ESTIMATES OF THE EFFECTIVE INTERACTION RADIUS OF PARTICLES

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Various methods for estimating the effective strong-interaction radius are discussed: 1) the estimate from the degree of the last term in the expansion of the angular distribution in Legendre polynomials; 2) the estimate from a known total cross section for a channel and the value of the angular distribution at one point; 3) the estimate from the total cross section for all channels and that for the elastic-scattering channel; 4) the estimate from the mean square of the transverse momentum and the uncertainty relation. The first two of these are extended to the case of inelastic reactions of the most general type, a + b → c + d + e + . . . (the spins of the particles are arbitrary). All of the methods have no connection with any particular model of the interaction (potential well, optical model).

THE effective radius r_0 for the interaction between particles[1)] is an important qualitative characteristic of the strong interactions. It is a matter of interest to get information about the interaction radii of π mesons and nucleons at various energies, and also about their interaction radii with strange particles, and those of strange particles with each other. The purpose of this paper is to discuss several methods for estimating r_0 from experimental data, which are among the first to present themselves: estimates from the total and elastic cross sections and from the angular and momentum distributions (as they are at first known only in broad outline). These methods involve either determination of the quantity l_0 (cf. footnote [1)]) or the use of the uncertainty relation. Therefore they do not depend on any ideas about a model for the process. They can be regarded as a first step in a phase-shift analysis—namely, as the obtaining of evidence about the minimum number of phase shifts necessary for the representation of the available experimental data. It is worth emphasizing that any additional data can only increase the required number of phases, that is, can only increase r_0.

1. For the case of arbitrary spins of the particles and unpolarized beams a and b the angular distribution of a reaction of the type a + b → c + d must be of the following form [cf. e.g., [1], Eqs. (4.5) and (4.6); [2], Eq. (5.1)]:

$$\sigma(\vartheta) = \sum_{L=0}^{L_0} B_L P_L(\cos\vartheta). \quad (1)$$

Furthermore L_0 must be equal to the smaller of the two numbers $2l_0$, $2l_0'$,[2)] where $l_0 \approx 4_0 p/h$ and $l_0' \approx r_0' p'/h$ are the maximum orbital angular momenta of particles a, b and particles c, d,[3)] respectively.

On the other hand, suppose the experimental $\sigma(\vartheta)$ is given. Since it is measured at a finite number of points and with some error, the coefficients

$$B_L = \frac{2L+1}{2}\int_0^\pi P_L(\cos\vartheta)\,\sigma(\vartheta)\sin\vartheta\,d\vartheta \quad (2)$$

in the expansion of $\sigma(\vartheta)$ in Legendre polynomials must all vanish to within the limits of error, beginning with a certain degree L_e. $L_e \le L_0$, since as the experimental errors decrease L_e may increase. Thus L_e possibly gives too low a value for the smaller of the quantities $2l_0$ and $2l_0'$.

To prove the correctness of Eq. (1) for arbitrary transitions a + b → c + d + e + . . ., let us take the amplitude of such a transition expressed in terms of the elements of the R matrix of the

[1)]One can define r_0, for example, by the classical relation $r_0 p = hl_0$, where l_0 is the largest orbital angular momentum of the relative motion that is still of any importance.

[2)]l_0' can be defined as the maximum number of partial waves that are of importance for the channel c + d → a + b of the reaction of c and d (the inverse reaction with the corresponding energy of c and d).

[3)]If the reaction is one with a threshold, such as π + p → Λ + K, for example, then because obviously $p' < p$, L_0 will be equal to $2l_0$. By determining L_0 (see the further discussion) we will then get an estimate of the interaction radius r_0' of the particles Λ and K (and a somewhat too low estimate for r_0).

transition in a representation which contains the orbital angular momentum (in terms of "generalized phase shifts"). For the case $a + b \rightarrow c + d + e$ such an expression is written out, for example, in Sec. 3 of [3] [cf. Eq. (9)]. Let us integrate the square of the absolute value of this amplitude (proportional to the angular distribution of the reaction products) over all of the angular (and energy) variables of particles d, e, etc., so that there remains only the angle ϑ of the relative momentum of particle c in the c.m.s. (or the momentum of the set of particles (d, e, . . .) in the c.m.s.). Then let us sum or average the resulting expression over all the spin projections. The result is of the form (1), and it turns out that the summation over L is taken not up to twice the value of the maximum total angular momentum, but up to the smaller of the quantities $2l_0$, $2l_0'$. Furthermore $l_0' = r_0' p_{c\,max}/\hbar$, where r_0' is to be interpreted as the effective radius of the volume in which particle c and the set (d, e, . . .) have been produced and interact, and $p_{c\,max}$ is the maximum value of the absolute value of the momentum of particle c in the c.m.s. which is allowed by the law of conservation of energy. We can of course take for c any one of the products of the reaction. For example, for the reaction $\pi + p \rightarrow p + N\pi$ one can take $\sigma(\vartheta)$ to be the angular distribution of the nucleon.

Equation (1) is true for each channel of the reaction $a + b \rightarrow \ldots$, for example, for the channels $\pi^- + p \rightarrow p + \pi^-$, $\rightarrow n + \pi^0$, $\rightarrow p + \pi^- + \pi^0$, and so on. When we write out relations (1) for all possible channels and add them together, we find that the result is again of the form (1). Therefore we can take $\sigma(\vartheta)$ in Eq. (2) to mean the angular distribution of the nucleon without regard to the number of other particles (to say nothing of their directions of emission). In other words, we can get information about L_0 and then about l_0 or l_0' without necessarily singling out any particular channel of the reaction in question.

Thus the method now being stated consists of finding the index L_e of the last nonvanishing coefficient in the expansion of the angular distribution of some chosen one of the products of the reaction in terms of Legendre polynomials. Of course it is not necessary first to find all of the preceding coefficients. To determine the value of L for which it is worthwhile to begin the calculation of the integrals (2) one can use the cruder estimates of r_0 to be explained later, which are based on inequalities which give a lower limit on r_0.

2. Ogievetskii and Grishin [4] have given an inequality for reactions of the type $a + b \rightarrow 1 + 2$ which can be of use for the estimation of r_0 or r_0' in cases in which one knows the value $\sigma(\vartheta')$ of the angular distribution at one point ϑ' and the total cross section of the reaction $a + b \rightarrow 1 + 2$. We write it in a form which is valid for arbitrary spins:

$$d\sigma(\vartheta')/d\Omega \leqslant (\sigma/4\pi)\,\Sigma(\vartheta'), \tag{3}$$

where $\Sigma(\vartheta')$ means the one of the expressions

$$\sum_{J=0}^{J_0} (2J+1)\,[d^J_{m_2+m_1,\; m_a+m_b}(\vartheta')]^2; \qquad -j_a \leqslant m_a \leqslant +j_a, \tag{4}$$

and so on (j_a is the spin of particle a) that has the largest value. The function $d^J_{m,n}$ is defined in [5,6]. J_0 denotes the maximum value of the total angular momentum, which is the smaller of the numbers $l_0 + j_a + j_b$ and $l_0' + j_1 + j_2$. Since the spins of the particles are usually not larger than 0 or 1, J is nearly equal to l if l is large.

To derive Eq. (3) one must use the expression for the amplitude for the reaction which is given as Eq. (31) in a paper by Jacob and Wick. [5] By using the Cauchy inequality one can get the estimate (3) for the differential cross section $d\sigma(\vartheta)/d\Omega$ (which is proportional to the square of the absolute value of the amplitude), by an argument like that used in [4]. The corresponding starting expressions for the case $a + b \rightarrow 1 + 2 + \ldots + N$ are given in [3]. It turns out that Eq. (3) also holds for such reactions, with the following changes. $d\sigma(\vartheta')/d\Omega_1$ is the angular distribution of a chosen particle, for example particle 1—see the preceding section. Instead of m_2 one must use in Eq. (3) the projection M_2 of the "spin" of the complex (2 . . . N) of all the other particles. If there is no information about this "spin," one must assume that M_2 takes all values allowed by the condition $|M_2 + m_1| \leq J$ (for $|M_2 + m_1| > J$ the function d^J vanishes). Furthermore in the case $N > 2$ the value J_e of the upper limit of the summation in Eq. (4) which is required for Eq. (3) to be satisfied can give an estimate only for l_0, and not for any of the orbital angular momenta of the products of the reaction.

We note that the form of Eq. (3) is the same for all N. Therefore for an estimate of the radius of the interaction of a π meson and a proton, for example, one can use Eq. (3), meaning by σ the total cross section of all channels with a proton in the final state ($\pi + p \rightarrow p + \pi$, $\rightarrow p + 2\pi$, . . .) and by $d\sigma(\vartheta')/d\Omega_1$ the differential cross-section for protons at the angle ϑ' (paying no attention to the other particles). In fact, we can write out a relation (3) for each channel and add these relations. The resulting expression will be of the form (3) with the indicated meanings of σ and $d\sigma(\vartheta')/d\Omega_1$, if Σ_1 means the largest of all the partial Σ_1's.

As can be seen from Eq. (3), this inequality is actually useful only if $\sigma(\vartheta')$ differs decidedly from $\sigma/4\pi$. Whether or not this method for estimating l_0 has an advantage in simplicity as compared with the preceding method can be judged only in concrete cases.[4)]

Hsien Ting-ch'ang has obtained[8] an estimate of the form (for arbitrary N)

$$d\sigma(\vartheta')/d\Omega \leqslant \sigma/4\pi\,(J_0+1)^2, \tag{5}$$

which is obtained from Eqs. (3) and (4) by replacement of all the functions d^J by unity. $d^J_{m,n}(\vartheta) \le 1$, since these functions satisfy the unitarity condition

$$1=\sum_n d^J_{m,n}(d^{J+})_{nm}=\sum_n [d^J_{m,n}(\vartheta)]^2. \tag{6}$$

Although this estimate is much too low, on the other hand it does not require the calculation of sums of the form (4), and can be useful for a simple preliminary estimate.

3. Rarita and Schwed[9] have indicated an interesting method for estimating the interaction radius in the elastic reaction $a+b\to a+b$, which requires only a knowledge of the elastic cross section σ^{el} and the total cross section σ^{tot} for the process $a+b\to$ (all channels). As Ogievetskii and Grishin have pointed out,[4] this estimate follows from the formula (3) and the so-called optical theorem (cf., e.g., Sec. 24 of [10]). We shall give the derivation for the case of arbitrary spins.

For $\vartheta' = 0$ the right member of Eq. (3) goes over into $\sigma^{el}(J_0+1)^2/4\pi$, since $d^J_{m,n}(0)=\delta_{mn}$. For the left member we have

$$\frac{d\sigma(0^\circ)}{d\Omega}[2j_a+1)(2j_b+1)]\equiv\sum_{m'_a,m'_b,m_a,m_b}|\langle m'_am'_b|R(0^\circ)|m_am_b\rangle|^2$$
$$\geqslant\sum_{m_a,m_b}|\langle m_am_b|R(0^\circ)|m_am_b\rangle|^2$$
$$\geqslant\sum_{m_a,m_b}[\mathrm{Im}\,\langle m_am_b|R(0^\circ)|m_am_b\rangle]^2$$
$$=\sum_{m_a,m_b}(p_a/4\pi h)^2(\sigma^{tot}_{m_a,m_b})^2\geqslant(p_a/4\pi h)^2(\sigma^{tot})^2[(2j_a+1)$$
$$\times(2j_a+1)]. \tag{7}$$

Use has been made of the relation

$$\sum_{i=1}^N (a_i)^2 \geqslant \left(\sum_{i=1}^N a_i\right)^2 \Big/ N$$

and the equation

$$(4\pi h/p_a)\,\mathrm{Im}\,\langle m_am_b|R(0^\circ)|m_am_b\rangle=\sigma^{tot}_{m_am_b}, \tag{8}$$

which follows from Eq. (24.14) of [10]; $\sigma^{tot}_{m_am_b}$ denotes the total cross section for interaction of particles a and b with fixed spin projections m_a and m_b. We get finally

$$(p_a/4\pi h)^2(\sigma^{tot})^2\leqslant\frac{\sigma^{el}}{4\pi}(J_0+1)^2. \tag{9}$$

4. To estimate r_0 one can also use the uncertainty relation[5)] $\Delta p_x \Delta x \ge h/2$.

Suppose the process of change of state of the particles (or of the production of new particles) effectively occurs in a limited volume of the relative coordinates with the radius r_0. This means that before the particles became free particles of the final state their relative coordinate was fixed with the accuracy r_0. Then in particular the component of the relative momentum perpendicular to the incident beam must have an uncertainty $\Delta p^0_\perp$ of the order $h/2r_0$. (The uncertainty $\Delta p^0_\perp$ is easier to detect than the uncertainty in the component parallel to the beam.) The experimental mean square transverse momentum $(\Delta p^e_\perp)^2_{av}$ can only be larger than $(\Delta p^0_\perp)^2$, since part of the quantity $(\Delta p^e_\perp)^2_{av}$ may be due to the concrete dynamical interaction and not to the fact that the range is limited. For example, the Coulomb interaction does not have a finite radius, but does scatter at nonzero angles, so that $(\Delta p^e_\perp)^2_{av} > 0$.

From

$$h/2\leqslant r_0\Delta p^0_\perp\leqslant[(\Delta p^e_\perp)^2_{av}]^{1/2}r_0$$

we get as a (too low) estimate for r_0:

$$r_0\geqslant\frac{h}{2}[(\Delta p^e_\perp)^2_{av}]^{-1/2} \tag{10}$$

(compare this with the estimate $r_0 \approx hl_0/p$ from the preceding methods). For this estimate we need to know only $(\Delta p^e_\perp)^2_{av}$ for any one of the particles that are products of the reaction.

[4)]We note that the estimate $l_0 \sim 16$ for pp scattering at 8.5 BeV[4] from the data used by Grishin and Ogievetskii can also be made rather easily by using Eq. (2). In fact, for $\vartheta_0 = 0$ all of the Legendre polynomials are equal to 1, and they then decrease to zero in such a way that the first root is at the point $\vartheta_0 \approx 2.4 \times 57^\circ/L$. In particular, the first root of P_{32} is at at 4.3°. Since more than half of all the scattered particles are scattered at angles smaller than 4°,[7] the coefficient B_{32} must still be different from zero. Calculation of the integral (2) indeed shows that the value of B_{32} differs from zero by more than twice the error.

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[5)]In particular, this was pointed out by M. I. Podgoretskiĭ at a seminar of the High-Energy Laboratory of the Joint Institute for Nuclear Research. We take this occasion to thank him for a discussion of this question.

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