.$$
$$\left.-x^2\frac{1-4\beta_\alpha}{3}\right]. \qquad (15)$$

Formula (15) contains terms of the form Ck_{il}^2, which are not expressed in terms of the scattering amplitudes. The $I_\alpha(x)$ yield the angular correlations and the energy distributions of the particles, other than Ck_{il}^2, while $K_\alpha(x)$ determine the dependence on the total energy.

For the differential cross section of the reaction we obtain

$$d\sigma/d\Gamma = |M_0|^2 \{1 + 2a_{12}a_{13} [k_{12}k_{13} + J_1(k_{12}) + J_1(k_{13}) + \mathscr{K}_1(k_{12}) + \mathscr{K}_1(k_{13})] + 2a_{12}a_{23} [k_{12}k_{23} + J_2(k_{12}) + J_2(k_{23}) + \mathscr{K}_2(k_{12}) + \mathscr{K}_2(k_{23})] + 2a_{13}a_{23} [k_{13}k_{23} + J_3(k_{13}) + J_3(k_{23}) + \mathscr{K}_3(k_{13}) + \mathscr{K}_3(k_{23})] + 2C_1k_{12}^2 + 2C_2k_{13}^2 + 2C_3k_{23}^2\},$$

$$d\Gamma = \delta(k_{12}^2/2\mu_{12} + p_3^2/2\mu_3 - E)\, d^3k_{12}d^3p_3. \tag{16}$$

After simple integration over the phase volume we obtain an expression for the total cross section

$$\sigma = \text{const} \cdot E^2 [1 + AE \ln(\mu/E) + BE],$$

$$A = -\frac{8}{\pi}\sqrt{\frac{m_1m_2m_3}{m_1+m_2+m_3}} \times [a_{12}a_{13}(1-\beta_1)/3 + a_{12}a_{23}(1-\beta_2)/3 + a_{13}a_{23}(1-\beta_3)/3]. \tag{17}$$

An analytic dependence of this type takes place in reactions where two π mesons are produced by a π^+ meson on a proton; on the other hand if π mesons are produced by π^- mesons and γ quanta, the total cross section contains terms proportional to $E^2\sqrt{E}$.

2. PRODUCTION OF TWO PIONS BY PIONS AND GAMMA QUANTA ON A NUCLEON

In this section we apply the theory developed for neutral particles to reactions in which two pions are produced by pions and γ quanta on a nucleon. The procedure differs from that of the neutral case only in the account of the charges of the produced particles, and we present therefore only the final formulas. Owing to possible charge exchange of the particles in the final state, terms linear in the momenta appear in the cross section if the matrix elements of channels of different charge have different phases at zero energy. This case takes place when a pion and a γ quantum collide with a proton; this case was discussed in detail earlier[1,2] and will not be discussed again here.

We consider first reactions in which two pions are produced by collision of a negative pion and a proton

$$\pi^- + p \to \pi^+ + \pi^- + n, \tag{18.1}$$

$$\pi^- + p \to \pi^0 + \pi^0 + n, \tag{18.2}$$

$$\pi^- + p \to \pi^- + \pi^0 + p. \tag{18.3}$$

In accordance with the notation of [2], we denote the amplitudes of these reactions at zero energy by

$$\lambda_1 = \rho_1 e^{i\varphi_1}, \qquad \lambda_2 = \rho_2 e^{i\varphi_2}, \qquad \lambda_3 = \rho_3 e^{i\varphi_3}$$

respectively and put

$$\alpha_{ik} = \rho_{ik}\sin\varphi_{ik}, \qquad \beta_{ik} = \rho_{ik}\cos\varphi_{ik},$$
$$\rho_{ik} = \rho_k/\rho_i, \qquad \varphi_{ik} = \varphi_i - \varphi_k. \tag{19}$$

If we introduce the pion zero-energy scattering amplitudes a_0 and a_1 for states with total isotopic spins T = 0 and T = 2, respectively, normalized as the limit of the quantity $k^{-1}e^{i\delta}\sin\delta$ as $k \to 0$, then the different pion scattering and charge-exchange amplitudes at zero energy are expressed in terms of a_0 and a_2 as follows

$$a_{++}^{++} = a_{--}^{--} = 2a_2, \quad a_{+0}^{+0} = a_{-0}^{-0} = a_2,$$

$$a_{+-}^{+-} = \frac{2}{3}a_0 + \frac{1}{3}a_2 \equiv a_s, \qquad a_{00}^{00} = \frac{2}{3}a_0 + \frac{4}{3}a_2 \equiv a_s^0,$$

$$a_{+-}^{00} = a_{00}^{+-} = \frac{2}{3}(a_2 - a_0) \equiv a_e. \tag{20}$$

Here a_{++}^{++} denotes the amplitude of the scattering $\pi^+ + \pi^+ \to \pi^+ + \pi^+$, a_{+-}^{00} denotes the amplitude of the charge exchange $\pi^+ + \pi^- \to \pi^0 + \pi^0$, etc.

The amplitudes are normalized to make the total cross section $\sigma = 4\pi|a|^2$ for the production of unlike pions and $\sigma = 2\pi|a|^2$ for the production of identical pions. Thus, for example,*

$$\sigma_{+-}^{00} = 2\pi|a_e|^2, \qquad \sigma_{00}^{+-} = 4\pi|a_e|^2,$$
$$\sigma_{++}^{++} = 2\pi|2a_2|^2 = 8\pi|a_2|^2, \qquad \sigma_{+0}^{+0} = 4\pi|a_2|^2. \tag{21}$$

The amplitudes of scattering and charge exchange of a pion on a nucleon can also be readily expressed in terms of the isotopic amplitudes $b_{1/2}$ and $b_{3/2}$:

$$b_{p+}^{p+} = b_{n-}^{n-} = b_{3/2}, \quad b_{p-}^{p-} = b_{n+}^{n+} = \frac{1}{3}b_{3/2} + \frac{2}{3}b_{1/2} \equiv b_s,$$

$$b_{p0}^{p0} = b_{n0}^{n0} = \frac{1}{3}b_{1/2} + \frac{2}{3}b_{3/2} \equiv b_s^0,$$

$$b_{p-}^{n0} = b_{n+}^{p0} = \frac{1}{3}\sqrt{2}(b_{3/2} - b_{1/2}) \equiv b_e. \tag{22}$$

Finally, we number the particles in the same sequence as they were written out in the reactions (18). Then, for example, k_{12} the first reaction will denote henceforth the relative-motion momentum of the first and second particles, i.e., π^+ and π^-, k_{23} pertains to the relative momentum of the neutral pion and the proton for the third reaction, etc.

If we disregard the dependence of the reaction cross sections on the energy of the incoming beam, i.e., the total energy of the produced particles, then terms such as $k_{il}^2 \ln(\mu/E)$ and $E\ln(\mu/E)$ are incorporated in the terms of the type Ck_{il}^2.

*Our definition differs by a factor of 2 from the corresponding definition in[6] for scattering (charge exchange) amplitudes of identical particles.

We then have for the squares of the matrix elements of the reactions (18), accurate to quadratic terms,

$$|\langle\pi^+\pi^-n|S|\pi^-p\rangle|^2 = \rho_1^2\{1 + k_{12}\alpha_{12}a_e + 2k_{13}\alpha_{13}b_e + \beta_1[k_{12}k_{13} + J_1(k_{12}) + J_1(k_{13})] + \beta_2[k_{12}k_{23} + J_1(k_{12}) + J_1(k_{23})] + \beta_3[k_{13}k_{23} + J_3(k_{13}) + J_3(k_{23})] + \beta_4[J_1(k_{12}) + J_1(k_{13})] + \beta_5 J_3(k_{13}) + C_1k_{12}^2 + C_2k_{13}^2\}; \quad (23.1)$$

$$\beta_1 = 2\left(a_s + \tfrac{1}{2}\beta_{12}a_e\right)(b_s + \beta_{13}b_e) + \alpha_{12}\alpha_{13}a_eb_e,$$

$$\beta_2 = 2\left(a_s + \tfrac{1}{2}\beta_{12}a_e\right)b_{3/2}, \qquad \beta_3 = 2(b_s + \beta_{13}b_e)b_{3/2},$$

$$\beta_4 = 2a_eb_e\beta_{13} - a_eb_e(\beta_{12}\beta_{13} + \alpha_{12}\alpha_{13}),$$

$$\beta_5 = 2(b_e)^2(\beta_{12} - \sqrt{2}\beta_{13}).$$

$$|\langle\pi^0\pi^0n|S|\pi^-p\rangle|^2 = \rho_2\{1 + 2k_{12}\alpha_{21}a_e + 2(k_{13} + k_{23})\alpha_{23}b_e + \gamma_1[k_{12}(k_{13} + k_{23}) + 2J_1(k_{12}) + J_1(k_{13}) + J_1(k_{23})] + \gamma_2[k_{13}k_{23} + J_3(k_{13}) + J_3(k_{23})] + \gamma_3[2J_1(k_{12}) + J_1(k_{13}) + J_1(k_{23})] + \gamma_4[J_3(k_{13}) + J_3(k_{23})] + D_1k_{12}^2\} \quad (23.2)$$

$$\gamma_1 = 2\alpha_{21}\alpha_{23}a_eb_e + 2\left(\tfrac{1}{2}a_s^0 + \beta_{21}a_e\right)(b_s^0 + \beta_{23}b_e),$$

$$\gamma_2 = 2(b_s^0 + \beta_{23}b_e)^2, \qquad \gamma_3 = -2(\alpha_{21}\alpha_{23} + \beta_{21}\beta_{23})a_eb_e + \beta_{23}a_eb_e, \qquad \gamma_4 = 2(b_e)^2(\beta_{21} - \beta_{23}^2).$$

$$|\langle\pi^-\pi^0p|S|\pi^-p\rangle|^2 = \rho_3^2\{1 + 2k_{13}\alpha_{32}b_e + 2k_{23}\alpha_{31}b_e + \delta_1[k_{12}k_{13} + J_1(k_{12}) + J_1(k_{13})] + \delta_2[k_{12}k_{23} + J_1(k_{12}) + J_1(k_{23})] + \delta_3[k_{13}k_{23} + J_3(k_{13}) + J_3(k_{23})] + \delta_4[J_1(k_{13}) - J_1(k_{23})] + \delta_5J_3(k_{13}) + \delta_6J_3(k_{23}) + F_1k_{12}^2 + F_2k_{13}^2; \quad (23.3)$$

$$\delta_1 = 2a_2(b_s + \beta_{32}b_e), \qquad \delta_2 = 2a_2(b_s^0 + \beta_{31}b_e),$$

$$\delta_3 = 2(b_s + \beta_{32}b_e)(b_s^0 + \beta_{31}b_e) + 2\alpha_{31}\alpha_{32}(b_e)^2,$$

$$\delta_4 = 2a_eb_e\left(\beta_{31} - \tfrac{1}{2}\beta_{32}\right),$$

$$\delta_5 = 2(b_e)^2[1 - \alpha_{31}\alpha_{32} - \beta_{31}\beta_{32}],$$

$$\delta_6 = -2(b_e)^2[\alpha_{31}\alpha_{32} + \beta_{31}\beta_{32}] + 2\beta_{31}(b_e)^2\sqrt{2}.$$

The functions J_1 and J_3 are defined as follows:

$$J_{1,3}(k_{il}) = I_{1,3}(x_{il}), \qquad x_{12} = k_{12}/\sqrt{E},$$

$$x_{13} = k_{13}\sqrt{(M+1)/2ME},$$

$$x_{23} = k_{23}\sqrt{(M+1)/2ME},$$

$$I_1(x) = -2E\sqrt{\frac{M}{M+2}}\frac{x}{\sqrt{1-x^2}}\left[\frac{M+2}{2(M+1)} - x^2\frac{M+3}{3(M+1)}\right]\frac{2}{\pi}\arccos x,$$

$$I_3(x) = -2E\sqrt{\frac{M}{M+2}}\frac{x}{\sqrt{1-x^2}}\left[\frac{M(M+2)}{(M+1)^2} - x^2\frac{3M^2+6M-1}{3(M+1)^2}\right]\frac{2}{\pi}\arccos x. \quad (24)$$

Here E — total kinetic energy of the three particles, M — mass of the nucleon, and the pion mass is set equal to unity.

Formula (23) contains terms of the form const $\cdot\, k_{il}^2$. In this case, however, there is no longer any need for including Ck_{23}^2 in (23.1) and (23.3), since k_{23}^2 is expressed in terms of k_{12}^2, k_{13}^2, and E, and the dependence on the total energy will not be written out in these formulas. In expression (23.1) there is no need for writing out both squares k_{13}^2 and k_{23}^2, since this expression contains only their sum, which is expressed in terms of k_{12}^2 and E.

We now indicate directly how to modify (23) and likewise separate the dependence on the total energy. We must add a term to all three formulas const $\cdot$ E and make the substitutions

$$I_1(x) \to I_1(x) + K_1(x), \qquad I_3(x) \to I_3(x) + K_3(x).$$

$$K_1(x) = -\frac{2E}{\pi}\ln\left(\frac{\mu}{E}\right)\sqrt{\frac{M}{M+2}}\left[\frac{-2M-3}{6(M+1)} + \frac{M+3}{3(M+1)}x^2\right],$$

$$K_3(x) = -\frac{2E}{\pi}\ln\left(\frac{\mu}{E}\right)\sqrt{\frac{M}{M+2}}\left[-\frac{3M^2+6M+1}{6(M+1)^2} + \frac{3M^2+6M-1}{3(M+1)^2}x^2\right]. \quad (25)$$

The expressions of the type $k_{12}k_{13} + J_1(k_{12}) + J_1(k_{13})$, which were separated in (23), behave like k_{12}^2 and k_{13}^2 when k_{12} and k_{13} are small.[6] They are more difficult to distinguish experimentally from the terms $C_1k_{12}^2$ and $C_2k_{13}^2$ than the terms that follow them. Therefore, for example, the determination of the coefficients β_1, β_2, and β_3 is more complicated than that of β_4 and β_5. The latter, however, are always proportional to the product of the charge-exchange amplitudes, and consequently they yield (like the linear terms) only information on the combination $a_{12} - a_0$.

Because of the unitarity condition, the quantities α_{ik} and β_{ik} in (23) can be expressed in terms of the πN scattering phase shifts δ_{11} and δ_{31} in the states $P_{1/2}$ with isotopic spins $1/2$ and $3/2$ at the energy corresponding to the threshold of the production of the two pions.[2]

If, as in [2], we write out the isotopically invariant matrix elements for the production of two pions in states T = 0 (total isotopic spin $1/2$) and T = 2 (total spin $3/2$) in the form

$$\langle\tfrac{1}{2}0|S|\tfrac{1}{2}\rangle = F_{11}e^{i\delta_{11}}, \qquad \langle\tfrac{3}{2}2|S|\tfrac{3}{2}\rangle = F_{31}e^{i\delta_{31}}, \quad (26)$$

then, as can readily be shown,*

$$\lambda_1 = -\frac{\sqrt{2}}{3}F_{11}e^{i\delta_{11}} + \frac{1}{3\sqrt{5}}F_{31}e^{i\delta_{31}},$$

$$\lambda_2 = \frac{\sqrt{2}}{3}F_{11}e^{i\delta_{11}} + \frac{2}{3\sqrt{5}}F_{31}e^{i\delta_{31}}, \qquad \lambda_3 = -\frac{1}{\sqrt{10}}F_{31}e^{i\delta_{31}}. \quad (27)$$

From this we can readily find, for example, that

*An error that has crept into[2] has been corrected in the last formula of (27).

$$\alpha_{12} = \frac{3 \sin(\delta_{31} - \delta_{11})}{x\sqrt{10} + 1/x\sqrt{10} - 2\cos(\delta_{31} - \delta_{11})},$$

$$\beta_{12} = \frac{(2/x\sqrt{10}) - x\sqrt{10} - \cos(\delta_{31} - \delta_{11})}{x\sqrt{10} + 1/x\sqrt{10} - 2\cos(\delta_{31} - \delta_{11})}, \qquad (28)$$

$x = F_{11}/F_{31}$, and also the connection between α_{12} and α_{13} or β_{12} and β_{13}:

$$\alpha_{12} = -\sqrt{2}\alpha_{13}, \quad 1 + \beta_{12} = -\sqrt{2}\beta_{13} \qquad (29)$$

Let us examine now the photoproduction of two pions:

$$\gamma + p \rightarrow \pi^- + \pi^+ + p, \qquad (30.1)$$

$$\gamma + p \rightarrow \pi^0 + \pi^0 + p, \qquad (30.2)$$

$$\gamma + p \rightarrow \pi^+ + \pi^0 + n. \qquad (30.3)$$

It is easy to note that since the charged states of the produced particles are obtained by replacing all the projections of the isotopic spin of the final states in reactions (18), formulas (23) and (25) can also be used in this case if we number the particles in the same sequence as in reactions (30). Now, for example, the momentum k_{13} for reaction (30.1) is the momentum of relative motion of the π^- meson and the proton; the momentum k_{23} in the third reaction pertains to the motion of the neutral pion relative to the neutron, etc. The numbering of the charged mesons in the first reaction of (30) differs from that in [1].

The zero-energy photoproduction amplitudes λ_i have of course nothing in common with the amplitudes of the reactions $\pi + N \rightarrow N + \pi + \pi$. They can be expressed in terms of the matrix elements of the photoproduction in states with total isotopic spin $1/2$ and $3/2$, namely $G_{11}\exp(i\alpha_{11})$ and $G_{31}\exp(i\alpha_{31})$, with total angular momentum $- 1/2$:

$$\lambda_1 = \frac{1}{\sqrt{3}} G_{11}e^{i\alpha_{11}} - \frac{1}{\sqrt{15}} G_{31}e^{i\alpha_{31}},$$

$$\lambda_2 = -\frac{1}{\sqrt{3}} G_{11}e^{i\alpha_{11}} - \frac{2}{\sqrt{15}} G_{31}e^{i\alpha_{31}}, \qquad \lambda_3 = \sqrt{\frac{3}{10}}\, G_{31}e^{i\alpha_{31}},$$

from which we have in this case

$$\alpha_{12} = \frac{3 \sin(\alpha_{31} - \alpha_{11})}{y\sqrt{5} + 1/y\sqrt{5} - 2\cos(\alpha_{31} - \alpha_{11})},$$

$$\beta_{12} = \frac{2/y\sqrt{5} - y\sqrt{5} - \cos(\alpha_{31} - \alpha_{11})}{y\sqrt{5} + 1/y\sqrt{5} - 2\cos(\alpha_{31} - \alpha_{11})}. \qquad (31)$$

The phase shifts α_{11} and α_{31} can be related with other processes by means of the unitarity condition.[2] It is easy to note that (29) holds also for α_{12} with α_{13} or β_{12} with β_{13}.

Let us proceed now to examine the production of a pion in collisions between a π^+ meson and a proton. In this case two reactions are possible:

$$\pi^+ + p \rightarrow \pi^+ + \pi^+ + n, \qquad (32.1)$$

$$\pi^+ + p \rightarrow \pi^+ + \pi^0 + p. \qquad (32.2)$$

Both amplitudes in (32) are readily expressed for zero energy in terms of the matrix elements introduced above for the production of a pion in a state with total isotopic spin $3/2$ (the pion spin is 2). We have

$$\lambda_1 = \frac{2}{\sqrt{5}} e^{i\delta_{31}} F_{31}, \quad \lambda_2 = -\frac{1}{\sqrt{10}} e^{i\delta_{31}} F_{31}. \qquad (33)$$

Since these amplitudes have no relative phase shift, no terms linear in k_{il} arise in the expression for the cross section. For this reason the reactions (32) were not considered in [1,2]. On the other hand, the expression for the cross section depends only on the zero-energy amplitudes of the $\pi\pi$ and πN scattering. We write down the results in terms of the isotopic amplitudes a_2 and $b_{1/2}$, $b_{3/2}$:

$$|\langle \pi^+\pi^+ n | S | \pi^+ p\rangle|^2 = \tfrac{4}{5}|F_{31}|^2\{1 + \beta_1[k_{12}(k_{13} + k_{23}) + 2J_1(k_{12}) + J_1(k_{13}) + J_1(k_{23})] + \beta_2[k_{13}k_{23} + J_3(k_{13}) + J_3(k_{23})] + \beta_3[J_3(k_{13}) + J_3(k_{23})] + C_1k_{12}^2\}; \qquad (34.1)$$

$$\beta_1 = 2a_2(\tfrac{1}{6}b_{3/2} + \tfrac{5}{6}b_{1/2}), \qquad \beta_2 = 2(\tfrac{1}{6}b_{3/2} + \tfrac{5}{6}b_{1/2})^2,$$

$$\beta_3 = -\tfrac{5}{18}(b_{3/2} - b_{1/2})^2.$$

$$|\langle \pi^+\pi^0 p | S | \pi^+ p\rangle|^2 = \tfrac{1}{10}|F_{31}|^2\{1 + \gamma_1[k_{12}k_{13} + J_1(k_{12}) + J_1(k_{13})] + \gamma_2[k_{12}k_{23} + J_1(k_{12}) + J_1(k_{23})] + \gamma_3[k_{13}k_{23} + J_3(k_{13}) + J_3(k_{23})] + \gamma_4 J_3(k_{23}) + D_1k_{12}^2 + D_2k_{13}^2; \qquad (34.2)$$

$$\gamma_1 = 2a_2b_{3/2}, \qquad \gamma_2 = 2a_2(\tfrac{5}{3}b_{1/2} - \tfrac{2}{3}b_{3/2}),$$

$$\gamma_3 = 2b_{3/2}(\tfrac{5}{3}b_{1/2} - \tfrac{2}{3}b_{3/2}), \quad \gamma_4 = \tfrac{20}{9}(b_{3/2} - b_{1/2})^2.$$

The functions J_1 and J_3 are defined as before by (24), while C and D are unknown constants.

If we are interested in the energy dependence of the cross section, we must again add to the right halves of (34) a term proportional to E, with an undetermined coefficient, and make the substitution (25). Of the two $\pi\pi$ scattering amplitudes, of course, only a_2 enters in (34).

Finally, let us write out the expressions for the total cross sections of the reactions (18), neglecting terms of order $E^{1/2}$ and $E\ln(\mu/E)$ compared with unity. The terms proportional to E include already the undetermined constants and are not expressed in terms of scattering amplitudes. For the reactions occurring when a π^- meson collides with a proton we have

$$\sigma(\pi^+\pi^- p \mid \pi^- p) = \rho_1^2E^2(1 + A_1\sqrt{E} + B_1E\ln(\mu/E)),$$

$$\sigma(\pi^0\pi^0 n \mid \pi^- p) = \rho_2^2E^2(1 + A_2\sqrt{E} + B_2E\ln(\mu/E)),$$

$$\sigma(\pi^-\pi^0 p \mid \pi^- p) = \rho_3^2E^2(1 + A_3\sqrt{E} + B_3E\ln(\mu/E));$$

$$A_1 = \frac{32}{15\pi}\left(\alpha_{12}a_e + 2\sqrt{\frac{2M}{M+1}}\,\alpha_{13}b_e\right),$$

$$A_2 = \frac{32}{15\pi}\left(2\alpha_{21}a_e + 4\sqrt{\frac{2M}{M+1}}\,\alpha_{23}b_e\right),$$

$$A_3 = \frac{32}{15\pi}\left(2\alpha_{32}b_e + 2\sqrt{\frac{2M}{M+1}}\,\alpha_{31}b_e\right),$$

$$B_1 = \frac{4}{3\pi}\sqrt{\frac{M}{M+2}}\frac{M}{M+1}\,[2a_s b_s^0 + \beta_{12}a_e b_s^0 + \beta_{13}a_2 b_e]$$

$$\frac{8}{3\pi}\sqrt{\frac{M}{M+2}}\frac{1}{(M+1)^2}\,[b_s b_{3/2} + \tfrac{1}{2}\beta_{12}(b_e)^2 + \beta_{13}\, b_s^0 b_e],$$

$$B_2 = \frac{4}{3\pi}\sqrt{\frac{M}{M+2}}\frac{M}{M+1}\,[a_s^0 b_s^0 + 2\beta_{21}a_e b_s^0 + 2\beta_{23}a_2 b_e]$$

$$\frac{8}{3\pi}\sqrt{\frac{M}{M+2}}\frac{1}{(M+1)^2}\,[(b_s^0)^2 + \beta_{21}(b_e)^2 + 2\beta_{23}b_s^0 b_e],$$

$$B_3 = \frac{4}{3\pi}\sqrt{\frac{M}{M+2}}\frac{M}{M+1}\,[a_2(b_s^0 + b_s) + \beta_{31}a_2 b_e + \beta_{32}a_2 b_e]$$

$$\frac{8}{3\pi}\sqrt{\frac{M}{M+2}}\frac{1}{(M+1)^2}\,[b_s b_s^0 + \tfrac{1}{2}(b_e)^2 + \beta_{31}b_s^0 b_e + \beta_{32}b_s^0 b_e]. \quad (35)$$

In collisions between a π^+ meson and a proton the total cross sections have the form

$$\sigma(\pi^+\pi^+ n \mid \pi^+ p) = \tfrac{2}{5}|F_{31}|^2 E^2\left(1 + BE\ln\frac{\mu}{E}\right),$$

$$\sigma(\pi^+\pi^0 p \mid \pi^+ p) = \tfrac{1}{20}|F_{31}|^2 E^2\left(1 + B'E\ln\frac{\mu}{E}\right);$$

$$B = \frac{4}{3\pi}\sqrt{\frac{M}{M+2}}\frac{M}{M+1},\ [2a_2(\tfrac{1}{6}b_{3/2} + \tfrac{5}{6}b_{1/2})]$$

$$\frac{8}{3\pi}\sqrt{\frac{M}{M+2}}\frac{1}{(M+1)^2}\,[\tfrac{1}{9}(5b_{1/2}^2 + 5b_{1/2}b_{3/2} - b_{3/2}^2)],\quad B' = B. \quad (36)$$

It is easy to show that the equality of the coefficients B and B′ is a simple consequence of isotopic invariance.

In order to determine B we must differentiate the experimental data with respect to the total cross sections near threshold. Speaking more accurately, we must plot the derivative $d(\sigma/E^2)/dE$ on a logarithmic scale. Then the slope of the resultant line is equal to B. Naturally, data very close to the reaction threshold are necessary in order to satisfy the condition $\ln(\mu/E) \gg 1$. On the other hand, since this method of determining the amplitudes calls for knowledge of the total cross sections only, it is possibly easier to accumulate the statistics.

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