COMMENT ON THE PROBLEM OF THREE PARTICLES WITH POINT INTERACTIONS

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An integral equation for the wave function of three particles with point interactions is considered. It is shown that the discrete spectrum of the equation is infinite and extends to $-\infty$.

SKORNYAKOV and Ter-Martirosyan[1] have obtained an integral equation for the determination of the wave function of a system of three particles interacting via point-like potentials. In the simplest case of scalar identical particles this equation has the form

$$\left(\alpha + i\sqrt{z - \tfrac{3}{4}\mathbf{k}^2}\right)\varphi(\mathbf{k}) + \frac{1}{\pi^2}\int \frac{\varphi(\mathbf{k}')\,d\mathbf{k}'}{\mathbf{k}^2 + \mathbf{k}\mathbf{k}' + \mathbf{k}'^2 - z} = 0. \quad (1)$$

However, as was first noted by Danilov,[2] the corresponding homogeneous equation has a solution for arbitrary z, so that Eq. (1) does not uniquely determine the required solution.

The asymptotic form of any solution $\varphi(\mathbf{k}, z)$ is

$$\varphi(\mathbf{k}, z) = A(z)\frac{\sin s_0 \ln|\mathbf{k}|}{\mathbf{k}^2} + B(z)\frac{\cos s_0 \ln|\mathbf{k}|}{\mathbf{k}^2} + o\left(\frac{1}{\mathbf{k}^2}\right), \quad (2)$$

where s_0 is some number (see below), and the ratio $A(z)/B(z)$ can be arbitrary. Starting from the orthogonality conditions for the eigenfunctions, Danilov arrives at the following recipe: Take those solutions of Eq. (1) for which

$$A(z) = \beta B(z), \quad (3)$$

where β is a fixed parameter independent of z. Danilov proposes to express this constant β in terms of the energy of the bound state of all three particles, E_0, assuming vaguely that only one bound state with energy $E(\beta)$ exists for this choice of the solution. The parameter β is determined by the condition $E(\beta) = E_0$.

We shall show in the present paper that the homogeneous equation corresponding to Eq. (1), together with condition (3), has a solution for an infinite set of negative values z_n, which extends to $-\infty$. In other words, the model of Ter-Martirosyan and Skornyakov, in the more precise form proposed by Danilov, has an infinite number of bound states, and there is no ground state.

For the proof we restrict ourselves to the spherically symmetric solutions. These satisfy the following equation:

$$\psi(k) + \frac{2}{\pi}\int_0^\infty \ln\left(\frac{k^2 + kk' + k'^2 + \lambda^2}{k^2 - kk' + k'^2 + \lambda^2}\right)\frac{\psi(k')}{\alpha - \sqrt{3k'^2/4 + \lambda^2}}\,dk' = 0. \quad (4)$$

Here we have used the notations $z = -\lambda^2$ and

$$\psi(|\mathbf{k}|) = |\mathbf{k}|\left(\alpha - \sqrt{3\mathbf{k}^2/4 + \lambda^2}\right)\varphi(\mathbf{k}).$$

Making a change of variables, $k = \lambda(t^2 - 1)/t\sqrt{3}$ and carrying out a Mellin transformation, we obtain the equation

$$f(s) = \frac{4}{\pi\sqrt{3}}L(s) \times \left\{f(s) + \frac{2\alpha}{\lambda}\int_0^\infty [M(s - s') - M(s + s')]\,f(s')\,ds'\right\};$$

$$L(s) = \frac{2\pi}{s}\frac{\operatorname{sh}(\pi s/6)}{\operatorname{ch}(\pi s/2)}, \quad M(s) = \frac{\operatorname{sh} s(\pi - \arccos(\alpha/\lambda))}{2\sqrt{1 - (\alpha/\lambda)^2}\operatorname{sh}\pi s}. \quad (5)^*$$

The solution $\psi(k)$ is related to $f(s)$ by the formula

$$\psi(k) = \int_0^\infty f(s)\sin s\,\ln\left[\frac{\sqrt{3}k}{2\lambda} + \frac{1}{\lambda}\sqrt{3k^2/4 + \lambda^2}\right]ds. \quad (6)$$

The term containing the integral in (5) becomes small for large λ. Hence the equation

$$1 - \frac{8}{\sqrt{3}}\operatorname{sh}\frac{\pi s}{6}\Big/ s\operatorname{ch}\frac{\pi s}{2} = 0$$

has a positive root s_0, the solution $f(s)$ exists for all sufficiently small α/λ and has the form

$$f(s) = \delta(s - s_0) + O(1/\lambda). \quad (7)$$

The corresponding function

$$\psi(k) = \sin s_0 \ln\left[\sqrt{3}k/2\lambda + \lambda^{-1}\sqrt{3k^2/4 + \lambda^2}\right] + O(1/\lambda)$$

has the asymptotic form (2) for large k, where

$$A(-\lambda^2) = \cos s_0 \ln(\sqrt{3}/\lambda) + O(\lambda^{-1}),$$

$$B(-\lambda^2) = \sin s_0 \ln(\sqrt{3}/\lambda) + O(\lambda^{-1}).$$

With these $A(z)$ and $B(z)$, Eq. (3) has an in-

*sh = sinh, ch = cosh.

finite number of roots z_n, no matter what the value of β is. These roots extend to $-\infty$ and have the asymptotic form

$$z_n = -3\exp\left(\frac{2\pi n}{s_0} - \frac{2\operatorname{arc\,ctg}\beta}{s_0}\right)[1 + O(\lambda^{-1})]. \qquad (8)^*$$

We note that a similar situation obtains in the so-called "fall of the particles to the center." [3]

In conclusion we should like to remark that the model of Ter-Martirosyan and Skornyakov is apparently not the only physically acceptable model for the description of a system of three particles with point interactions. A more general scheme is discussed in a mathematical paper of the authors.[4]

*arc ctg = cot^{-1}.

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