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TRANSPORT EQUATION FOR A DEGENERATE SYSTEM OF FERMI PARTICLES

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Applying the temperature-dependent diagram technique and the method of analytic continuation, we give a derivation of the transport equation for the distribution function of excitations in a degenerate Fermi system. The analytic properties of the four-vertex part are studied and the equation for it is extended. We consider briefly whether the results obtained can be applied to the microscopic theory of a Fermi liquid.

1. It is well known that a degenerate Fermi system has an excitation branch of the Fermi type, and the excitations are weakly damped when their momenta are sufficiently close to the limiting Fermi momentum. This property is the basis of the semi-phenomenological theory of a Fermi liquid given by Landau. An application of the quantum field-theoretical methods made it possible to obtain a microscopic interpretation of the most important quantities in the Fermi-liquid theory.[1-3]

The present paper is devoted to a derivation of a transport equation for a degenerate Fermi system. To be specific, we consider the electric conductivity of a normal metal. An application of the temperature-dependent diagram technique and the method of analytic continuation[4] yielded an equation for the distribution function of the excitations in which the collision integral was expressed in terms of the four-vertex part Γ. The connection between the collision integral and Γ is, of course, independent of the character of the transport problem. The results obtained can thus be helpful for studying different problems about the kinetics of a Fermi liquid.

2. When we use the temperature-dependent diagram technique to calculate the conductivity, it is convenient to start from the expression (see [5])

$$\sigma_{\mu\nu}(\mathbf{k}, \omega) = \left(\frac{e}{m}\right)^2 \iint \frac{d^3\mathbf{p}\, d^3\mathbf{p}'}{(2\pi)^6}\, p_\mu \frac{K^R_{\mathbf{pp}'}(\mathbf{k}, \omega) - K^R_{\mathbf{pp}'}(\mathbf{k}, 0)}{i\omega}\, p'_\nu, \tag{1}$$

where e and m are the electronic charge and mass,* $\mathbf{k}$ and ω the wave vector and the frequency of the external field, and $K^R_{\mathbf{pp}'}(\mathbf{k}, \omega)$ the Fourier component of the retarded commutator

$$\widetilde{K}^R_{\mathbf{pp}'}(\mathbf{k}, t) = i\langle [e^{i(H-\mu N)t} a^+_{\mathbf{p}'-\mathbf{k}/2}\, a_{\mathbf{p}'+\mathbf{k}/2}\, e^{-i(H-\mu N)t},\ a^+_{\mathbf{p}+\mathbf{k}/2}\, a_{\mathbf{p}-\mathbf{k}/2}]\rangle \theta(t). \tag{2}$$

*We use a system of units in which $\hbar = 1$.

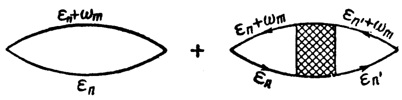


FIG. 1

The average is over a grand canonical ensemble.

We now introduce the function

$$\widetilde{K}_{\mathbf{pp}'}(\mathbf{k}, \tau) = \langle T_\tau\, (e^{(H-\mu N)\tau}\, a^+_{\mathbf{p}'-\mathbf{k}/2}\, a_{\mathbf{p}'+\mathbf{k}/2}\, e^{-(H-\mu N)\tau}) a^+_{\mathbf{p}+\mathbf{k}/2} a_{\mathbf{p}-\mathbf{k}/2}\rangle.$$

The Fourier component of this function

$$K_{\mathbf{pp}'}(k, \omega_m) = \frac{1}{2}\int_{-1/T}^{1/T} e^{\omega_m \tau}\, \widetilde{K}_{\mathbf{pp}'}(\mathbf{k}, \tau)\, d\tau; \qquad \omega_m = 2m\pi i T \tag{3}$$

and the quantity $K^R_{\mathbf{pp}'}(\mathbf{k}, \omega)$ are values of the same function, which is analytic in the upper half-plane, respectively in the points ω_m $(m > 0)$ on the imaginary axis and on the real axis. Moreover,

$$K^R_{\mathbf{pp}'}(\mathbf{k}, 0) = K_{\mathbf{pp}'}(\mathbf{k}, 0). \tag{4}$$

One can check this by performing a Lehmann expansion of the functions K^R and K.

The function $K_{\mathbf{pp}'}(\mathbf{k}, \omega_m)$ can be represented by a sum of the diagrams depicted in Fig. 1. These diagrams correspond to the expression

$$\begin{aligned} K_{\mathbf{pp}'}(\mathbf{k}, \omega_m) = &-T\sum_n G_{\mathbf{p}+\mathbf{k}/2}(\varepsilon_n + \omega_m)\, G_{\mathbf{p}-\mathbf{k}/2}(\varepsilon_n)\, \delta_{\mathbf{p}-\mathbf{p}'} \\ &- T^2 \sum_{nn'} G_{\mathbf{p}+\mathbf{k}/2}(\varepsilon_n + \omega_m)\, G_{\mathbf{p}-\mathbf{k}/2}(\varepsilon_n)\, \Gamma_{\mathbf{pp}'\mathbf{k}}(\varepsilon_n, \varepsilon_{n'}; \omega_m) \\ &\times G_{\mathbf{p}'+\mathbf{k}/2}(\varepsilon_{n'} + \omega_m)\, G_{\mathbf{p}'-\mathbf{k}/2}(\varepsilon_{n'}). \end{aligned} \tag{5}$$

Here $G_{\mathbf{p}}(\epsilon_n)$ is the temperature-dependent Green's function defined at the set of points $\epsilon_n = (2n+1)\pi i T$, and $\Gamma_{\mathbf{pp}'\mathbf{k}}(\epsilon_n, \epsilon_{n'}; \omega_m)$ is the four-vertex part.

3. In order to carry out an analytic continuation in (5) it is necessary to elucidate the analytic properties of Γ. To do this we consider the Lehmann

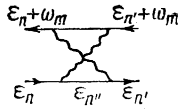


FIG. 2

expansion (given in the Appendix) of the two-particle Green's function $K(\epsilon_n, \epsilon_{n'}; \omega_m)$ with which $\Gamma(\epsilon_n, \epsilon_{n'}; \omega_m)$ is connected in the well-known way. (We dropped the momentum subscripts as they will not interest us in this section.)

It is clear from that expansion that $K(\epsilon, \epsilon'; \omega)$ as function of the complex variables ϵ, ϵ', and ω has singularities when

a) $\mathrm{Im}\,\varepsilon = 0, \quad \mathrm{Im}\,(\varepsilon + \omega) = 0, \quad \mathrm{Im}\,\varepsilon' = 0,$
$\mathrm{Im}\,(\varepsilon' + \omega) = 0;$
b) $\mathrm{Im}\,(\varepsilon + \varepsilon' + \omega) = 0;$
c) $\mathrm{Im}\,\omega = 0, \quad \mathrm{Im}\,(\varepsilon - \varepsilon') = 0.$ (6)

These singularities correspond to cuts parallel to the real axis in the complex planes of each argument. The whole space of the variables ϵ, ϵ', and ω is thus divided into several regions in each of which Γ is an analytic function of any of its arguments, while the values of the other arguments are fixed.

Green's functions which are external end points of Γ and which are included in K also possess singularities of the type a). Therefore the presence of these singularities in the function K still does not mean that they also occur in the function Γ. However, a study of the separate diagrams of the vertex part shows that Γ has singularities of the type a). As an example we consider the diagram given in Fig. 2. This diagram corresponds to the expression

$$\Gamma_1(\varepsilon_n, \varepsilon_{n'}; \omega_m) = T \sum_{n''} G(\varepsilon_{n''}) G(\varepsilon_n + \varepsilon_{n'} + \omega_m - \varepsilon_{n''}) \times D(\varepsilon_n - \varepsilon_{n''}) D(\varepsilon_{n''} - \varepsilon_{n'}).$$

Here D is some boson Green's function corresponding to the interaction. It is most convenient to study the analytic properties of this diagram by substituting for the summation over n'' an integration:

$$\Gamma_1(\varepsilon_n, \varepsilon_{n'}; \omega_m) = \frac{1}{4\pi i} \int_C dz \,\mathrm{th}\, \frac{z}{2T} G(z) G(\varepsilon_n + \varepsilon_{n'} + \omega_m - z) D(\varepsilon_n - z) D(z - \varepsilon_{n'}) + T [G(\varepsilon_n) G(\varepsilon_{n'} + \omega_m) D(0) D(\varepsilon_n - \varepsilon_{n'}) + G(\varepsilon_{n'}) G(\varepsilon_n + \omega_m) \times D(\varepsilon_n - \varepsilon_{n'}) D(0)]. \qquad (7)^*$$

The contour C is depicted in Fig. 3; it goes around all poles of $\tanh(z/2T)$ except $z = \epsilon_n$ and $z = \epsilon_{n'}$ and does not contain other singularities of the integrand. The integrals over the arcs of the large circle are equal to zero. The integrals over the small circles are compensated by terms outside the integral. Taking into account that

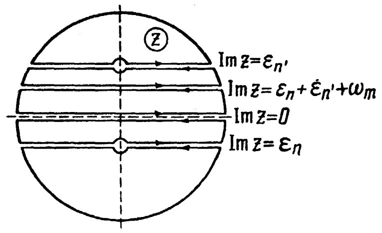


FIG. 3

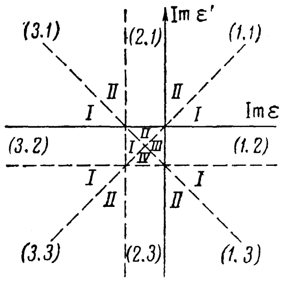


FIG. 4

$$\mathrm{th}\,\frac{\varepsilon + \varepsilon_n}{2T} = \mathrm{cth}\,\frac{\varepsilon}{2T} \text{ and } \mathrm{th}\,\frac{\varepsilon + \omega_m}{2T} = \mathrm{th}\,\frac{\varepsilon}{2T},$$

we get thus for Γ_1 the expression

$$\Gamma_1(\varepsilon_n, \varepsilon_{n'}; \omega_m) = \frac{1}{4\pi i} \int_{-\infty}^{\infty} d\varepsilon'' \Big\{ \mathrm{th}\,\frac{\varepsilon''}{2T} [G^R(\varepsilon'') - G^A(\varepsilon'')] G(\varepsilon_n + \varepsilon_{n'} + \omega_m - \varepsilon'') \times D(\varepsilon_n - \varepsilon'') D(\varepsilon'' - \varepsilon_{n'}) + \mathrm{th}\,\frac{\varepsilon''}{2T} [G^A(-\varepsilon'') - G^R(-\varepsilon'')] G(\varepsilon'' + \varepsilon_n + \varepsilon_{n'} + \omega_m) D(-\varepsilon'' - \varepsilon_{n'} - \omega_m) \times D(\varepsilon'' + \varepsilon_n + \omega_m) + \mathrm{cth}\,\frac{\varepsilon''}{2T} [D^A(-\varepsilon'') - D^R(-\varepsilon'')] G(\varepsilon'' + \varepsilon_n) G(\varepsilon_{n'} + \omega_m - \varepsilon'') D(\varepsilon'' + \varepsilon_n - \varepsilon_{n'}) + \mathrm{cth}\,\frac{\varepsilon''}{2T} [D^R(\varepsilon'') - D^A(\varepsilon'')] G(\varepsilon'' + \varepsilon_{n'}) \times G(\varepsilon_n + \omega_m - \varepsilon'') D(\varepsilon_n - \varepsilon_{n'} - \varepsilon'') \Big\}. \qquad (8)$$

The integration in the vicinity of $\epsilon'' = 0$ must refer to the principal value. Since the functions G and D have singularities when the imaginary parts of their arguments tend to zero, the diagram considered here will give all singularities (6), as can be seen from (8).

We shall in the following be interested in the properties of $\Gamma(\epsilon, \epsilon'; \omega)$ as functions of ϵ and ϵ' for fixed values of ω, with $\mathrm{Im}\,\omega > 0$. It is convenient in that case to represent the analytic properties of Γ as in Fig. 4, where the singularities (6) are plotted in the $\mathrm{Im}\,\epsilon$, $\mathrm{Im}\,\epsilon'$ plane. The lines drawn in the figure divide the plane into 16 regions, each of which corresponds to a function Γ which is analytic in that region in any of its arguments. The rectangular regions in the figure are numbered by two indices (i, k), each of which takes on three values. Some of these regions are divided into parts by the diagonal cuts. These parts are denoted by Roman numbers. Such a system of notation is also used for the Γ functions (for instance, $\Gamma^{I}_{1,1}$).

4. We can now carry out the analytic continuation in Eq. (5). To do this we first replace the

*th = tanh, cth = coth, ch = cosh.

sums over n and n′ by integrals, in the same way as was done in the case of the diagram of Fig. 2. We can write

$$T\sum_{n'}\Gamma(\varepsilon_n,\varepsilon_{n'};\omega_m)\,G(\varepsilon_{n'}+\omega_m)\,G(\varepsilon_{n'})$$
$$=\frac{1}{4\pi i}\int_{L'}dz'\,\mathrm{th}\frac{z'}{2T}\Gamma(\varepsilon_n,z';\omega_m)\,G(z'+\omega_m)\,G(z'),$$

where the integration is performed in the positive direction along the edges of the cuts: Im $z'=0$, Im $z'=-\omega_m$, Im $z'=\epsilon_n$, and Im $z'=-\epsilon_n-\omega_m$, while near the points $z'=\epsilon_n$ and $z'=-\epsilon_n-\omega_m$ the principal value of the integral must be taken. One verifies easily, by writing out explicitly the integrals over the different parts of the contour L′, that the expression obtained has singularities at Im z = 0 and Im $z=-\omega_m$, as function of the complex variable z corresponding to ϵ_n. Using this fact we can replace the sum over n in Eq. (5) by an integral. As a result we obtain an expression which is an analytic function of ω in the upper half-plane. Performing the analytic continuation with respect to ω on the real axis we get finally

$$K^R(\omega)=-\frac{1}{4\pi i}\int_{-\infty}^{\infty}d\varepsilon\Big[\mathrm{th}\frac{\varepsilon}{2T}K_1(\varepsilon,\omega)+\Big(\mathrm{th}\frac{\varepsilon+\omega}{2T}-\mathrm{th}\frac{\varepsilon}{2T}\Big)K_2(\varepsilon,\omega)-\mathrm{th}\frac{\varepsilon+\omega}{2T}K_3(\varepsilon,\omega)\Big],\qquad(9)$$

where

$$K_i(\varepsilon,\omega)=g_i(\varepsilon,\omega)\Big\{1+\frac{1}{4\pi i}\int d\varepsilon'\,\mathcal{T}_{ik}(\varepsilon,\varepsilon';\omega)\,g_k(\varepsilon',\omega)\Big\},\qquad(10)$$

$$g_1(\varepsilon,\omega)=G^R(\varepsilon+\omega)\,G^R(\varepsilon),$$
$$g_2(\varepsilon,\omega)=G^R(\varepsilon+\omega)\,G^A(\varepsilon),$$
$$g_3(\varepsilon,\omega)=G^A(\varepsilon+\omega)\,G^A(\varepsilon).\qquad(11)$$

The quantities $\mathcal{T}_{ik}$ are connected with the functions Γ_{ik} which arise because of the analytic continuation of the vertex part, as follows:

$$\mathcal{T}_{11}(\varepsilon,\varepsilon';\omega)=\mathrm{th}\frac{\varepsilon'}{2T}\Gamma_{11}^{\mathrm{I}}(\varepsilon,\varepsilon;\omega)+\mathrm{cth}\frac{\varepsilon'-\varepsilon}{2T}[\Gamma_{11}^{\mathrm{II}}(\varepsilon,\varepsilon';\omega)-\Gamma_{11}^{\mathrm{I}}(\varepsilon,\varepsilon';\omega)].$$

$$\mathcal{T}_{12}(\varepsilon,\varepsilon';\omega)=\Big(\mathrm{th}\frac{\varepsilon'+\omega}{2T}-\mathrm{th}\frac{\varepsilon'}{2T}\Big)\Gamma_{12}(\varepsilon,\varepsilon';\omega),$$

$$\mathcal{T}_{13}(\varepsilon,\varepsilon';\omega)=-\mathrm{th}\frac{\varepsilon'+\omega}{2T}\Gamma_{13}^{\mathrm{I}}(\varepsilon,\varepsilon';\omega)-\mathrm{cth}\frac{\varepsilon'+\varepsilon+\omega}{2T}[\Gamma_{13}^{\mathrm{II}}(\varepsilon,\varepsilon';\omega)-\Gamma_{13}^{\mathrm{I}}(\varepsilon,\varepsilon';\omega)],$$

$$\mathcal{T}_{21}(\varepsilon,\varepsilon';\omega)=\mathrm{th}\frac{\varepsilon'}{2T}\Gamma_{21}(\varepsilon,\varepsilon';\omega),$$

$$\mathcal{T}_{22}(\varepsilon,\varepsilon';\omega)=\Big(\mathrm{cth}\frac{\varepsilon'-\varepsilon}{2T}-\mathrm{th}\frac{\varepsilon'}{2T}\Big)\Gamma_{22}^{\mathrm{II}}(\varepsilon,\varepsilon';\omega)+\Big(\mathrm{cth}\frac{\varepsilon'+\varepsilon+\omega}{2T}-\mathrm{cth}\frac{\varepsilon'-\varepsilon}{2T}\Big)\Gamma_{22}^{\mathrm{III}}(\varepsilon,\varepsilon';\omega)+\Big(\mathrm{th}\frac{\varepsilon'+\omega}{2T}-\mathrm{cth}\frac{\varepsilon'+\varepsilon+\omega}{2T}\Big)\Gamma_{22}^{\mathrm{IV}}(\varepsilon,\varepsilon';\omega),$$

$$\mathcal{T}_{23}(\varepsilon,\varepsilon';\omega)=-\mathrm{th}\frac{\varepsilon'+\omega}{2T}\Gamma_{23}(\varepsilon,\varepsilon';\omega),$$

$$\mathcal{T}_{31}(\varepsilon,\varepsilon';\omega)=\mathrm{th}\frac{\varepsilon'}{2T}\Gamma_{31}^{\mathrm{I}}(\varepsilon,\varepsilon';\omega)+\mathrm{cth}\frac{\varepsilon'+\varepsilon+\omega}{2T}[\Gamma_{31}^{\mathrm{II}}(\varepsilon,\varepsilon';\omega)-\Gamma_{31}^{\mathrm{I}}(\varepsilon,\varepsilon';\omega)],$$

$$\mathcal{T}_{32}(\varepsilon,\varepsilon';\omega)=\Big(\mathrm{th}\frac{\varepsilon'+\omega}{2T}-\mathrm{th}\frac{\varepsilon'}{2T}\Big)\Gamma_{32}(\varepsilon,\varepsilon';\omega);$$

$$\mathcal{T}_{33}(\varepsilon,\varepsilon';\omega)=-\mathrm{th}\frac{\varepsilon'+\omega}{2T}\Gamma_{33}^{\mathrm{I}}(\varepsilon,\varepsilon';\omega)-\mathrm{cth}\frac{\varepsilon'-\varepsilon}{2T}[\Gamma_{33}^{\mathrm{II}}(\varepsilon,\varepsilon';\omega)-\Gamma_{33}^{\mathrm{II}}(\varepsilon,\varepsilon';\omega)].\qquad(12)$$

The problem of the analytic continuation of (5) is solved by Eqs. (9) to (12).

5. For the following it is necessary to elucidate some properties of the Green's functions

$$G^R(x,x')=-i\langle\{\psi(x),\psi^+(x')\}\rangle\theta(t-t'),$$
$$G^A(x,x')=i\langle\{\psi(x),\psi^+(x')\}\rangle\theta(t'-t).$$

In the momentum representation we can write

$$G_{\mathbf{p}}^R(\varepsilon)=[\varepsilon-\varepsilon_{\mathbf{p}}^0-\Sigma_{\mathbf{p}}^R(\varepsilon)]^{-1},\qquad(13)$$

where $\epsilon_{\mathbf{p}}^0=(p^2/2m)-\mu$. The fact that there are weakly damped fermion excitations present in the system corresponds to a well-defined small imaginary part of $\Sigma_{\mathbf{p}}^R(\epsilon)$, namely such that, if the temperature is sufficiently low and $\epsilon\sim T$, $\epsilon_{\mathbf{p}}^0\sim T$; Im $\Sigma_{\mathbf{p}}^R(\epsilon)\ll T$. It follows from this that if $\epsilon\sim T$ and $v|p-p_0|\sim T$ (v is the velocity on the Fermi surface),

$$\mathrm{Im}\,G_p^R(\varepsilon)=-\pi a\delta(\varepsilon-\varepsilon_p),\qquad(14)$$

where $\epsilon_{\mathbf{p}}$ is the root of the equation $\epsilon-\epsilon_{\mathbf{p}}^0-\mathrm{Re}\,\Sigma_{\mathbf{p}}^R(\epsilon)=0$, and

$$a=\Big[1-\frac{\partial}{\partial\varepsilon}\mathrm{Re}\,\Sigma_{\mathbf{p}}^R(\varepsilon)|_{\varepsilon=\varepsilon_{\mathbf{p}}}\Big]^{-1}\qquad(15)$$

We consider now the quantities g_i defined by Eqs. (11). When $\omega\ll T$ and $vk\ll T$,

$$g_1(P,K)\approx[G^R(P)]^2;\qquad P=(\varepsilon,\mathbf{p}),\qquad K=(\omega,\mathbf{k}),$$

i.e., we can assume that g_1 is in that case independent of ω and **k**. If this quantity occurs in an integral where values $\epsilon\sim T$ and $v|p-p_0|\sim T$ are important, we can use for it the simple expression

$$g_1(P,K)\approx a^2(\varepsilon-\varepsilon_{\mathbf{p}}+i\delta)^{-2},\qquad\delta=+0.\qquad(16)$$

The quantity $g_3=g_1^*$ has the same properties. Only the function $g_2(P,K)=G_{\mathbf{p}+\mathbf{k}/2}^R(\epsilon+\omega)\cdot G_{\mathbf{p}-\mathbf{k}/2}^R(\epsilon)$ depends appreciably on ω and **k** for small values of ω and k.

We shall see that in all integrals which contain g_2 the domain of integration is limited by the values $\epsilon\sim T$ and $v|p-p_0|\sim T$. We can thus write for $\omega\ll T$ and $vk\ll T$

$$g_2(P,K)\approx 2\pi ia^2\delta(\varepsilon-\varepsilon_{\mathbf{p}})/(\omega-\mathbf{vk}+2i\gamma_{\mathbf{p}})\qquad(17)$$

where

$$\mathbf{v} \equiv \mathbf{p}/m^* = a\,[\mathbf{p}/m + \nabla_\mathbf{p}\,\mathrm{Re}\,\Sigma_\mathbf{p}(\varepsilon)|_{\varepsilon=\varepsilon_\mathbf{p}}], \quad (18)$$

$$\gamma_\mathbf{p} = -a\,\mathrm{Im}\,\Sigma_\mathbf{p}^R(\varepsilon_\mathbf{p}) > 0. \quad (19)$$

In accordance with what was said before, $\gamma_\mathbf{p} \ll T$ when $v\,|p - p_0| \sim T$.

6. We consider now in somewhat greater detail the properties of the quantities $\mathcal{T}_{ik}$. We note first that some graphic representation of these quantities is possible. We introduce the irreducible parts $\mathcal{T}_{ik}^{(1)}$ which are obtained, as the result of the analytic continuation and of applying Eq. (12), from all diagrams $\Gamma^{(1)}(\epsilon_n, \epsilon_{n'}; \omega_m)$ which do not contain a pair of lines of the type $G(\epsilon_n + \omega_m)\,G(\epsilon_n)$. One can then easily verify that $\mathcal{T}_{ik}$ satisfies the equation

$$\mathcal{T}_{ik}(P, P'; K) = \mathcal{T}_{ik}^{(1)}(P, P'; K) + \frac{1}{2i\,(2\pi)^4}\int d^4P''\,\mathcal{T}_{il}^{(1)}(P, P''; K)\,g_l(P'', K)\,\mathcal{T}_{lk}(P'', P'; K). \quad (20)$$

This means that $\mathcal{T}_{ik}$ can be written as the sum of diagrams containing different numbers of irreducible parts $\mathcal{T}^{(1)}$, which we depict by shaded rectangles and which are joined by pairs of lines g_l which we shall call sections l.

We saw that from among the three functions g_i only g_2 depends appreciably on ω and k when ω and k are small. It is thus expedient to introduce for each of the functions $\mathcal{T}_{ik}$ the totality of diagrams $\mathcal{T}_{ik}^{(0)}$ which does not contain the section 2. We shall then have instead of the set of Eqs. (20) one equation for $\mathcal{T}_{22}$:

$$\mathcal{T}_{22}(P, P'; K) = \mathcal{T}_{22}^{(0)}(P, P'; K) + \frac{1}{2i\,(2\pi)^4}\int d^4P''\,\mathcal{T}_{22}^{(0)}(P, P''; K)\,g_2(P'', K) \times \mathcal{T}_{22}(P'', P'; K), \quad (21)$$

whereas all other quantities $\mathcal{T}_{ik}$ can be expressed in terms of $\mathcal{T}_{22}$ and $\mathcal{T}_{ik}^{(0)}$ as is shown in Fig. 5. If $vk \ll T$ we can assume that all $\mathcal{T}_{ik}^{(0)}$ are independent of k. The dependence of these quantities on ω occurs in practice only because of the hyperbolic functions in (12). In particular, the functions $\mathcal{T}_{12}$ and $\mathcal{T}_{32}$ are proportional to $\tanh[(\epsilon+\omega)/2T] - \tanh(\epsilon/2T)$ and tend to zero for $\omega = 0$. Therefore, when $\omega = 0$ the functions $\mathcal{T}_{ik}$, with the exception of $\mathcal{T}_{22}$, do not contain diagrams which have at least one section 2.

We need relations connecting the derivatives of the Green's functions with the quantities $\mathcal{T}_{ik}$ at $K = 0$. These relations can be obtained in a way similar to the one used for $T = 0$ [2] or through an analytic continuation of the relations for the temperature-dependent diagram technique. [6] We shall therefore give them without derivation

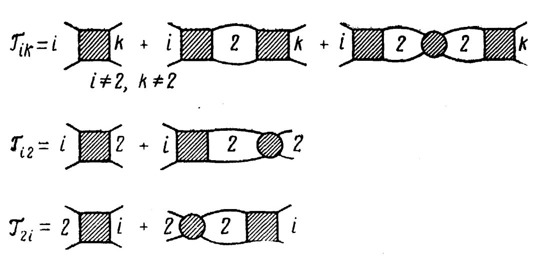


FIG. 5

$$\frac{\partial}{\partial\varepsilon}[G_\mathbf{p}^R(\varepsilon)]^{-1} = 1 + \frac{1}{2i\,(2\pi)^4}\int d^4P'\,\{\mathcal{T}_{11}(P, P')\,[G^R(P')]^2 + \mathcal{T}_{13}(P,P')\,|G^A(P')|^2\}, \quad (22)$$

$$\frac{\partial}{\partial p}[G_\mathbf{p}^R(\varepsilon)]^{-1} = -\frac{\mathbf{p}}{m} - \frac{1}{2i\,(2\pi)^4}\int d^4P'\,\mathbf{p}'\,\{\mathcal{T}_{11}(P, P')\,[G^R(P')]^2 + \mathcal{T}_{13}(P, P')\,[G^A(P')]^2\}. \quad (23)$$

7. We show now that the conductivity $\sigma_{\mu\nu}$ can be expressed in terms of the single function $\mathcal{T}_{22}$ only, while the other $\mathcal{T}_{ik}$ determine the values of renormalization constants.

Bearing in mind the case $\omega \ll T$, $kv \ll T$ we retain a dependence on ω and k only in g_2 and $\mathcal{T}_{i2}$. It then follows from (1) and (9) that we need only be interested in those diagrams $K_1(\epsilon, \omega)$ and $K_3(\epsilon, \omega)$ which contain at least one section 2. All those diagrams and also the diagrams forming $K_2(\epsilon, \omega)$ are illustrated in Fig. 6, in which the rectangles correspond to the quantities $\mathcal{T}^{(0)}$ which do not contain sections 2, and a circle represents $\mathcal{T}_{22}$. Substituting the expressions for the K_i corresponding to these diagrams into (9) and (1), and applying Eq. (17), we get

$$\sigma_{\mu\nu}(\mathbf{k}, \omega) = \frac{i}{2}\left(\frac{e}{m}\right)^2 a^2 \left\{\int \frac{d^3p}{(2\pi)^3}\,Q_\mu^{(1)}(\mathbf{p})\,\frac{(1/2T)\,\mathrm{ch}^{-2}(\varepsilon_\mathbf{p}/2T)}{\omega - \mathbf{vk} + 2i\gamma_\mathbf{p}} \times Q_\nu^{(2)}(\mathbf{p}) + \frac{a^2}{2}\int \frac{d^3\mathbf{p}\,d^3\mathbf{p}'}{(2\pi)^6}\,Q_\mu^{(1)}(\mathbf{p}) \times \frac{(1/2T)\,\mathrm{ch}^{-2}(\varepsilon_\mathbf{p}/2T)\,\mathcal{T}_{22}(p, p'; k, \omega)}{[\omega - \mathbf{vk} + 2i\gamma_\mathbf{p}](\omega - \mathbf{v}'\mathbf{k} + 2i\gamma_{p'})}\,Q_\nu^{(2)}(\mathbf{p}'), \quad (24)$$

where $Q_\mu^{(1)}(\mathbf{p})$ and $Q_\mu^{(2)}(\mathbf{p})$ are the values at $\epsilon = \epsilon_\mathbf{p}$ of the following quantities

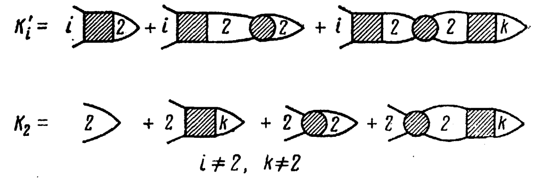


FIG. 6

$$Q_\mu^{(1)} = p_\mu + \frac{2T}{\omega}\,\mathrm{ch}^2\,\frac{\varepsilon}{2T}\,\frac{1}{2i\,(2\pi)^4}\int d^4P''\,p''_\mu\,\mathrm{th}\,\frac{\varepsilon''}{2T}\,[g_1(P'')$$
$$\times\,\mathcal{T}_{12}^{(0)}(P'', P;\,\omega) - g_3(P'')\,\mathcal{T}_{32}^{(0)}(P'', P';\,\omega)], \qquad (25)$$

$$Q_\mu^{(2)} = p_\mu + \frac{1}{2i\,(2\pi)^4}\int d^4P''\,[\mathcal{T}_{21}^{(0)}(P, P'')\,g_1(P'')$$
$$+\,\mathcal{T}_{23}^{(0)}(P, P'')\,g_3(P'')]\,p''_\nu. \qquad (26)$$

We took here into account that $\mathcal{T}_{2i}^{(0)}(P, P''; K) \approx \mathcal{T}_{2i}^{(0)}(P, P''; 0)$. The quantities $\mathcal{T}_{i2}(P'', P; \omega)$ contain a factor $(\omega/2T)\cosh^{-2}(\epsilon_p/2T)$, and are otherwise independent of ω.

Using Eqs. (12) we can write

$$Q_\mu^{(1)} = p_\mu + \frac{1}{2i\,(2\pi)^4}\int d^4P''p''_\mu\,\mathrm{th}\,\frac{\varepsilon''}{2T}\,[g_1(P'')\,\Gamma_{12}^{(0)}(P'', P)$$
$$-\,g_3(P'')\,\Gamma_{32}^{(0)}(P'', P)],$$
$$Q_\mu^{(2)} = p_\mu + \frac{1}{2i\,(2\pi)^4}\int d^4P''\,p''_\mu\,\mathrm{th}\frac{\varepsilon''}{2T}\,[\Gamma_{21}^{(0)}(P, P'')\,g_1(P'')$$
$$-\,\Gamma_{23}^{(0)}(P, P'')\,g_3(P'')].$$

By considering the separate diagrams of the vertex part or the Lehmann expansion for the two-particle Green's function (see Appendix) one can easily check that

$$\Gamma_{i2}(P'', P) = \Gamma_{2i}(P, P''), \qquad \Gamma_{12}(P, P'') = [\Gamma_{32}(P, P'')]^*.$$

Thus, $\mathbf{Q}^{(1)} = \mathbf{Q}^{(2)} = \mathbf{Q}$, where $\mathbf{Q}$ is a real quantity. It is clear that

$$\mathbf{Q}(\mathbf{p}) = \alpha\mathbf{p}. \qquad (27)$$

From our earlier considerations it follows that

$$\alpha^2 = (m/am^*)^2. \qquad (28)$$

We write the quantities $\mathcal{T}_{21}^{(0)}$ and $\mathcal{T}_{23}^{(0)}$ which occur in (26) in the following form:

$$\mathcal{T}_{2i}^{(0)}(P, P'') = \tfrac{1}{2}\left[\mathcal{T}_{1i}(P, P'') + \mathcal{T}_{3i}(P, P'')\right] + \left\{\mathcal{T}_{2i}^{(0)}(P, P'')\right.$$
$$\left.-\tfrac{1}{2}\left[\mathcal{T}_{1i}(P, P'') + \mathcal{T}_{3i}(P, P'')\right]\right\}.$$

A simple, though rather tedious study of the separate diagrams shows that when $\epsilon \sim T$ the second term differs from zero only in an interval $\epsilon'' \sim T$, while outside that interval it decreases exponentially. Assuming that all diagrams have this property we get the result that one can in (26) replace $\mathcal{T}_{2i}^{(0)}$ by $\frac{1}{2}[\mathcal{T}_{1i} + \mathcal{T}_{3i}]$. Indeed, if we substitute the difference $\mathcal{T}_{2i}^{(0)} - \frac{1}{2}[\mathcal{T}_{1i} + \mathcal{T}_{3i}]$ into (26), we find that in the integral the values $\epsilon'' \sim T$, $v\,|p'' - p_0| \sim T$ are the important ones. Since we are, moreover, interested in the value of $\mathbf{Q}(\mathbf{p})$ when $v\,|p - p_0| \sim T$ one may assume that the above-mentioned difference depends only on $p - p''$. We get thus when we use Eq. (16) and integrate over $\epsilon_{p''} = v(p'' - p_0)$ the result that the integral vanishes. Equation (28) follows then from (23) and (18). Equations (24), (27), and (28) completely determine the connection between $\sigma_{\mu\nu}$ and $\mathcal{T}_{22}$.

8. We now introduce the quantity $f_{\mathbf{p}}(\mathbf{k}, \omega)$, which is the change in the excitation distribution function which is linear in the external field $\mathbf{E}$, starting from the equation

$$j_\mu(\mathbf{k}, \omega) = \frac{e}{m^*}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\,p_\mu\,f_{\mathbf{p}}(\mathbf{k}, \omega).$$

It then follows from (24), (27), and (28) that

$$f_{\mathbf{p}}(\mathbf{k}, \omega) = \frac{i}{2}\,\frac{e}{m^*}\,\frac{(1/2T)\mathrm{ch}^{-2}(\varepsilon_{\mathbf{p}}/2T)}{\omega - \mathbf{vk} + 2i\gamma_{\mathbf{p}}}$$
$$\times\,E_\nu\left\{p_\nu + \frac{1}{2}\int\frac{d^3\mathbf{p}'}{(2\pi)^3},\ \frac{a^2\mathcal{T}(\mathbf{p}, \mathbf{p}'; \mathbf{k}, \omega)}{\omega - \mathbf{v}'\mathbf{k} + 2i\gamma_{\mathbf{p}'}}\,p_\nu'\right\}. \qquad (29)$$

(Here and henceforth we drop the subscripts in $\mathcal{T}_{22}$.) The equation for $f_{\mathbf{p}}(\mathbf{k}, \omega)$ follows directly from Eq. (21) for $\mathcal{T}$. Introducing, as usual, instead of $f_{\mathbf{p}}$ a function $\varphi_{\mathbf{p}}$ such that $f_{\mathbf{p}}(\mathbf{k}, \omega) = \varphi_{\mathbf{p}}(\mathbf{k}, \omega)\,dn_{\mathbf{p}}/d\epsilon_{\mathbf{p}}$, where $n_{\mathbf{p}} = (e^{\epsilon_{\mathbf{p}}/T} + 1)^{-1}$, we get

$$i\,(\omega - \mathbf{vk})\,\varphi_{\mathbf{p}}(\mathbf{k}, \omega)$$
$$= e\,\mathbf{vE} + i\frac{a^2}{2}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\,\mathcal{T}^{(0)}(\mathbf{p}, \mathbf{p}'; \mathbf{k}, \omega)\,\varphi_{\mathbf{p}'}(\mathbf{k}, \omega)$$
$$+\,2\gamma_{\mathbf{p}}\varphi_{\mathbf{p}}(\mathbf{k}, \omega). \qquad (30)$$

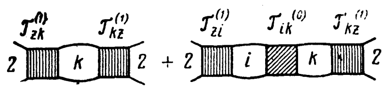
The quantity $\mathcal{T}^{(0)}$ representing all diagrams that do not contain sections 2 consists of an irreducible part $\mathcal{T}^{(1)}$ which neither contains sections 1 nor 3 and of diagrams which have different numbers of sections 1 and 3. Such diagrams are illustrated in Fig. 7. It is clear that they all contain the quantity $\mathcal{T}_{i2}(P, P')$ and it then follows from (12) that they are all proportional to $(\omega/2T)\cosh^{-2}(\epsilon_{\mathbf{p}}/2T)$.

If we also split off from $\mathcal{T}^{(1)}$ the part proportional to ω we write $\mathcal{T}^{(0)}$ in the form

$$\mathcal{T}^{(0)}(\mathbf{p}, \mathbf{p}'; \omega) = \mathcal{T}^{(1)}(\mathbf{p}, \mathbf{p}') + \frac{\omega}{2T}\,\mathrm{ch}^{-2}\,\frac{\varepsilon_{\mathbf{p}}}{2T}\,\mathcal{T}^{(2)}(\mathbf{p}, \mathbf{p}'). \qquad (31)$$

By studying separate diagrams or from the Lehmann expansion for the two-particle Green's function we can check that $\mathcal{T}(P, P'; K)$ is a purely imaginary quantity for $K = 0$. Since $g_2(P, K)$ is real at $K = 0$, $\mathcal{T}^{(1)}(P, P')$ is also a purely imaginary quantity. The expression

$$I = \frac{ia^2}{2}\int\frac{d^3p'}{(2\pi)^3}\,\mathcal{T}^{(1)}(\mathbf{p}, \mathbf{p}')\,\varphi_{\mathbf{p}'}(\mathbf{k}, \omega) + 2\gamma_{\mathbf{p}}\,\varphi_{\mathbf{p}}(\mathbf{k}, \omega) \qquad (32)$$



$i \neq 2,\ k \neq 2$

FIG. 7

is thus the collision integral in the transport equation (30). We explain the significance of the second term in (31) in the following.

9. There is, of course, a direct connection between the transport equation and Eq. (21) for the vertex part $\mathcal{T}$ not only for the problem of electrical conductivity considered here. In order to establish the connection between the quantity $\mathcal{T}$ introduced by us and the vertex part occurring in the zero-temperature diagram technique, we show how one can obtain from (21) the equation for zero sound in a Fermi liquid. [1]

We consider Eq. (21) in the limiting case $\omega\tau \gg 1$, where τ is the relaxation time which is of the order of magnitude of $\gamma_{\bar{p}}^{-1}$. Since the collision integral (32) can be written in the form φ/τ for estimating purposes, it follows that $\mathcal{T}^{(1)}$ is of the order of magnitude of $\mu/p_0^3\tau T$. On the other hand, if we estimate the simplest diagrams we get easily the result that $\mathcal{T}^{(2)}$ in (31) is of the order of magnitude μ/p_0^3. Therefore, when $\omega\tau \ll 1$, the quantity $\mathcal{T}^{(0)} \approx \mathcal{T}^{(1)}$, whereas for $\omega\tau \gg 1$

$$\mathcal{T}^{(0)}(\mathbf{p}, \mathbf{p}'; \omega) \approx \frac{\omega}{2T}\,\mathrm{ch}^{-2}\frac{\varepsilon_{\mathbf{p}}}{2T}\,\mathcal{T}^{(2)}(\mathbf{p}, \mathbf{p}').$$

We shall be interested just in that case. Neglecting in (21) terms of the order of magnitude $(\omega\tau)^{-1}$, we get

$$\mathcal{T}(\mathbf{p}, \mathbf{p}'; \mathbf{k}, \omega) = \frac{\omega}{2T}\,\mathrm{ch}^{-2}\frac{\varepsilon_{\mathbf{p}'}}{2T}\,\mathcal{T}^{(2)}(\mathbf{p}, \mathbf{p}') + \frac{a^2}{2}\int\frac{d^3p''}{(2\pi)^3}\frac{1}{2T}\,\mathrm{ch}^{-2}\frac{\varepsilon_{\mathbf{p}''}}{2T}\,\frac{\omega\mathcal{T}^{(2)}(\mathbf{p}, \mathbf{p}'')}{\omega - \mathbf{v}''\mathbf{k} + i\delta}\,\mathcal{T}(\mathbf{p}'', \mathbf{p}'; k, \omega). \quad (33)$$

Introducing the notation

$$\mathcal{T}(\mathbf{p}, \mathbf{p}'; 0, \omega) = \frac{\omega}{2T}\,\mathrm{ch}^{-2}\frac{\varepsilon_{\mathbf{p}'}}{2T}\,\Gamma^{\omega}(\mathbf{p}, \mathbf{p}') \quad (\omega\tau \gg 1), \quad (34)$$

we get for $k = 0$ for Γ^{ω} the equation

$$\Gamma^{\omega}(\mathbf{p}, \mathbf{p}') = \mathcal{T}^{(2)}(\mathbf{p}, \mathbf{p}') + \frac{a^2}{2}\int\frac{d\mathbf{p}''}{(2\pi)^3}\frac{1}{2T} \times \mathrm{ch}^{-2}\frac{\varepsilon_{\mathbf{p}''}}{2T}\,\mathcal{T}^{(2)}(\mathbf{p}, \mathbf{p}'')\,\Gamma^{\omega}(\mathbf{p}'', \mathbf{p}'). \quad (35)$$

In the integral term in this equation and also in Eq. (33) practically only the quantity $\cosh^{-2}(\epsilon_{\mathbf{p}''}/2T)$ depends on the magnitude of the vector $\mathbf{p}''$. Integrating over p'' and eliminating $\mathcal{T}^{(2)}$ from (35) and (33) we get thus

$$\mathcal{T}(\mathbf{p}, \mathbf{p}'; \mathbf{k}, \omega) = \frac{\omega}{2T}\,\mathrm{ch}^{-2}\frac{\varepsilon_{\mathbf{p}'}}{2T}\,\Gamma^{\omega}(\mathbf{p}, \mathbf{p}') + \frac{a^2p_0}{(2\pi)^3}\int do_{\mathbf{p}''} \times \frac{\mathbf{p}''\mathbf{k}\Gamma^{\omega}(\mathbf{p}, \mathbf{p}'')}{\omega - \mathbf{v}''\mathbf{k} + i\delta}\,\mathcal{T}(\mathbf{p}'', \mathbf{p}'; \mathbf{k}, \omega). \quad (36)$$

To find the natural vibrations corresponding to zero sound we must drop the free term. The equation we obtain then is the same as Landau's equation. [1] We can thus conclude that the quantity Γ^{ω} is the same as the corresponding quantity occurring in the zero-temperature diagram technique.

(The case $\omega\tau \gg 1$ corresponds for $T = 0$ to the limiting transition $k = 0$, $\omega \to 0$.) When deriving Eq. (36) we completely neglected terms of the order $(\omega\tau)^{-1}$. When such terms are taken into account it is possible to obtain the damping of zero sound and also to consider other phenomena connected with damping. To do this, it is, however, necessary to study in detail the structure of the quantities $\mathcal{T}^{(1)}$ and $\gamma_{\mathbf{p}}$, which is outside the framework of the present paper.

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APPENDIX

We perform the spectral expansion of the two-particle temperature-dependent Green's function

$$K(x_1x_2; x_3x_4) = \langle T\psi(x_1)\,\psi(x_2)\,\psi^+(x_3)\,\psi^+(x_4)\rangle,$$

where

$$\psi(x) = e^{(H-\mu N)\tau}\,\psi(r)\,e^{-(H-\mu N)\tau}.$$

This function consists of 24 parts corresponding to different permutations of the ψ operators. All permutations fall into six cycles with four permutations in each.

The contribution from the cycle created by the order 1-2-3-4 is equal to

$$K_1(x_1x_2; x_3x_4) = \sum_{\alpha_1\alpha_2\alpha_3\alpha_4}\langle\alpha_1|\psi(r_1)|\alpha_2\rangle\langle\alpha_2|\psi(r_2)|\alpha_3\rangle \times \langle\alpha_3|\psi^+(r_3)|\alpha_4\rangle\langle\alpha_4|\psi^+(r_4)|\alpha_1\rangle\exp\{E_1(\tau_1-\tau_4) + E_2(\tau_2-\tau_1) + E_3(\tau_3-\tau_2) + E_4(\tau_4-\tau_3)\}$$
$$\times[e^{-E_1/T}\theta(\tau_1-\tau_2)\,\theta(\tau_2-\tau_3)\,\theta(\tau_3-\tau_4) - e^{-E_2/T}\,\theta(\tau_2-\tau_3)\,\theta(\tau_3-\tau_4)\,\theta(\tau_4-\tau_1) + e^{-E_3/T}\,\theta(\tau_3-\tau_4)\,\theta(\tau_4-\tau_1)\,\theta(\tau_1-\tau_2) - e^{-E_4/T}\theta(\tau_4-\tau_1)\,\theta(\tau_1-\tau_2)\,\theta(\tau_2-\tau_3)]. \quad (A.1)$$

Here all energies E_i are calculated from μN_i. We choose from the four differences $\tau_i - \tau_k$ any three independent ones, for instance $t_1 = \tau_1 - \tau_2$, $t_2 = \tau_2 - \tau_3$, and $t_3 = \tau_3 - \tau_4$ and carry out in them a periodic continuation from the interval $(-1/T, 1/T)$ onto the whole axis of imaginary times. To do this one needs expand (A.1) in a triple Fourier series in terms of t_1, t_2, and t_3, taking into account that we must have $|t_1 + t_2 + t_3| \equiv |\tau_1 - \tau_4| \le 1/T$. Carrying out next a Fourier-series expansion in all four τ_i, we get for the Fourier components the expression

$$K_1(\varepsilon_1\varepsilon_2; \varepsilon_3\varepsilon_4) = T^{-1}\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\,K_1'(\varepsilon_1\varepsilon_2; \varepsilon_3\varepsilon_4).$$

The function K_1' depends in actual fact on three variables for which we choose $\epsilon = \epsilon_4$, $\epsilon' = \epsilon_2$, and $\omega = \epsilon_1 - \epsilon_4 = \epsilon_3 - \epsilon_2$. In terms of these variables

$$K_1'(\varepsilon, \varepsilon'; \omega) = \sum_{\alpha_1\alpha_2\alpha_3\alpha_4} A(1, 2, 3, 4) \times \left\{ \frac{e^{-E_1/T}}{(E_2-E_1-\varepsilon-\omega)(E_3-E_1-\varepsilon-\varepsilon'-\omega)(E_4-E_1-\varepsilon)} - \frac{e^{-E_2/T}}{(E_1-E_2+\varepsilon+\omega)(E_3-E_2-\varepsilon')(E_4-E_2+\omega)} + \frac{e^{-E_3/T}}{(E_1-E_3+\varepsilon+\varepsilon'+\omega)(E_2-E_3+\varepsilon')(E_4-E_3+\varepsilon'+\omega)} - \frac{e^{-E_4/T}}{(E_1-E_4+\varepsilon)(E_2-E_4-\omega)(E_3-E_4-\varepsilon')} \right\}, \quad \text{(A.2)}$$

where $\epsilon = (2n+1)\pi iT$; $\omega = 2m\pi iT$; the quantity $A(1,2,3,4)$ is the product of matrix elements of the ψ operators occurring in (A.1). The corresponding formulae for the other cycles are obtained from (A.2) by simple permutations of the indices.

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