# ANALYTIC PROPERTIES OF THE SQUARE DIAGRAM WITH DECAY MASSES

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Submitted to JETP editor April 28, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 1215-1220 (October, 1961)

The analytical properties of the simplest square diagram of perturbation theory are investigated for the case in which the masses of the external particles do not satisfy the condition of stability.

1. In a previous paper by the writers[1]* a study was made of the analytical properties of the simplest square diagram with arbitrary masses (Fig. 1). The specific form of the dispersion representations with respect to energy and momentum transfer ($\mu_{13}$ and $\mu_{24}$) for the amplitude A of the diagram was found for any values of the external masses that satisfy the condition of stability:

$$\mu_{ik} = (m_i^2 + m_k^2 - p_{ik}^2)/2m_i m_k > -1.$$

From the study made in I it could be seen that as soon as one of the masses (for example, $\mu_{12}$) has a value that makes decay possible, the corresponding singularity of $\Delta^{(-)}(\mu_{12}, \mu_{23})$ [Eq. I (12)], from which the integration over $\mu_{13}$ starts in the dispersion integral for A [Eq. I (29)], goes off into the complex plane; this means that there is a complex singularity of $A(\mu_{ik})$ as a function of $\mu_{13}$, and therefore Im A $\neq 0$ for all real $\mu_{13}$ and $\mu_{24}$.

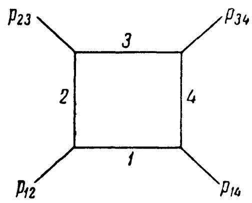


FIG. 1

In the present paper we shall show that in the decay case the imaginary part of A is determined not only by the usual absorption part of the scattering $A_1(\mu_{ik})$ [Eq. I (2)], but also by the decay absorption part $A_d(\mu_{ik})$. The latter is different from 0 for all $\mu_{13}$ and $\mu_{24}$. This result, obtained by analytic continuation with respect to the external masses, obviously agrees with the unitarity condition, from which it follows that for a decay value of the mass $\mu_{12}$ the imaginary part of the process comes not only from the fact that a real intermediate state with particles 1 and 3 exists, but also from the existence of a real state with particles 1 and 2, through which the process can go (see below, Fig. 3). Thus in the decay case it is possible to write A either in the form of a dispersion integral over a complex path of $\mu_{13}$ (or of $\mu_{24}$), or in the form of the integral along the real axis of $\mu_{13}$ ($\mu_{24}$) from $-\infty$ to $\infty$ of the absorption parts $A_1$ and $A_d$ (or $A_2$ and $A_d$).*

It is obvious that the results obtained not only are valid for the diagram considered, but to some extent characterize the analytic properties of any decay process involving four particles. Moreover, since in the decay case a given diagram can be regarded as the simplest diagram with five external lines [two external lines at the vertex (12)], the existence of complex singularities and absorption parts with respect to different variables (of the type of energy and mass) is for five-branch vertices (and for many-branch vertices) already a general rule, which does not arise as an exception for certain masses of the particles involved, but always exists for definite values of the external parameters (including physical values of these parameters).

2. We consider the case in which only one mass $\mu_{12}$ has a decay value, and shall set the others, $\mu_{23}$, $\mu_{34}$, and $\mu_{14}$, equal to each other [the definition of the amplitude A is given by Eq. I (1)], that is,

$$1 > \mu_{23} = \mu_{34} = \mu_{14} = \mu > 0. \tag{1}$$

According to the analysis carried out in I, the condition (1) means that the singularities of $A_1$ that correspond to these masses— $\Delta^{\pm}(\mu_{34}, \mu_{14})$ and $\Delta^{\pm}(\mu_{23}, \mu_{34})$ [cf. Eqs. I (12) and I (13)] —are not

*In the present paper we use the notations of[1], which is hereafter cited as I.

*$A_2$ is the absorption part that corresponds to conditions in which $\mu_{24}$ is the energy and $\mu_{13}$ is the momentum transfer (Fig. 5, a).

singularities of A (normal case with respect to these masses).

This simple symmetrical case contains within it all the essential features of the decay diagram, and at the same time is free from the superfluous complications characteristic of the case of different masses, which are of no importance in principle. Later we shall indicate the main differences that arise in the general case; they will follow directly from the analysis of I and the considerations given here. It follows from Eqs. I (12) and I (13) that for equal masses the singularities of the absorption part $A_1$ are of the forms

$$\Delta^{(\pm)} = \Delta^{(\pm)}(\mu_{34}, \mu_{14}) = \Delta^{(\pm)}(\mu_{23}, \mu_{34}) = 1 \quad \text{or} \quad 2\mu^2 - 1, \qquad (2)$$

for the upper and lower signs, respectively, and

$$\Delta^{(\pm)}(\mu_{12}) = \Delta^{(\pm)}(\mu_{12}, \mu_{23}) = \Delta^{(\pm)}(\mu_{12}, \mu_{14})$$

$$= \mu\mu_{12} \pm \sqrt{(1-\mu^2)(1-\mu_{12}^2)}. \qquad (3)$$

Analogous simplifications can also be made for the singularities of $\square(\mu_{13})$ and $\square(\mu_{24})$ (cf. I), but their analytic form is not important, and they are shown graphically for the case in which we are interested in Fig. 2 (curves I—V).

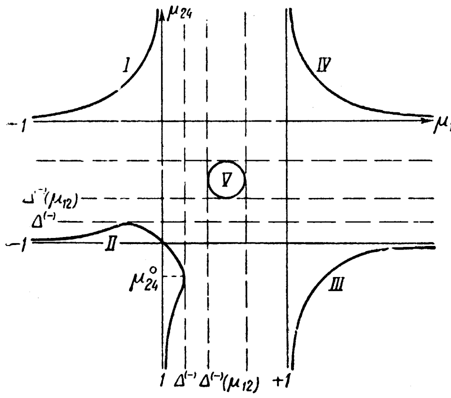


FIG. 2

Let us use the results of I for the case $\theta_{12} + \theta_{23} + \theta_{34} + \theta_{14} < 2\pi$, $\theta_{23} = \theta_{34} = \theta_{14} < \pi/2$, $\theta_{12} < \pi$. Figure 2 shows the positions of the singularities of the absorption part $A_1$ in this case. For $\mu_{24} > 1$ we have the dispersion relation [Eq. I (29)]:

$$A(\mu_{13}, \mu_{24}) = -\frac{1}{\pi}\int_{-\infty}^{-1} \frac{A_1(\mu_{24}, \mu_{13}')\, d\mu_{13}'}{\mu_{13}' - \mu_{13}} + A_{\text{an}}(\mu_{13}, \mu_{24}), \qquad (4)$$

$$A_{\text{an}}(\mu_{13}, \mu_{24}) = -i\frac{1}{\pi}\int_{-1}^{\Delta^{(-)}(\mu_{12})} \frac{\rho(\mu_{24}, \mu_{13}')\, d\mu_{13}'}{\mu_{13}' - \mu_{13}}, \qquad (5)$$

$$\rho = 2\pi\,[\sqrt{K(\mu_{ik})}]^{-1}. \qquad (6)$$

$A_1(\mu_{ik})$ and $K(\mu_{ik})$ are defined by the formulas:

$$A_1 = \frac{1}{\sqrt{K(\mu_{ik})}} \ln \frac{V + \sqrt{(\mu_{13}^2 - 1)K(\mu_{ik})}}{V - \sqrt{(\mu_{13}^2 - 1)K(\mu_{ik})}},$$

$$V = (\mu_{13} - 1)\,[\mu_{24}(\mu_{13} + 1) - \mu(\mu_{12} + \mu)]; \qquad (7)$$

$$K(\mu_{ik}) = (\mu_{13}^2 - 1)(\mu_{24}^2 - 1) - 2\mu(\mu_{13} - 1)(\mu_{24} - 1)(\mu + \mu_{12})$$

$$+ (\mu^2 - 1)(\mu - \mu_{12})^2. \qquad (8)$$

As compared with the analogous formulas I (3) and I (5) we have changed the sign in Eq. (7) and in the first term of Eq. (4). In the integral (5) $\sqrt{K} = i\,|\sqrt{K}\,|$.

For $\mu_{24} < \Delta^{(-)}(\mu_{12})$ [cf. Eq. I (55)] we have

$$A(\mu_{13}, \mu_{24}) = \frac{1}{\pi^2}\iint d\mu_{13}'\, d\mu_{24}' \frac{\rho(\mu_{13}', \mu_{24}')}{(\mu_{13}' - \mu_{13})(\mu_{24}' - \mu_{24})}$$

$$+ A_{\text{an}}'(\mu_{13}, \mu_{24}), \qquad (9)$$

$$A_{\text{an}}'(\mu_{13}, \mu_{24}) = -\frac{i}{\pi}\int_{\Delta^{(-)}}^{\Delta^{(-)}(\mu_{12})} \frac{\rho(\mu_{13}', \mu_{24})\, d\mu_{13}'}{\mu_{13}' - \mu_{13}}$$

$$-\frac{2i}{\pi}\int_{\square^{(-)}(\mu_{24})}^{\Delta^{(-)}} \frac{\rho(\mu_{13}', \mu_{24})\, d\mu_{13}'}{\mu_{13}' - \mu_{13}} \qquad (10)$$

for $\mu_{24} > \mu_{24}^0$ [$\mu_{24}^0$ is the point of reversal of the curve of $\square_{II}(\mu_{13})$, cf. Fig. 2], and

$$A_{\text{an}}' = -i\frac{1}{\pi}\int_{\Delta^{(-)}}^{\Delta^{(-)}(\mu_{12})} \frac{\rho(\mu_{13}', \mu_{24})\, d\mu_{13}'}{\mu_{13}' - \mu_{13}} \qquad (11)$$

with $\mu_{24} < \mu_{24}^0$.

In the first term of Eq. (9) the integration is taken over the region inside the curve II and $\sqrt{K} > 0$. In Eqs. (10) and (11) $\sqrt{K} = -i\,|\sqrt{K}\,|$. As was shown in I, with these definitions curve II is not a singularity of the amplitude A.

**3.** The formulas (4) — (11) are analytic in $\mu_{12}$, and therefore we can continue them in this variable and obtain a representation of the amplitude A for the decay case. The point $\mu_{12} = -1$ is obviously a singularity of the function A, and therefore the passages around this point by the paths $\mu_{12} \pm i\epsilon$ lead to different results. The Feynman amplitude (with negative imaginary quantities added to the internal masses) corresponds to the path $\mu_{12} - i\epsilon$. With this path all of the formulas (4) — (11) hold as before, but the point $\Delta^{(-)}(\mu_{12})$ of Eq. (3) goes off into the upper half-plane. This means that in the variable $\mu_{13}$ (and by symmetry also in $\mu_{24}$) the Feynman amplitude A for the diagram of Fig. 1 with the decay mass has a complex singularity in the upper half-plane. Thus there is no region on the real axis of $\mu_{13}$ in which

Im A = 0, and the dispersion relation in $\mu_{13}$ contains an integration over a complex path. Since the singularity lies in the upper half-plane and there are no complex singularities in the lower half-plane, we can write instead of the relations (4) — (11) with a complex path relations on the real axis of $\mu_{13}$, but with infinite limits.

Let us examine the case $\mu_{24} > 1$. Then we have from Eqs. (4) and (5)

$$A(\mu_{13}, \mu_{24}) = -\frac{1}{\pi}\int_{-\infty}^{-1}\frac{A_1(\mu'_{13}, \mu_{24})\,d\mu'_{13}}{\mu'_{13}-\mu_{13}} - \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{A_d(\mu'_{13}, \mu_{24})\,d\mu'_{13}}{\mu'_{13}-\mu_{13}}, \qquad (12)$$

$$A_d(\mu_{ik}) = \operatorname{Im} A_{\mathrm{an}} \quad (\mu_{13} = \mu_{13} - i\varepsilon). \qquad (13)$$

The specific expression for $A_d(\mu_{ik})$ for the diagram of Fig. 1 can be obtained directly from Eq. (13) if we note that it follows from Eqs. (5) and (13) that:

$$A_d(\mu_{13}, \mu_{24}) = -\frac{i}{\pi}\int_{\Delta^{*(-)}(\mu_{12})}^{\Delta^{(-)}(\mu_{12})}\frac{\rho(\mu'_{13}, \mu_{24})\,d\mu'_{13}}{\mu'_{13}-\mu_{13}}, \qquad (14)$$

if in Eq. (5) the path from the point $-1$ runs off at once into the complex region.

The calculation of the integral (14) is not especially difficult, though it is somewhat cumbersome. We are much more interested, however, in the physical meaning of $A_d(\mu_{ik})$ than in the concrete expression, and therefore we shall conduct the calculation of $A_d(\mu_{ik})$ in a different way, which leads to the same result as Eq. (14), but is much more closely connected with the physics.

The region $\mu_{24} > 1$ and $\mu_{13} > -1$ (region I of Fig. 2) is the physical region of the reaction in which particles with momenta $p_{12}$ and $p_{23}$ are converted into particles with momenta $p_{34}$ and $p_{14}$ ($p_{12}, p_{23} \to p_{34}, p_{14}$). In this region, for $\mu_{12} < -1$, the imaginary part of the amplitude A can be found from the relation of unitarity and is equal to the sum of the absorption parts which correspond to the fact that the process goes through real intermediate states with particles (1, 3) and (1, 2) (Fig. 3, a and b). (In Fig. 3 the lines that separate the initial and final states in the unitarity condition are marked with crosses.)

The state (1, 3) leads to the appearance of the absorption part $A_1$, which is given by Eq. (7) in the region $\mu_{13} < -1$ and is zero for $\mu_{13} > -1$ (Fig. 3, a). Therefore $A_d$ in the physical region $\mu_{13} < -1$ is the decay absorption part with the intermediate state (1, 2) (Fig. 3, b). For $\mu_{13} > -1$,

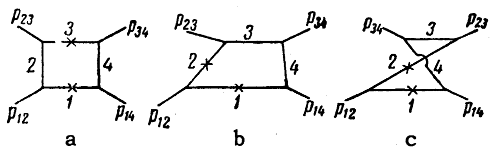

FIG. 3

$\mu_{24} > 1$ there is no physical region of the direct reaction ($p_{12}, p_{23} \to p_{34}, p_{14}$), but here (in region IV, Fig. 2) there is a physical region of the reaction ($p_{14}, p_{23} \to p_{34}, p_{12}$). The diagram of Fig. 1 has no absorption part in the energy of the crossing reaction, $(p_{23} + p_{14})^2$ but in this region it does have a decay absorption part $A_d$ (Fig. 3, c), which is then the imaginary part of A.

In the two physical regions we can find $A_d$ from the unitarity condition, and of course get the same expression in both,

$$A_d = \frac{1}{\sqrt{K(\mu_{ik})}}\ln\frac{V_d + \sqrt{(\mu_{12}^2-1)K(\mu_{ik})}}{V_d - \sqrt{(\mu_{12}^2-1)K(\mu_{ik})}},$$

$$V_d = \mu_{12}\mu_{13}\mu_{24} + \mu(\mu_{13}+\mu_{24}) + \mu[\mu_{12}(\mu-\mu_{12})+1]. \qquad (15)$$

Outside the physical regions the imaginary part of A will coincide with the sum of the absorption parts wherever the analytic continuations of $A_1$ and $A_d$ are real. As is shown in I, $A_1$ is real right up to the curve II (Fig. 2). As for $A_d$, if the expression (15) is defined as real in one of the regions I or IV, it must be complex in the other. Since in the physical regions $A_d$ must be a real function, we cannot define the decay absorption part as the expression (15) with the principal value of the logarithm in region I (or IV) and as its analytic continuation in region IV (or I). But if we take $A_d$ as real in both regions I and IV, as follows from the unitarity condition, then in the intermediate region $-1 < \mu_{13} < 1$ we get different values of $A_d$, depending on which region, $\mu_{13} > 1$ or $\mu_{13} < -1$, we start from for the continuation of the expression (15).

This, however, is only a seeming ambiguity. The formulas (4), (5), (12), and (14) do not contain it. The ambiguity is due to the fact that in the term $A_{\mathrm{an}}$ in Eqs. (4) and (5) we can integrate over paths leaving the real axis at different points, i.e., in Eq. (5) we can separate the integrals from the real axis $-1$ to some point $\tilde{\mu}_{13}$. In Eq. (14) this would correspond to different points of intersection of the path of integration with the real axis. This would also lead to redefinition of $A_1$ and $A_d$ and to different points $\tilde{\mu}_{13}$, up to which the decay absorption part (15) is defined as the analytic continuation from the right or from the left. If we have the formulas (4), (5), and (10), then the redefinition of $A_d$, Eq. (15), occurs at the point $\mu_{13} = -1$.* In the decay case the point $\mu_{13} = -1$ is the only remaining real singularity of A. We can redefine the point $\mu_{13} = \Delta^{(-)}$. Then

$$A(\mu_{13}, \mu_{24}) = -\frac{1}{\pi}\int_{-\infty}^{-1} \frac{A_1(\mu_{13}', \mu_{24})\, d\mu_{13}'}{\mu_{13}' - \mu_{13}} - \frac{i}{\pi}\int_{-1}^{\Delta^{(-)}} \frac{\rho(\mu_{13}', \mu_{24})\, d\mu_{13}'}{\mu_{13}' - \mu_{13}}$$

$$-\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{A_d'(\mu_{13}', \mu_{24})\, d\mu_{13}'}{\mu_{13}' - \mu_{13}}. \qquad (16)$$

The formula (16) has the convenience that its first two terms are easily continued into the region $\mu_{24} < 1$, and give the first term in Eq. (9). Since all of the formulas remain analytic up to $\mu_{24} \geq \Delta^{(-)}$, Eqs. (12) and (16), with $A_1$ and $A_d$ defined by Eqs. (7) and (15), remain valid in this entire region.

4. In the region $\mu_{24} < \Delta^{(-)}$ let us first consider values $\mu_{24} < -1$, since they are of the greatest interest. Using the same transformation as in Eq. (12) on the integral (10) over a complex path, to convert it into an integral over $\mu_{13}$ from $-\infty$ to $+\infty$, we can identify the various terms in the imaginary part of A with the absorption parts that arise in the physical regions.

The physical region for the reaction of decay of particle $p_{12}$ into $p_{14}$, $p_{23}$, $p_{34}$ is in the lower left quadrant of Fig. 2. In this region the imaginary part of A is given by three absorption parts $A_d$, $A_1$, and $A_2$ (Fig. 4, a, b, c). The concrete forms of $A_1$ and $A_d$ for the diagram in question are given by Eqs. (7) and (15), where the logarithm is defined as the principal value inside curve II of Fig. 2. The quantity $A_2$ (Fig. 4, c) is the absorption part that arises owing to the fact that for $\mu_{24} < -1$ the decay can go through a real intermediate state with particles 2 and 4. The concrete form of $A_2$ is obtained from Eq. (7) by the interchange $\mu_{13} \rightleftarrows \mu_{24}$. In the lower right quadrant there is the physical region of the reaction $(p_{23}, p_{34} \to p_{12}, p_{14})$, in which the nonvanishing absorption parts are $A_2$ and $A_d$ (Fig. 5, a, b). Thus for $\mu_{24} < -1$, Eq. (9) is equivalent to the following formula:

$$A(\mu_{13}, \mu_{24}) = -\frac{1}{\pi}\int_{-\infty}^{-1} \frac{A_1(\mu_{13}', \mu_{24})\, d\mu_{13}'}{\mu_{13}' - \mu_{13}}$$

$$-\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{[A_d(\mu_{13}', \mu_{24}) + A_2(\mu_{13}', \mu_{24})]\, d\mu_{13}'}{\mu_{13}' - \mu_{13}}. \qquad (17)$$

*As we go around the point $\mu_{13} = -1$ the integral (5) gets just the added imaginary part that returns the logarithm (15) to its principal value.

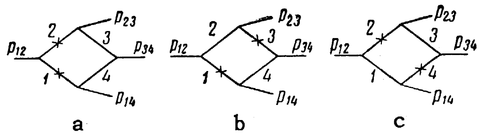

FIG. 4

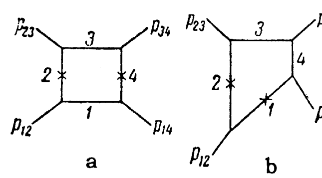

FIG. 5

It can be shown that the formulas (9) — (11) correspond to the definition of $A_d$ and $A_2$ in which the analytic continuation of Eqs. (7) and (15) from both physical regions is taken up to the point $\mu_{13} = \Delta^{(-)}$.

Finally, in the region $\Delta^{(-)} > \mu_{24} > -1$ the formula (12) is retained, with $A_1$ and $A_d$ defined by continuation with respect to $\mu_{24}$ from the regions II and III; for $A_d$ the redefinition in terms of $\mu_{13}$ is accomplished just as in Eq. (17) for $\mu_{13} = \Delta^{(-)}$.

**5.** If the masses of the particles do not satisfy Eq. (1), but, as before, only one mass has a value permitting decay, then there is nothing new in principle as compared with the case treated in I. The changes correspond entirely to the complications that arise in I, and are explained by the unsymmetrical position of the curves I — IV and the appearance of singularities $\Delta^{(-)}$ of A, which correspond to anomalous masses. This leads to a change of the points of redefinition of the absorption parts (for example, for $\mu_{24} > 1$ the point in question does not coincide with the point $\mu_{13} = -1$, but is equal to one of the $\Delta^{(-)}$). In formulas of the type of Eq. (12) or Eq. (17) there are anomalous added quantities from the other masses (see I).

If we have a diagram with two masses $\mu_{ik} < -1$, another decay absorption part appears, and makes a contribution in Eqs. (12) and (17). If the decay masses are $\mu_{12}$ and $\mu_{23}$ (two adjacent masses), then the singularity of $\Delta(\mu_{12}, \mu_{23})$ in the variable $\mu_{13}$ is on the real axis, and the singularities in $\mu_{24}$ go off into the complex plane. As before, the integral (5) will be taken along a real path, but is the integral of a complex function, since in this case the singularity of $\rho$ lies on the path of integration in Eq. (5). The integral (5) can again be transformed into an integral from $-\infty$ to $+\infty$ of the sum of two decay absorption parts. The same result is obtained in the case of masses located along a diagonal.

In a certain sense the decay case is even simpler and more intuitive than the anomalous case (cf. I). This is due to the fact that it is more closely connected with the unitarity relation, and it is from this relation that all of the dispersion parts that occur in the dispersion relations for the decay masses are obtained. This connection is possible owing to the fact that there is no region of real $\mu_{13}$ and $\mu_{24}$ which would be separated by singularities from all the physical regions. In the anomalous case there is such a region, and the analytic continuation of the unitarity condition into it is difficult.

It is just for this reason that the physical interpretation[2] of the anomalous added terms is not directly connected with the unitarity condition.

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[2] Y. Nambu, Nuovo cimento **9**, 610 (1958). R. Oehme, Phys. Rev. **111**, 1430 (1958).

Translated by W. H. Furry
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