ON THE THEORY OF QUANTIZATION OF SPACE-TIME

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The hypothesis is proposed that the geometric structure of x-space "in the small" and correspondingly of p-space "in the large" is closely connected with weak interactions of elementary particles. Furthermore, a scheme is investigated in which momentum space is one of constant curvature [4] and x-space is quantized.[3] It is shown that there are reasons in the new geometry for rejecting the requirement of invariance under space inversion and "strong" time inversion, providing that the CPT theorem is correct.

1. INTRODUCTION

IN this work, as in the previous one,[1] we shall assume that the hypothetical "fundamental length," without whose introduction it is impossible to give a correct description of the physical processes taking place at high energies or, correspondingly, in small space-time scales, is simultaneously a constant which determines the intensity of the weak interaction.* This means that the geometrical structure of x-space "in the small" and of p-space "in the large" must be closely connected with the weak interactions of elementary particles,† and possibly we now have a right to expect an explanation of such unusual "geometric" properties of weak processes as the nonconservation of spatial parity‡ and non-invariance under "strong" time inversion. It is assumed in what follows that the new geometry in x- and p-space should not contradict the basic principles of quantum mechanics and the general theory of relativity, i.e., as before, the states of physical systems can be described by vectors in some Hilbert space, and the observed quantities can be put in correspondence with the operators of this space; also, the invariance under Lorentz transformations is preserved.

In the mathematical methods used, the given research borders on the research of Snyder [3] and Gol'fand,[4] since the constructions carried out below are equivalent to the introduction in p-space of a geometry of space of constant curvature. Our goal is a more detailed exposition of the mathematical problems related thereto.

*In[1], l is chosen equal to the "β-decay length" $\sqrt{G/\hbar c}$ $= 6 \times 10^{-17}$ cm (G is the Fermi constant).

†Perhaps in the spirit of the general theory of relativity (see[2]).

‡The possibility of nonconservation of parity in weak interactions on the basis of the new representation of the structure of space in scales of the order of $\sqrt{G/\hbar c}$ was first pointed out by Shapiro.[2]

2. A NEW GEOMETRY OF p-SPACE

As is clear from what was said above, we shall call p-space the four-dimensional momentum space which figures in quantum field theory. In the usual theory, this space is pseudo-Euclidean, and therefore possess a 10-parameter group of motions (L_{10}), consisting of the group of Lorentz transformations (rotations) (L_6), translations (T_4) and reflections. In the new geometry, definite limitations must be placed on the components of these 4-momenta p_m ($m = 0, 1, 2, 3$) when they are near $1/l$.* By analogy with the theory of relativity, we assume that these limitations can be written in the form of some inequalities that connect the components p_m and $1/l$. The simplest relativistically invariant inequalities of the required type (with account of a reasonable limit as $l \to 0$) have the form

$$p^2 \equiv p_0^2 - p_1^2 - p_2^2 - p_3^2 \leqslant 1/l^2, \qquad (2.1)$$

$$p^2 \geqslant -1/l^2. \qquad (2.2)$$

We now construct the geometry of the 4-space in which all the vectors satisfy either (2.1) or (2.2).† It will be convenient to consider the two cases together if the hypersurfaces $p^2 = l^{-2}$ and $p^2 = -l^{-2}$,

*The system of units is used in which $\hbar = c = 1$.

†Naturally, it does not then follow that all 4-momenta in the theory (on the basis of which this geometry will be constructed) will obey one of these inequalities. For example, the limitation $v \leq c$ does not extend to the magnitude of the phase velocity in the theory of relativity.

corresponding to (2.1) and (2.2), which limit the admissible values of the squares of the 4-momenta, can be described by the one equation

$$p^2 = \varepsilon/l^2, \tag{2.3}$$

where $\epsilon = \pm 1$.

In the presence of a limiting hypersurface (2.3), the translation transformations T_4 in the group of motions L_{10} should be replaced by some new transformations $\tilde{T}_4$, which transform this hypersurface into itself; the remaining transformations of the group L_{10} (rotation and reflection) obviously transform without change into the new group of motions ($\tilde{L}_{10}$), since they leave (2.3) invariant.

Thus the new group of motions $\tilde{L}_{10}$ can be defined as the set of transformations which leave the hypersurface (2.3) unmoved. In the usual p-space, this region was infinitely far removed from the region which is fixed relative to the group of motions. Therefore, it is natural to consider the points of the hypersurface (2.3) as infinitely distant points of p-space in the sense of the new geometry. Infinitely distant points in the previous sense of this word [except for those which belong to (2.3)] are now seen to be not at all isolated, and therefore must be regarded in a fashion completely equivalent to the rest. The so-called homogeneous coordinates [5] are most useful for this purpose. In our case, we shall introduce them in the following form:

$$p_m = l^{-1}\eta_m/\eta_4 \qquad (m = 0,\ 1,\ 2,\ 3). \tag{2.4}$$

To each choice of coordinates (p_0, p_1, p_2, p_3) there corresponds a choice ($\rho\eta_0$, $\rho\eta_1$, $\rho\eta_2$, $\rho\eta_3$, $\rho\eta_4$), where ρ is a non-vanishing factor [and therefore the set of values (0, 0, 0, 0, 0) is excluded]. It is clear that the choice ($\rho\eta_0$, $\rho\eta_1$, $\rho\eta_2$, $\rho\eta_3$, 0) corresponds to an infinitely distant (in the foregoing sense) point in p-space. In homogeneous coordinates, (2.3) takes the form

$$\eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \varepsilon\eta_4^2 = 0 \tag{2.5}$$

or $g^{\mu\nu}\eta_\mu\eta_\nu = 0$, where $\mu\nu = 0, 1, 2, 3, 4$ and $g^{\mu\nu} = 0$ for $\mu \neq 0$, $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$, $g^{44} = -\epsilon$. Equation (2.5) remains invariant for all linear orthogonal transformations of the form $\eta'_\mu = a_\mu^\nu \eta_\nu$. These transformations form a group (G_{10}) of the hypersphere of the pseudo-Euclidean 5-space with the variables η_μ. As follows from (2.4), the group $\tilde{L}_{10}$ is isomorphic to the factor-group G_{10} over its subgroup, which consists of two transformations: $\eta_\mu \to \eta_\mu$ and $\eta_\mu \to -\eta_\mu$ ($\mu = 0, 1, 2, 3, 4$). This circumstance makes it possible to consider the five-dimensional hyperspace, whose diametrically opposite points are identical, as a model of our p-space.[5,6] Thus the p-space now represents a space of constant curvature.

The transformation group G_{10} decomposes into four related components, which are distinguished by the following marks:[7]

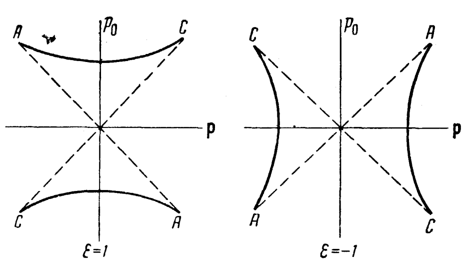
For $\epsilon = +1$:

$$\begin{aligned} &\text{a) } \det a_\mu^\nu = 1, && \partial\eta_0'/\partial\eta_0 > 0;\\ &\text{b) } \det a_\mu^\nu = 1, && \partial\eta_0'/\partial\eta_0 < 0;\\ &\text{c) } \det a_\mu^\nu = -1, && \partial\eta_0'/\partial\eta_0 > 0;\\ &\text{d) } \det a_\mu^\nu = -1, && \partial\eta_0'/\partial\eta_0 < 0. \end{aligned} \tag{2.6}$$

For $\epsilon = -1$:

$$\begin{aligned} &\text{a) } \det a_\mu^\nu = 1, && \partial(\eta_0', \eta_4')/\partial(\eta_0, \eta_4) > 0;\\ &\text{b) } \det a_\mu^\nu = 1, && \partial(\eta_0', \eta_4')/\partial(\eta_0, \eta_4) < 0;\\ &\text{c) } \det a_\mu^\nu = -1, && \partial(\eta_0', \eta_4')/\partial(\eta_0, \eta_4) > 0;\\ &\text{d) } \det a_\mu^\nu = -1, && \partial(\eta_0', \eta_4')/\partial(\eta_0, \eta_4) < 0. \end{aligned} \tag{2.7}$$

Because of the correspondence between the groups G_{10} and $\tilde{L}_{10}$ pointed out above, one need use only the components a) and b) from (2.6) and (2.7) for the description of $\tilde{L}_{10}$; the components c) and d) need not be considered at all. All the transformations b) can be obtained from the transformations a) if we multiply the latter by special transformations of the form $\eta_0 \to -\eta_0$, $\eta_4 \to -\eta_4$ in the case $\epsilon = 1$, and $\eta_\alpha \to -\eta_\alpha$, $\eta_4 \to -\eta_4$ in the case $\epsilon = -1$ ($\alpha = 1, 2, 3$). By virtue of (2.4), the first of these two transformations leads to the inversion of 3-space $\mathbf{p} \to -\mathbf{p}$, the second to the inversion of the time axis $p_0 \to -p_0$. The Lorentz rotations are contained in the components a), corresponding to rotations in the planes ($\eta_m\eta_n$) about the η_4 axis, and rotations in the planes ($\eta_4\eta_m$) which should obviously be identified with the transformations $\tilde{T}_4$, wherein m is the index of that axis of p-space in whose direction the "displacement" is carried out.* Rotation in the plane ($\eta_4\eta_0$) will be hyperbolic for $\epsilon = 1$ and elliptic for $\epsilon = -1$; on the other hand, rotations in



*It then follows that the transformations $\tilde{T}_4$ do not form a group.

the planes (η_4, η_α) will be elliptic for $\epsilon = 1$ and hyperbolic for $\epsilon = -1$. The product of three rotations of angle π in the planes (η_4, η_α) leads, for $\epsilon = 1$, to inversion of the axes η_4, η_α, i.e., in accord with (2.4), to the transformation $p_0 \to -p_0$. Similarly, rotation through π in the plane (η_0, η_4), for $\epsilon - 1$, is a reflection of the axes η_0 and η_4, which is equivalent to $\mathbf{p} \to -\mathbf{p}$.

Thus, with the aid of the translation $\widetilde{T}_4$, one can transform (in continuous fashion) a right-hand four-dimensional coordinate system into a left-hand system, and conversely. Inasmuch as rotation in the plane (η_4, η_0) corresponds to a "displacement" along the p_0 axis, then, in the case $\epsilon = -1$, it is necessary to regard the p-space as closed on itself in the direction of the p_0 axis, and topologically equivalent to a four-dimensional Möbius sheet. The p-space has a similar structure for $\epsilon = 1$, only with the obvious difference that here it is closed on itself in the space-like direction (see the drawing; A and C are identical points).

We now find the explicit form of the transformation of the vector p_m by an arbitrary vector k_m. By virtue of (2.4), the desired transformation will be bilinear in the components p_m. Therefore, taking the requirements of relativistic covariance into account, we can write

$$p'_m = \frac{f_1(k^2)\, p_m + f_2(k^2)\,(kp)\, k_m + f_3(k^2)\, k_m}{f_4(k^2)\,(kp) + 1}; \qquad (2.8)$$

$k^2 = (kk)$, $(kp) = k_0 p_0 - \mathbf{k}\cdot\mathbf{p}$. The unknown functions $f_1, \ldots, f_4$ are uniquely determined from the conditions:

$$\begin{aligned} &1)\ p'^2 = \varepsilon l^{-2}\ \text{for}\ p^2 = \varepsilon l^{-2},\\ &2)\ p' = 0,\ \text{for}\ p = -k,\\ &3)\ p' = p\ \text{for}\ k = 0. \end{aligned}$$

In sum, Eq. (2.8) takes the form*

$$p'_m = p_m(+)k_m = \frac{p_m\sqrt{1-\varepsilon k^2 l^2} + k_m\left(1 + \varepsilon (pk)\, l^2 / [1 + \sqrt{1-\varepsilon k^2 l^2}]\right)}{1 + \varepsilon (pk)\, l^2} \qquad (2.9)$$

The symbol $(-)$ is introduced, in accord with[4], in the following natural fashion: $p(-)k = p(+)(-k)$. Here it is shown that $(p(+)k)(-)k = p$. In contrast with the usual translation transformation, the operations $(\pm)$ do not commute. We can establish the fact that

$$\varepsilon (p(\pm)k)^2 l^2 = 1 - \frac{(1-\varepsilon p^2 l^2)(1-\varepsilon k^2 l^2)}{(1 \pm \varepsilon (pk)\, l^2)^2}. \qquad (2.10)$$

The commutability of the operations $(\pm)$ for collinear 4-vectors is evident from (2.10). For example, if $p = p_0$, $k = k_0$, then, for $\epsilon = 1$,

$$p_0(+)\,k_0 = (p_0 + k_0)/(1 + p_0 k_0 l^2). \qquad (2.11)$$

As $l \to 0$, Eqs. (2.9)—(2.11) transform into the corresponding expressions of ordinary geometry.

*We have used the notation of[4] for the operation $\widetilde{T}_4$.

3. QUANTIZATION OF SPACE-TIME

In accord with the supposition made above on the transfer of the principles of quantum mechanics over into the new geometry, we preserve the previous interpretation of infinitesimal operators of the group of motions. Then the transformation of the wave functions for translation by a small vector k_m in p-space has the form

$$\varphi(p(+)k) = (1 - i x^m k_m)\,\varphi(p), \qquad (3.1)$$

where the x^m are by definition the operators of the coordinates and time (summation is carried out over the index m from 0 to 3. With account of (2.9), we get the following as operators of the scalar wave functions $\varphi(p)$:

$$\begin{aligned} x^\alpha &= i\,(\partial/\partial p_\alpha + \varepsilon l^2 p_\alpha p_m \partial/\partial p_m), \qquad \alpha = 1, 2, 3;\\ t &= i\,(\partial/\partial p_0 - \varepsilon l^2 p_0 p_m \partial/\partial p_m). \end{aligned} \qquad (3.2)$$

For $\epsilon = 1$, (3.2) are the operators of the coordinates and time, considered in the Snyder theory.[3] We introduce the variables η_μ into (3.2). Since, because of (2.4),

$$\eta_4 \frac{\partial}{\partial \eta_n} = \frac{1}{l}\frac{\partial}{\partial p_n}, \qquad \frac{\partial}{\partial \eta_4} = -\frac{1}{\eta_4} p_m \frac{\partial}{\partial p_m},$$

then

$$x^\alpha = il\left(\eta_4 \frac{\partial}{\partial \eta_\alpha} - \varepsilon \eta_\alpha \frac{\partial}{\partial \eta_4}\right), \qquad t = il\left(\eta_4 \frac{\partial}{\partial \eta_0} + \varepsilon \eta_0 \frac{\partial}{\partial \eta_4}\right). \qquad (3.3)$$

These operators were first obtained in such a form (for $\epsilon = 1$) by Snyder.*

With the help of (3.3), it is easy to establish that when $\epsilon = 1$ the spatial coordinates have a discrete spectrum of eigenvalues of the form nl, where n is an integer, since the spectrum of the time coordinate remains continuous; for $\epsilon = -1$, the situation is reversed: the time is discrete while the spectrum of the operators x is continuous. The commutation relations between the operators of coordinates, momenta, moments and other features of the Snyder theory are changed in trivial fashion for $\epsilon = -1$, and we shall therefore not consider them. We only note that our formalism is free of the arbitrariness in the determination of the 4-momentum, pointed out by Snyder, for any ϵ, since the quantities $p_m f(\eta_4/\eta)$ are incorrectly transformed in the translations of (2.9).

*It is obvious that apart from a numerical factor Eqs. (3.3) for x^m can be found immediately from five-dimensional consideration of the translations T_4.

4. SPINOR REPRESENTATION OF THE GROUP OF MOTIONS OF p-SPACE

According to Sec. 2, the group L_{10} is isomorphic to the group of five-dimensional transformations $\eta'_\mu = a_\mu^\nu \eta_\nu$ for which $\det a_\mu^\nu = 1$ [components a) and b) in (2.6) and (2.7)]. Therefore, the spinor representation of this latter group will be the sought representation of the group $\tilde{L}_{10}$. The matrix of the spinor transformation S corresponding to an arbitrary motion $\eta'_\mu = a_\mu^\nu \eta_\nu$ of the components a) and b) can be found from the condition

$$S^{-1}\Gamma^\lambda S = a_\mu^\lambda \Gamma^\mu, \tag{4.1}$$

where Γ^μ are five four-row matrices satisfying the following anticommutation relations:

$$\Gamma^\mu\Gamma^\lambda + \Gamma^\lambda\Gamma^\mu = 2g^{\mu\lambda} \tag{4.2}$$

[$g^{\mu\lambda}$ is the metric tensor from (2.5)].

We choose for Γ^μ the matrices $\gamma^m\gamma^5$, $\sqrt{\epsilon}\gamma^5$ (m = 0, 1, 2, 3)* where

$$\gamma^0 = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}, \quad \gamma^\alpha = \begin{pmatrix} 0 & \sigma_\alpha \\ -\sigma_\alpha & 0 \end{pmatrix}, \quad \gamma^5 = -i\begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}. \tag{4.3}$$

Here $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, σ_α are the Pauli matrices (α = 1, 2, 3).

Let us introduce the spin tensor of second rank $H(\eta) = \eta_\mu\Gamma^\mu$, which transforms as

$$H' = SHS^{-1}. \tag{4.4}$$

It is easy to establish that $\det H = (\eta^2)^2 = \det H' = (\eta'^2)^2$, i.e., the matrices S actually form a representation of the given group. We can investigate not only linear but also antilinear spinor transformations with the example of the spin tensor H, owing to the simplicity of its structure. Initially, we shall find all the matrices S corresponding to reflections of the η_μ axes. Since $\det a_\mu^\nu = 1$, then the number of reflected axes will always be even.† From the form of H, it is easy to prove [7] that the matrix $S = \lambda\Gamma^{i_1}\Gamma^{i_2}\ldots\Gamma^{i_{2k}}$, where i_m is the number of the coordinate axes and λ is the phase factor, corresponds to the axes $\eta_{i_1}, \eta_{i_2}, \ldots, \eta_{i_{2k}}$.

The matrices obtained in this fashion are shown in Table I. As phase factors in the case of matrices of the time inversion for the case $\epsilon = 1$, and of space inversion for $\epsilon = -1$, imaginary units were chosen so that the square of these transformations are rotations through 2π (see Sec. 2). The remaining phase factors in the first two rows of the table are omitted for simplicity; in the third row, i again appears, since the latter transformation is a product of the first two.

Table I

	$\varepsilon = 1$	$\varepsilon = -1$
$\eta_0 \to -\eta_0$ $\eta_4 \to -\eta_4$	γ^0	$i\gamma^0$
$\eta_\alpha \to -\eta_\alpha$ $\eta_4 \to -\eta_4$	$i\gamma^0\gamma^5$	$\gamma^0\gamma^5$
$\eta_m \to -\eta_m$	$i\gamma^5$	$i\gamma^5$

Now let the spinor of first rank ψ be given, transformed according to the law $\psi' = S\psi$ for motions from $\tilde{L}_{10}$. We shall determine how the covariant quadratic forms of the type $\psi^+ O\psi$ can be constructed from ψ and ψ^+ (the "plus" indicates the Hermitian conjugate). We shall first prove that for $\epsilon = 1$,

$$\begin{aligned} \Gamma^0 S^+ \Gamma^0 &= S^{-1} \quad \text{for the transformations a),} \\ \Gamma^0 S^+ \Gamma^0 &= -S^{-1} \quad \text{for the transformations b);} \end{aligned} \tag{4.5}$$

and for $\epsilon = -1$,

$$\begin{aligned} (i\Gamma^0\Gamma^4)\, S^+\, (i\Gamma^0\Gamma^4) &= S^{-1} \quad \text{for the transformations a)} \\ (i\Gamma^0\Gamma^4)\, S^+\, (i\Gamma^0\Gamma^4) &= -S^{-1} \quad \text{for the transformations b).} \end{aligned} \tag{4.6}$$

The proofs of the relations (4.5) and (4.6) are entirely similar, and we shall therefore carry out discussions only for the case $\epsilon = -1$. Inasmuch as $H' = SHS^{-1}$, we get $H^+ = (S^{-1})^+ H^+ S^+$. But it follows from (4.3) and Table I that

$$H^+(\eta) = -H(-\eta_0, \eta_\alpha, -\eta_4)$$

$$= -(\Gamma^0\Gamma^4)\, H\, (\Gamma^0\Gamma^4)^{-1} = -(i\Gamma^0\Gamma^4)\, H\, (i\Gamma^0\Gamma^4).$$

Therefore,

$$H'^+ = -(i\Gamma^0\Gamma^4)\, H'\, (i\Gamma^0\Gamma^4) = -(S^{-1})^+ (i\Gamma^0\Gamma^4)\, H\, (i\Gamma^0\Gamma^4)\, S^+,$$

or,

$$H' = (i\Gamma^0\Gamma^4)(S^{-1})^+ (i\Gamma^0\Gamma^4)\, H\, (i\Gamma^0\Gamma^4)\, S^+ (i\Gamma^0\Gamma^4).$$

Thus, $(i\Gamma^0\Gamma^4)S^+(i\Gamma^0\Gamma^4) = \alpha S^{-1}$, where α is a numerical factor. The matrices S, corresponding to the transformations a), can be reduced by a continuous change in the group parameters to the unitary matrix E, as a result of which we find $(i\Gamma^0\Gamma^4)E(i\Gamma^0\Gamma^4) = \alpha E$, i.e., $\alpha = 1$. If now S corresponds to the component b), then such matrices can be reduced to a transformation of time inversion, whence $\alpha = -1$.

*Yu. A. Gol'fand pointed out the necessity of the choice of the basis Γ^μ in such a form. In this case, as $l \to 0$, the five-dimensional spinor representation considered transforms into the usual four-dimensional spinor representation in the basis γ^m.

†The matrix S corresponding to transformations with $\det a_\mu^\nu = -1$, for example, $a_\mu^\nu = -\delta_\mu$, does not generally exist, since it is impossible to find six anticommuting four-row matrices.

Instead of the spinor ψ^+, it is convenient to consider the quantity $\bar{\psi} = \psi^+ L^0$, where $L^0 = \Gamma^0$ for $\epsilon = 1$ and $L^0 = i\Gamma^0\Gamma^4$ for $\epsilon = -1$. Obviously, for transformations of the type a), the law for the transformation of $\bar{\psi}$ will be the following: $\bar{\psi}' = \bar{\psi}S^{-1}$, and for the transformation b): $\bar{\psi}' = -\bar{\psi}S^{-1}$. As can easily be seen, only three independent quadratic forms can be constructed from the quantities ψ and $\bar{\psi}$, which are covariant under the transformations a):

$$1)\ \text{scalar}\ \bar{\psi}\psi, \quad 2)\ \text{vector}\ \bar{\psi}\Gamma^\mu\psi, \quad 3)\ \text{tensor}\ \bar{\psi}\Gamma^\mu\Gamma^\nu\psi. \tag{4.7}$$

If we introduce the usual definition of a conjugate spinor, i.e., if we set $\bar{\psi} = \psi^+\gamma^0$, then the quantities 1)—3) from (4.7) are written (for $\epsilon = 1$) as $\bar{\psi}\gamma^5\psi$, $\bar{\psi}\gamma^5\Gamma^\mu\psi$, $\bar{\psi}\gamma^5\Gamma^\mu\Gamma^\nu\psi$, respectively, and for $\epsilon = -1$, they keep their own form. For Lorentz rotations (rotations about the η_4 axis), five components of the vectors ($\bar{\psi}\psi$ for $\epsilon = 1$ and $\bar{\psi}\gamma^5\psi$ for $\epsilon = -1$) are transformed independently of the first four components ($\bar{\psi}\gamma^m\psi$ for $\epsilon = 1$ and $\bar{\psi}\gamma^m\gamma^5\psi$ for $\epsilon = -1$), i.e., the 5-vector decomposes into an ordinary 4-vector (pseudovector) and a scalar (pseudoscalar). Similarly, the 5-tensor decomposes for Lorentz transformations into an ordinary 4-tensor and a 4-vector for $\epsilon = -1$, or a 4-pseudovector for $\epsilon = 1$.

Concluding the consideration of linear transformations of the spinors of p-space, we shall give (without derivation) the expressions for the operators x^m in the case when the wave function is a spinor:*

$$x^\alpha = i\,(\partial/\partial p_\alpha + \varepsilon l^2 p_\alpha p_m \partial/\partial p_m) + \tfrac{1}{2}\varepsilon\sqrt{-\varepsilon}\, l\gamma^\alpha,$$

$$t = i\,(\partial/\partial p_0 - \varepsilon l^2 p_0 p_m \partial/\partial p_m) - \tfrac{1}{2}\varepsilon\sqrt{-\varepsilon}\, l\gamma^0. \tag{4.8}$$

In order to find all the antilinear spinor transformations, it is necessary to determine such a matrix K that, for an arbitrary transformation $\psi' = S\psi$ the following relation holds:

$$(K\psi^*)' = SK\psi^*. \tag{4.9}$$

It then follows from the equality $\psi'^* = S^*\psi^*$ and (4.9) that

$$K^{-1}SK = S^*. \tag{4.10}$$

With the aid of (4.4), we find

$$H^{*\prime} = S^*H^*(S^{-1})^*. \tag{4.11}$$

*The reason for the appearance of the matrices γ^m in (4.8) is clear if we take into consideration the obvious circumstance that the infinitesimal operators of the spinor representation group G_{10} are proportional to the expressions $(\Gamma^\mu\Gamma^\nu - \Gamma^\nu\Gamma^\mu)$.

Since the matrices Γ^0, Γ^1, Γ^3, Γ^4 for $\epsilon = 1$ and the matrices Γ^0, Γ^1, Γ^3 for $\epsilon = -1$ are pure imaginaries, the operation of complex conjugation, applied to H, means, in the first case, the reflection of the axes η_0, η_1, η_3, η_4, and, in the second, the reflection of $\eta_0, \eta_1, \ldots, \eta_3$. That is, one can write that for $\epsilon = 1$

$$H^* = (\Gamma^0\Gamma^1\Gamma^3\Gamma^4)\, H\, (\Gamma^0\Gamma^1\Gamma^3\Gamma^4) = \Gamma^2 H\, (\Gamma^2)^{-1}, \tag{4.12}$$

and for $\epsilon = -1$

$$H^* = -(\Gamma^2\Gamma^4)\, H\, (\Gamma^2\Gamma^4)^{-1}.$$

Substituting (4.12) in (4.11), we shall have, for $\epsilon = 1$,

$$\Gamma^2 H'(\Gamma^2)^{-1} = S^*\Gamma^2 H(\Gamma^2)^{-1}(S^{-1})^*; \tag{4.13}$$

and for $\epsilon = -1$,

$$(\Gamma^2\Gamma^4)\, H'(\Gamma^2\Gamma^4)^{-1} = S^*\Gamma^2\Gamma^4 H(\Gamma^2\Gamma^4)^{-1}(S^{-1})^*. \tag{4.14}$$

Comparing (4.13) with (4.14), we finally obtain

$$K = (\Gamma^2)^{-1} = \gamma^2\gamma^5 \quad \text{for}\ \varepsilon = 1,$$
$$K = (\Gamma^2\Gamma^4)^{-1} = i\gamma^2 \quad \text{for}\ \varepsilon = -1. \tag{4.15}$$

In the case $\epsilon = -1$, the transformation $\psi' = K\psi^+(p) = K\gamma^0\bar{\psi}(p) = -iC\bar{\psi}(p)$, where $C = \gamma^0\gamma^z$, is identical with the ordinary operation of charge conjugation.

For $\epsilon = 1$, we have, correspondingly,

$$\psi' = \gamma^2\gamma^5\gamma^0\bar{\psi} = \gamma^5 C\bar{\psi}(p). \tag{4.16}$$

That is, one can show that in this variant the role of charge conjugation should be filled by the new transformation (4.16).

We now write out (see Table II) all the antilinear spinor transformations for reflections of the coordinate axes ($\epsilon = \pm 1$; the phase factors in the operations of charge conjugation are omitted).

To complete consideration in this section of the group of motions of p-space, we shall show on what basis, within the framework of the new geometry, one could remove the requirement of invariance of the theory under space inversion and "strong" time inversion (see Sec. 1). The discussions which we shall give apply fundamentally to transformations of a displacement in p-space. In

Table II

	$\varepsilon = 1$	$\varepsilon = -1$
$\eta_0 \to -\eta_0$ $\eta_4 \to -\eta_4$	$\psi' = \gamma^0\gamma^5 C\bar{\psi}(-\mathbf{p})$	$\psi' = i\gamma^0 C\bar{\psi}(-\mathbf{p})$
$\eta_\alpha \to -\eta_\alpha$ $\eta_4 \to -\eta_4$	$\psi' = -i\gamma^0 C\bar{\psi}(-p_0)$	$\psi' = \gamma^0\gamma^5 C\bar{\psi}(-p_0)$
$\eta_m \to -\eta_m$	$\psi' = -iC\bar{\psi}(-p)$	$\psi' = i\gamma^5 C\bar{\psi}(-p)$

the ordinary theory, these transformations are not connected with any physical symmetry, and therefore their invariance is not required. It is natural to assume that even in the given scheme, there should not be requirements of invariance under displacement in momentum space. Then, in accord with what was pointed out above, the theory will not be invariant relative to the transformations* $\psi' = i\gamma^0\gamma^5\psi(-p_0)$ in the case $\epsilon = 1$ ("strong" time inversion) and $\psi' = i\gamma^0\psi(-\mathbf{p})$ in the case $\epsilon = -1$ (space inversion). If the CPT theorem remains valid, i.e., if invariance under the transformation $\psi' = i\gamma^5\psi(-p)$ is preserved, then in the case $\epsilon = 1$ this will mean the absence of invariance under space inversion and, correspondingly, for $\epsilon = -1$, invariance under "strong" time inversion.

5. TRANSFORMATION OF TRANSLATION IN x-SPACE AND THE ADDITION OF MOMENTA

In the ordinary theory of commuting operators $x^m = i\partial/\partial p_m$, the common eigenfunctions have the form $e^{-i(pa)}$ where a^m are the eigenvalues. If one gives the quantities a^m the meaning of parameters of the translation group, i.e., if we assume that $x'^m = x^m + a^m$, then, as is well known, one can regard the corresponding set of exponentials $e^{-i(pa)}$ as a representation of this group realized by the wave functions $\Psi(p)$:

$$\Psi'(p) = e^{-i(pa)}\Psi(p). \tag{5.1}$$

Any other arbitrary wave function $\Phi(q)$ will be similarly transformed in a displacement by a^m:

$$\Phi'(q) = e^{-i(qa)}\Phi(q). \tag{5.2}$$

If the systems described by $\Psi(p)$ and $\Phi(p)$ are regarded as non-interacting, and as a single integral system, then the product $X = \Psi(p)\Phi(q)$ will be the wave function of the compound system that is obtained.† It follows from (5.1) and (5.2) that

$$X' = e^{-i(p+q,\, a)}X, \tag{5.3}$$

i.e., the four-momentum of the compound system is a vector with components $p_m + q_m$.

The operators x^m from (3.2) do not commute with one another and therefore do not have a common set of eigenfunctions. It then follows that in quantized space-time it is not possible to carry out a translation by an arbitrary 4-vector a^m.[3] But displacements in the direction of any one of the axes are possible, since each of the operators x^m separately has eigenfunctions. For example, for the operator t (at $\epsilon = 1$), the eigenfunction is $\exp(ia\tanh^{-1}p_0 l)$, where a is the characteristic number (see [3]). If $\Psi(p)$ is the wave function, then for $t' = t + a$, we have*

$$\Psi' = \exp(-ia\,\mathrm{Arth}\,p_0 l)\,\Psi(p).$$

Similarly, we can write for (5.2) and (5.3)

$$\Phi' = \exp(-ia\,\mathrm{Arth}\,q_0 l)\,\Phi(q);$$

$$X' = \exp\left[-ia\,\mathrm{Arth}\frac{(p_0+q_0)\,l}{1+p_0q_0l^2}\right]X,$$

where $X = \Psi\Phi$. Thus, in the compound system, $p_0(+)q_0$ appears as an invariant quantity under time translations [see (2.11)]. It can therefore be assumed that the four-dimensional "sum" $p(+)q$ [see (2.9)] is in correspondence with the sum $p + q$ which figures in (5.3), i.e., one can regard $p(+)q$ as the analogue of the total 4-momentum of the system. The noncommutability of the components in this sum and the non-singlevaluedness of its determination (which follows therefrom) correspond to the impossibility of an arbitrary transformation in quantized x-space pointed out above. We shall also show that, although the components of the 4-momentum p_m are not now infinitesimal operators of the displacements in x-space, the permutation relations among them remain as before: $[p_m, p_n] = 0$. Together with the relations of the structure group of Lorentz rotations, they form the usual system of permutation relations of the inhomogeneous Lorentz group (see, for example, [8]).

CONCLUDING REMARKS

As was shown in Sec. 2, after the introduction of the new group of motions L_{10}, p-space can be regarded as a space of constant curvature. In this case, it is shown that the distance between two of its points $(p_0, \mathbf{p})$ and $(q_0, \mathbf{q})$ is determined by the formula[5]

$$\rho(p, q) = l^{-1}\,\mathrm{Arth}\,l\sqrt{\varepsilon\,(p\,(-)\,q)^2}. \tag{6.1}$$

The idea of the replacement of the ordinary pseudo-Euclidean space of momenta by a four dimensional space of constant curvature was recently expressed by Gol'fand.[4] For a description of the processes of interaction, Gol'fand proposed a Feynman diagram technique, generalized in the manner of the new geometry, while for writing down the laws of conservation of energy-momentum, use is made of the addition rule (2.9) (in [4] the case $\epsilon = 1$ was considered). Also, as in the theory of Snyder, invariant integration was introduced in p-space with the volume element $d\Omega = \sqrt{g}\,d_p^4$, where g is the determinant of the metric tensor. For arbitrary ϵ,

*We write out these transformations here in spinor form.

†For simplicity, it is assumed that p and q are the only dynamic variables in the states Ψ and Φ.

*$\mathrm{Arth} = \tanh^{-1}$.

$$\sqrt{g} = (1 - \varepsilon p^2 l^2)^{-5/2}. \qquad (6.2)$$

The method of taking integrals in curved p-space[4] leads to a finite result only for $\epsilon = 1$. In connection with going out into the region $p^2 \geq l^{-2}$, which is accomplished in this case, we note the following. The motion group $\widetilde{L}_{10}$ which we obtained by starting from the inequalities (2.1) and (2.2), transforms into itself not only the external, but also the internal, region, relative to the hypersurface (2.3).* Therefore, the new geometry also appears in the internal region.[5] All the necessary formulas can easily be obtained from those set down above. It is then clear that going beyond the limit of the inequalities (2.1) and (2.2) has at least a geometric meaning. The physical meaning of this operation can be revealed only in the process of the further development of the theory.

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*A region is called external in relation to the finite cross section if one can draw from it a real tangent plane to this cross section.

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