CONTRIBUTION TO THE THEORY OF RELATIVISTIC COULOMB SCATTERING I.

V. G. GORSHKOV

Leningrad Physico-Technical Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 28, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 977-984 (September, 1961)

The cross section for scattering of charged spinor particles in the Coulomb field of a nucleus is calculated by taking into account the finite size of the nucleus and screening. The relative accuracy of the calculation is of the order of $(aZ)^2$. The calculations are all based on the Furry-Sommerfield-Maue function and lead to an analytic formula for the cross section.

1. INTRODUCTION

THE cross section of elastic scattering of a relativistic spinor particle in a Coulomb field can be expressed in the form of a series in the partial moments.[1] This series is not summed in the general case, and leads to numerical results with considerable difficulty. If only terms of order $(\alpha Z)^3$ are retained, this series can be summed,[2] the sum being an exceedingly simple analytic function. It proved possible to obtain the same formula in the Born approximation.[3]

Attempts to include terms of order $(\alpha Z)^4$ were made by Mitter and Urban,[4] by perturbation theory, but the infrared divergences were incorrectly eliminated. The results obtained in that paper are presented in the form of cumbersome double integrals. In the nonrelativistic case, a proof of the elimination of the infrared divergences in terms of order $(\alpha Z)^4$ was given by Kacser.[5]

In the present paper we develop a method in which terms of order $(\alpha Z)^4$ can be accounted for by using the Furry-Sommerfeld-Maue function. The radiation corrections, which are significant only at very small values of Z $(Z \sim 1)$, are neglected. In part I of the paper, the finite nuclear dimensions and screening are taken into account in first approximation. We consider the conditions under which this is sufficient for the employed accuracy in $(\alpha Z)^4$. In part II of the article the finite nuclear dimensions and the screening will also be accounted for in the second approximation, and the results of numerical calculations will be given for certain values of the parameters. The final expression for the cross section can be represented in both cases in analytic form.

In a recent paper by Johnson, Weber, and Mullin[6] analogous calculations are made for a pure Coulomb potential using a somewhat different method. In the region covered by both investigations, the results are in agreement apart from algebraic transformations.

2. GENERAL EXPRESSION FOR THE SCATTERING AMPLITUDE

The scattering amplitude of a Dirac particle in an external statistical field V has the following form (see [7])

$$F(\mathbf{k}, \mathbf{p}) = 2\pi^2 \bar{u}_{\mathbf{k}} \langle \mathbf{k} | \hat{V} | \psi_{\mathbf{p}} \rangle u_{\mathbf{p}} = \bar{u}_{\mathbf{k}} f(\mathbf{k}, \mathbf{p}) u_{\mathbf{p}}, \qquad (1)$$

where $\hat{V} = \gamma_4 V$, $k^2 = p^2 = E^2 - m^2$, $\hbar = c = 1$; and p and q are the momenta of the incoming and outgoing particles. The function $|\mathbf{k}\rangle u_{\mathbf{k}}$ is a solution of the free Dirac equation, while the function $|\psi_{\mathbf{p}}\rangle u_{\mathbf{p}}$ is the solution of the Dirac equation in a field V asymptotically represented by a plane and diverging wave

$$|\psi_{\mathbf{p}}\rangle = |\mathbf{p}\rangle + G\hat{V} | \psi_{\mathbf{p}}\rangle = R | \mathbf{p}\rangle,$$
$$\langle \mathbf{f} | G | \mathbf{s}\rangle = \frac{m - i\hat{f}}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon}\, \delta(\mathbf{f} - \mathbf{s}), \qquad (2)$$
$$R = (1 - G\hat{V})^{-1} = \sum_{n=0}^{\infty} (G\hat{V})^n;$$
$$\hat{V}R = R^+\hat{V} \quad (f_0 = E; \quad \hat{f} = \gamma_i f_i). \qquad (2a)$$

Assume that the field V can be represented as a sum of a strong field V_1 and a weak field V_2. We introduce the solution of the Dirac equation in the field V_1:

$$|\varphi_{\mathbf{p}}\rangle = |\mathbf{p}\rangle + G\hat{V}_1 |\varphi_{\mathbf{p}}\rangle = R_1 |\mathbf{p}\rangle, \qquad (3)$$

where

$$R_1 = 1 + G\hat{V}_1 R_1 = (1 - G\hat{V}_1)^{-1}, \quad \hat{V}_1 R_1 = R_1^+ \hat{V}_1. \qquad (3a)$$

It is then easy to verify that the following equation holds:

$$R = R_1 + R_1 G\hat{V}_2 R = R_1 + RG\hat{V}_2 R_1. \quad (4)$$

Taking (2a), (3a), and (4) into account we can obtain the following expression for the amplitude (1):

$$f(\mathbf{k}, \mathbf{p}) = f_1(\mathbf{k}, \mathbf{p}) + f_2(\mathbf{k}, \mathbf{p}), \quad (5)$$

$$f_1(\mathbf{k}, \mathbf{p}) = 2\pi^2 \langle \mathbf{k} | \hat{V}_1 | \varphi_\mathbf{p} \rangle, \quad (5a)$$

$$f_2(\mathbf{k}, \mathbf{p}) = 2\pi^2 \{\langle \varphi_\mathbf{k} | \hat{V}_2 | \varphi_\mathbf{p} \rangle + \langle \varphi_\mathbf{k} | \hat{V}_2 R_1 G \hat{V}_2 | \varphi_\mathbf{p} \rangle + \langle \mathbf{k} | \hat{V}_2 RG\hat{V}_2 R_1 G \hat{V}_2 | \varphi_\mathbf{p} \rangle\}, \quad (5b)$$

where $\langle \psi_\mathbf{k} | = \langle \mathbf{k} | R_1^+$ corresponds to a function asymptotically represented by a plane and converging wave.

As shown earlier,[8] the function $| \varphi_\mathbf{p} \rangle$ can be represented in the first potential by

$$| \varphi_\mathbf{p} \rangle = | \varphi_\mathbf{p}^0 \rangle + | \Phi_\mathbf{p} \rangle, \quad (6)$$

with

$$| \varphi_\mathbf{p}^0 \rangle = | \mathbf{p} \rangle + G_0 V_1 | \varphi_\mathbf{p}^0 \rangle,$$

$$\langle \mathbf{f} | G_0 | \mathbf{s} \rangle = \frac{2E}{\mathbf{f}^2 - \mathbf{p}^2 - i\varepsilon} \delta(\mathbf{f} - \mathbf{s}), \quad (6a)$$

$$| \Phi_\mathbf{p} \rangle = | \varphi_\mathbf{p}^I \rangle + G\hat{V}_1 | \Phi_\mathbf{p} \rangle = R_1 | \varphi_\mathbf{p}^I \rangle, \quad (6b)$$

where

$$\langle \mathbf{f} | \varphi_\mathbf{p}^I \rangle = \frac{\tilde{q}_{fp}}{2E} \langle \mathbf{f} | \varphi_\mathbf{p}^0 \rangle,$$

$$\mathbf{q}_{fp} = \mathbf{f} - \mathbf{p}, \quad \tilde{q} = \boldsymbol{\alpha}\mathbf{q}.$$

We can then obtain for (5a) the expression

$$f_1(\mathbf{k}, \mathbf{p}) = 2\pi^2 \langle \mathbf{k} | \hat{V}_1 | \varphi_\mathbf{p} \rangle = 2\pi^2 \{\langle \mathbf{k} | \hat{V}_1 | \varphi_\mathbf{p}^0 \rangle + \langle \mathbf{k} | \hat{V}_1 | \Phi_\mathbf{p} \rangle\}. \quad (7)$$

The second term of (7) can be transformed with the aid of (3a) and (6b):

$$\langle \mathbf{k} | \hat{V}_1 | \Phi_\mathbf{p} \rangle = \langle \mathbf{k} | \hat{V}_1 R_1 | \varphi_\mathbf{p}^I \rangle = \langle \mathbf{k} | R^+ \hat{V}_1 | \varphi_\mathbf{p}^I \rangle = \langle \varphi_\mathbf{k} | \hat{V}_1 | \varphi_\mathbf{p}^I \rangle = \langle \varphi_\mathbf{k}^0 | \hat{V}_1 | \varphi_\mathbf{p}^I \rangle + \langle \varphi_\mathbf{k}^I | \hat{V}_1 R_1 | \varphi_\mathbf{p}^I \rangle. \quad (8)$$

Substituting (8) in (7) we obtain

$$f_1(\mathbf{k}, \mathbf{p}) = 2\pi^2 \{\langle \mathbf{k} | \hat{V}_1 | \varphi_\mathbf{p}^0 \rangle + \langle \varphi_\mathbf{k}^0 | \hat{V}_1 | \varphi_\mathbf{p}^I \rangle + \langle \varphi_\mathbf{k}^I | \hat{V} R_1 | \varphi_\mathbf{p}^I \rangle\}. \quad (9)$$

3. CHOICE OF POTENTIALS

As the total electrostatic potential of the nucleus with account of its finite dimensions and screening we can choose a potential in the form

$$V = (V_\mathrm{s} - V_\mathrm{C}) + V_\mathrm{n}, \quad (10)$$

where V_S is the screened potential of the point-like nucleus V_C is the Coulomb potential of the point-like nucleus and V_n is the potential of the nucleus with distributed charge.

Following Moliere,[9] we use for the first term of (10) in momentum space the expression

$$\langle \mathbf{f} | V_\mathrm{s} - V_\mathrm{C} | \mathbf{s} \rangle = -\alpha Z \sum_{i=1}^{4} a_i / 2\pi^2 (q_{fs}^2 + \lambda_i^2), \quad \lambda_i = \eta b_i, \quad (11)$$

$$a_1 = 0.10, \quad a_2 = 0.55, \quad a_3 = 0.35,$$

$$a_4 = -1, \quad \sum_{i=1}^{4} a_i = 0, \quad (11a)$$

$$b_1 = 6.0, \quad b_2 = 1.20, \quad b_3 = 0.30, \quad b_4 = 0,$$

$$\eta = mZ^{1/3}/121, \quad \alpha = e^2 = 1/137. \quad (11b)$$

Using, in analogy with Uberall,[10] the Laplace transform for the nuclear charge density*

$$r\rho(r) = \int_0^\infty \varphi(\lambda) e^{-\lambda r} d\lambda, \quad (12)$$

we can obtain an expression for V_n in the form

$$\langle \mathbf{f} | V_\mathrm{n} | \mathbf{s} \rangle = -\frac{\alpha}{2\pi^2} \frac{1}{q^2} \int \rho(r) e^{-i\mathbf{qr}} d^3r = -\frac{\alpha}{2\pi^2} \frac{1}{q^2} 4\pi \int_0^\infty \frac{\varphi(\lambda)}{q^2 + \lambda^2} d\lambda$$

$$= -\frac{\alpha}{2\pi^2} \left\{\frac{1}{q^2} 4\pi \int_0^\infty \frac{\varphi(\lambda)}{\lambda^2} d\lambda - 4\pi \int_0^\infty \frac{\varphi(\lambda)}{\lambda^2} \frac{1}{q^2 + \lambda^2} d\lambda\right\},$$

$$\mathbf{q} = \mathbf{f} - \mathbf{s}. \quad (13)$$

Recognizing that

$$4\pi \int \frac{\varphi(\lambda)}{\lambda^2} d\lambda = 4\pi \int_0^\infty \rho(r) r^2 dr = Z,$$

substituting (13) and (11) in (10), and choosing for V_1 the Coulomb potential of the nucleus, we obtain

$$\langle \mathbf{f} | V | \mathbf{s} \rangle = \langle \mathbf{f} | V_1 | \mathbf{s} \rangle + \langle \mathbf{f} | V_2 | \mathbf{s} \rangle, \quad (14)$$

$$\langle \mathbf{f} | V_1 | \mathbf{s} \rangle = -V(0), \quad \langle \mathbf{f} | V_2 | \mathbf{s} \rangle = -\gamma_i V(\lambda_i),$$

$$V(\lambda) = \frac{\alpha Z}{2\pi^2} \frac{1}{q_{fs}^2 + \lambda^2}, \quad (14a)$$

$$\gamma_i \equiv \sum_{i=1}^{4} a_i - \frac{4\pi}{Z} \int_0^\infty d\lambda_i \frac{\varphi(\lambda_i)}{\lambda_i^2}. \quad (14b)$$

Confining ourselves to a relative accuracy of order $(\alpha Z)^2$, we must include in the amplitude terms up to order $(\alpha Z)^3$ [since the zeroth term, which is the first term in (9), is of order (αZ)]. We can therefore replace R_1 by unity in the third term of (9).

In this part of the paper we confine ourselves to the case when the norm of the operator (14b) satisfies the inequality

$$\alpha Z > \| \gamma \| > (\alpha Z)^2. \quad (15)$$

In the computational accuracy employed it is therefore sufficient to include only the first (linear)

*The screening can, generally speaking, be accounted for in the same manner.

term of (5b). In the second part of the paper we shall consider the case

$$1 > \|\gamma\| > \alpha Z, \qquad (15a)$$

when the second term of (5b) must also be included.

Taking (15) into account, we can rewrite the first term of (5b) as

$$f_2(\mathbf{k}, \mathbf{p}) = 2\pi^2 \{\langle \varphi_\mathbf{k}^0 | \hat{V}_2 | \varphi_\mathbf{p}^0 \rangle + \langle \varphi_\mathbf{k}^0 | \hat{V}_2 | \varphi_\mathbf{p}^\mathrm{I} \rangle + \langle \varphi_\mathbf{k}^\mathrm{I} | \hat{V}_2 | \varphi_\mathbf{p}^0 \rangle\}. \qquad (16)$$

4. CALCULATION OF THE SCATTERING AMPLITUDE

The functions $\varphi_\mathbf{p}^0$ and $\varphi_\mathbf{p}^\mathrm{I}$ in momentum space can be represented in the form[8]

$$\langle \mathbf{f} | \varphi_\mathbf{p}^0 \rangle = -\frac{\partial}{\partial \varepsilon} \Phi(\mathbf{q}, \mathbf{p}, \varepsilon)\Big|_{\varepsilon \to 0}, \quad \langle \varphi_\mathbf{p}^0 | \mathbf{f} \rangle = \langle \mathbf{f} | \varphi_\mathbf{p}^0 \rangle, \qquad (17a)$$

$$\langle \mathbf{f} | \varphi_\mathbf{p}^\mathrm{I} \rangle = \frac{\tilde{q}}{2E} \langle \mathbf{f} | \varphi_\mathbf{p}^0 \rangle = \alpha Z \frac{1}{-2i\xi} \tilde{\nabla}_p \Phi(\mathbf{q}, \mathbf{p}, 0),$$

$$\langle \varphi_\mathbf{p}^\mathrm{I} | \mathbf{f} \rangle = -\langle \mathbf{f} | \varphi_\mathbf{p}^\mathrm{I} \rangle, \qquad (17b)$$

$$\Phi(\mathbf{q}, \mathbf{p}, \varepsilon) = \frac{N}{2\pi^2} \frac{1}{2\pi i} \oint^{(0^+, 1^+)} \frac{dx}{x} \left(\frac{-x}{1-x}\right)^{i\xi} \frac{1}{(\mathbf{q} + \mathbf{p}x)^2 - (px + i\varepsilon)^2}; \qquad (17c)$$

$$\mathbf{q} = \mathbf{f} - \mathbf{p}, \quad \xi = \alpha Z E / p = \alpha Z / \beta, \quad N = e^{\pi\xi/2} \Gamma(1 - i\xi).$$

The integration contour in (17c) is a closed curve circling about the points 0 and 1 once in counterclockwise direction. With the aid of (17) we obtain the following expressions for the matrix elements (9) and (16) [see (14)]:

$$2\pi^2 \langle \varphi_\mathbf{k}^0 | \hat{V}(\lambda) | \varphi_\mathbf{p}^0 \rangle = \alpha Z N^2 \gamma_4 \left(-\frac{\partial}{\partial \varepsilon_2}\right)\left(-\frac{\partial}{\partial \varepsilon_1}\right) H(\mathbf{q}, \mathbf{k}, \mathbf{p}; \varepsilon_1, \varepsilon_2; \lambda)\big|_{\varepsilon_1, \varepsilon_2 \to 0}, \qquad (18a)$$

$$2\pi^2 \langle \varphi_\mathbf{k}^0 | \hat{V}(\lambda) | \varphi_\mathbf{p}^\mathrm{I} \rangle = (\alpha Z)^2 N^2 \gamma_4 \left(-\frac{\partial}{\partial \varepsilon_2}\right)\left(\frac{1}{-2i\xi_1} \tilde{\nabla}_p\right) H(\mathbf{q}, \mathbf{k}, \mathbf{p}; 0, \varepsilon_2; \lambda)\big|_{\varepsilon_2 \to 0}, \qquad (18b)$$

$$2\pi^2 \langle \varphi_\mathbf{k}^\mathrm{I} | \hat{V}(\lambda) | \varphi_\mathbf{p}^\mathrm{I} \rangle = (\alpha Z)^3 N^2 \gamma_4 \left(-\frac{1}{2i\xi_2} \tilde{\nabla}_k\right)\left(\frac{1}{-2i\xi_1} \tilde{\nabla}_p\right) H(\mathbf{q}, \mathbf{k}, \mathbf{p}; 0, 0; \lambda), \qquad (18c)$$

$$H(\mathbf{q}, \mathbf{k}, \mathbf{p}; \varepsilon_1, \varepsilon_2; \lambda) = \frac{1}{(2\pi i)^2} \oint^{(0^+, 1^+)} \frac{dx_2}{x_2} \left(\frac{-x_2}{1-x_2}\right)^{i\xi_2} \oint^{(0^+, 1^+)} \frac{dx_1}{x_1} \left(\frac{-x_1}{1-x_1}\right)^{i\xi_1} \frac{1}{(2\pi^2)^2}$$

$$\times \int \frac{d^3 f d^3 s}{[(\mathbf{q}_{kf} - \mathbf{k}x_2)^2 - (kx_2 + i\varepsilon_2)^2] (q_{fs}^2 + \lambda^2) [(\mathbf{q}_{sp} + \mathbf{p}x_1)^2 - (px_1 + i\varepsilon_1)^2]}. \qquad (18d)$$

In the appendix we transform the expression for $H(\mathbf{q}, \mathbf{k}, \mathbf{p}; \epsilon_1, \epsilon_2; \lambda)$ to the form

$$H(\mathbf{q}, \mathbf{k}, \mathbf{p}; \varepsilon_1, \varepsilon_2; \lambda) = \int_{\varepsilon_2}^{\infty} d\varepsilon_2' \int_{\varepsilon_1}^{\infty} d\varepsilon_1' K(\mathbf{q}, \mathbf{k}, \mathbf{p}, \eta), \qquad (19)$$

$$K(\mathbf{q}, \mathbf{k}, \mathbf{p}; \eta) = \frac{1}{2\pi i} \oint^{(0^+, 1^+)} \frac{dx}{x} \left(\frac{-x}{1-x}\right)^{i\xi_2} \left\{\frac{a + bx}{\alpha + \beta x}\right\}^{i\xi_2} \frac{1}{a + bx}, \qquad (19a)$$

where

$$a = q^2 + \eta^2, \quad \alpha = (\mathbf{q} + \mathbf{p})^2 - (p + i\eta)^2,$$

$$b = -2(\mathbf{qk} + ki\eta), \quad \beta = -2\{(\mathbf{q} + \mathbf{p})\mathbf{k} + k(p + i\eta)\};$$

$$\eta = \varepsilon_1' + \varepsilon_2' + \lambda \qquad (19b)$$

We note that $\mathbf{q}$ is constant when it comes to differentiation in (18b) and (18c).

The matrix element (18a) enters only in (16), and within the framework of the accuracy employed we should therefore confine ourselves in (18a) and (18b) to terms linear in ξ and in (18c) to the zeroth term. When $\xi_1 = 0$ the value of the integral in (19a) is given by the residue of the last term of the integrand. When $\xi_2 = 0$, the contour of integration can be contracted to zero and the integral becomes equivalent to $\delta(x)$. Taking these circumstances into account, we obtain

$$K(\mathbf{q}, \mathbf{k}, \mathbf{p}; \eta) = \frac{1}{a}\left(1 + 2i\xi \ln \frac{a}{\alpha}\right). \qquad (20a)$$

Calculating the gradient with respect to p under the integral sign in (18b) and then calculating the residues, we obtain

$$-\frac{\tilde{\nabla}_p}{2i\xi_1} K = \frac{1}{a\alpha}\left\{\tilde{C}_p + i\xi_1 \tilde{C}_p \ln \frac{a}{\alpha} - i\xi_2 \left[\tilde{C}_p \ln(1 + B) + \frac{\tilde{C}_p A + \tilde{D}}{A - B} \ln \frac{1 + A}{1 + B}\right]\right\}, \qquad (20b)$$

$$\mathbf{C}_p = \mathbf{q} - i\eta \mathbf{p}/p, \quad \mathbf{D} = \mathbf{k} + k\mathbf{p}/p, \quad A = b/a, \quad B = \beta/\alpha. \qquad (20c)$$

Finally, calculating the gradient with respect to k of (20b), we get

$$\left(-\frac{\tilde{\nabla}_k}{2i\xi_2}\right)\left(-\frac{\tilde{\nabla}_p}{2i\xi_1}\right) K = \frac{1}{2} \frac{1}{a\alpha} \left\{2 + \frac{1}{A - B} \ln \frac{1 + A}{1 + B}\right\}\left(1 - \frac{\tilde{k}\tilde{p}}{kp}\right). \qquad (20d)$$

Taking (20c) and (19b) into account, we substitute (20a), (20b), and (20c) into (18a), (18b), and (18c). Then, recognizing that the first term in (9) is the Schrödinger scattering amplitude in a Coulomb field* divided by 2E, and taking account of the identities

$$\bar{u}_\mathbf{k} \gamma_4 \tilde{k} u_\mathbf{p} = \bar{u}_\mathbf{k} \gamma_4 \tilde{p} u_\mathbf{p} = u_\mathbf{k} (\gamma_4 E - m) u_\mathbf{p}, \qquad (21a)$$

$$\bar{u}_\mathbf{k} \gamma_4 \tilde{k} \tilde{p} u_\mathbf{p} = \bar{u}_\mathbf{k} \gamma_4 (\gamma_4 E - m)^2 u_\mathbf{p}, \qquad (21b)$$

---

*We note that the scattering amplitude in a Coulomb field is multiplied by the distorting phase factor $\exp[-i\xi \ln(2pr)]$, but if an infinitesimally small screening is introduced, a factor $\exp[i\xi \ln(2p/\lambda)]$ appears in front of the entire wave function (λ is the screening parameter).[8] These two factors cancel each other.

we obtain the following expressions for the amplitudes (9) and (10)

$$f_1(\mathbf{k},\mathbf{p}) = \gamma_4\frac{\alpha Z}{q^2}\Big\{\exp(i\xi\, 2\ln\varepsilon) + \alpha Z M\frac{E-\gamma_4 m}{p}K_1(0) + (\alpha Z)^2\frac{M}{p^2}[2E(E-\gamma_4 m)K_2(0) + m(\gamma_4 E - m)K_3(0)]\Big\}e^{i\varphi}, \quad (22)$$

$$f_2(\mathbf{k},\mathbf{p}) = \gamma_4\frac{\alpha Z}{q^2}\gamma_i M\Big\{\frac{\varepsilon^2}{\varepsilon^2+\mu_i^2} + 2\alpha Z\Big[i\frac{E}{p}\frac{\varepsilon^2}{\varepsilon^2+\mu_i^2}\ln\frac{\varepsilon^2+\mu_i^2}{-i\mu_i(1+i\mu_i)} + \frac{E-\gamma_4 m}{p}K_1(\mu_i)\Big]\Big\}e^{i\varphi}, \quad (23)$$

where

$$\varepsilon = \frac{q}{2p} = \sin\frac{\vartheta}{2}, \quad \mu_i = \frac{\lambda_i}{2p},$$

$$M = -N^2 e^{-i\varphi} = 2\pi\xi/(1-e^{-2\pi\xi}),$$

$$e^{i\varphi} = -\frac{\Gamma(1-i\xi)}{\Gamma(1+i\xi)},$$

and γ_i is given by (14b). Furthermore,

$$K_1(\mu) = \varepsilon^2\int_\mu^\infty\frac{d\eta}{(1+i\eta)(\varepsilon^2+\eta^2)} = \frac{i\varepsilon}{2}\Big(\frac{1}{1-\varepsilon}\ln\frac{\varepsilon+i\mu}{1+i\mu} - \frac{1}{1+\varepsilon}\ln\frac{-\varepsilon+i\mu}{1+i\mu}\Big), \quad (24)$$

$$K_2(\mu) = \varepsilon^2\int_\mu^\infty\frac{id\mu}{(1+i\eta)(\varepsilon^2+\eta^2)}\ln\frac{\varepsilon^2+\eta^2}{-i\eta(1+i\eta)}$$
$$= \frac{\varepsilon^2}{1-\varepsilon^2}\Big\{L_2\Big(\frac{1+\varepsilon}{1+i\mu}\Big) + L_2\Big(\frac{1-\varepsilon}{1+i\mu}\Big) - L_2\Big(\frac{1}{1+i\mu}\Big)\Big\} + \frac{1}{2\varepsilon}\Big\{\frac{1}{1-\varepsilon}L_2\Big(\frac{\varepsilon}{\varepsilon+i\mu}\Big) - \frac{1}{1+\varepsilon}L_2\Big(\frac{\varepsilon}{\varepsilon-i\mu}\Big)$$
$$+\frac{1}{1-\varepsilon}L_2\Big(\frac{\varepsilon-1}{\varepsilon+i\mu}\Big) - \frac{1}{1+\varepsilon}L_2\Big(\frac{1+\varepsilon}{\varepsilon-i\mu}\Big) - \frac{1}{1-\varepsilon}L_2\Big(\frac{2\varepsilon}{\varepsilon+i\mu}\Big) + \frac{1}{1+\varepsilon}L_2\Big(\frac{2\varepsilon}{\varepsilon-i\mu}\Big)\Big\}, \quad (25)$$

$$K_3(\mu) = -\int_\mu^\infty\eta d\eta\Big(\frac{2\varepsilon^2}{i\eta(1+i\eta)(\varepsilon^2+\eta^2)} + \frac{1}{(1+i\eta)^2}\ln\frac{\varepsilon^2+\eta^2}{\eta^2}\Big)$$
$$= 2L_2\Big(\frac{1}{1+i\mu}\Big) - L_2\Big(\frac{1+\varepsilon}{1+i\mu}\Big) - L_2\Big(\frac{1-\varepsilon}{1+i\mu}\Big). \quad (26)^*$$

The integrals (24) and (26) are calculated by making the substitution

$$i\eta\to t, \quad \ln\frac{t+b}{t+a} = \int_a^b\frac{dx}{x+t},$$

where the integration along the imaginary axis in the complex t plane is replaced by integration along the real axis with all the singularities circuited from above. The function $L_2(z)$ contained in (25) and (26) is the logarithmic Euler function[11] (see also [5]):

$$L_2(z) = -\int_0^z\frac{\ln(1-t)}{t}dt = \sum_{n=0}^\infty\frac{z^n}{n^2}. \quad (27)$$

All the multiple-valued functions in (24) — (26) are specified in terms of their principal values on a plane with a cut from $z = 1$ to $z = +\infty$ along the positive axis, where $2\pi > \arg(z-1) > 0$.

The function $L_2(z)$ has the following properties, which make it readily possible to calculate its imaginary and real parts

$$L_2(z) + L_2\Big(\frac{1}{z}\Big) = \pi^2/3 - \tfrac{1}{2}\ln^2 z + \pi i\ln z, \quad (28a)$$

$$L_2(1-z) + L_2(z) = \pi^2/6 - \ln z\ln(1-z), \quad (28b)$$

$$\tfrac{1}{2}L_2(z^2) = L_2(z) + L_2(-z), \quad (28c)$$

$$\operatorname{Im}L_2(z) = 0 \quad \text{for} \quad -1\leqslant z\leqslant 1. \quad (28d)$$

5. SCATTERING CROSS SECTION IN THE PRESENCE OF INITIAL POLARIZATION

The cross section in the presence in the initial polarization ζ of the incoming particle is obtained from (1) with the aid of the formula[7]

$$\sigma(\vartheta) = \tfrac{1}{2}\operatorname{Sp}f(\mathbf{k},\mathbf{p})(1-i\hat{a}\gamma_5)(m-i\hat{p})\overline{f(\mathbf{k},\mathbf{p})}(m-i\hat{k}),$$
$$\bar{f} = \gamma_4 f^+\gamma_4, \quad \mathbf{a} = \boldsymbol{\zeta} + \frac{\mathbf{p}\boldsymbol{\zeta}}{m(m+E)}\mathbf{p}, \quad a_0 = \frac{\mathbf{p}\boldsymbol{\zeta}}{m}; \quad (29)$$

The expression for $f(k, p)$ is given by (5), (22), and (23).

After calculating the trace in (29) and transform (25) and (26) with the aid of (28a) and (28b), we obtain the following expression for the cross section:

$$\sigma = \sigma_R\{Q_1 + Q_2 + (R_1+R_2)\mathbf{n}\boldsymbol{\zeta}\},$$
$$\sigma_R = \Big(\frac{\xi}{2p}\Big)^2\frac{1}{\varepsilon^4}, \quad \mathbf{n} = \frac{[\mathbf{K}\,\mathbf{p}]}{\sin\vartheta}, \quad (30)^*$$

$$Q_1 = 1 - \beta^2\varepsilon^2 + \alpha Z\pi M\beta\varepsilon(1-\varepsilon) + 2(\alpha Z)^2 M\varepsilon\Big\{L_2(\varepsilon) - L_2(-\varepsilon) + \ln\varepsilon\Big(\ln\frac{1-\varepsilon}{1+\varepsilon} + \varepsilon\ln\varepsilon\Big) + \frac{\pi^2}{6}(2\varepsilon-3)$$
$$-\beta^2\varepsilon\Big[L_2(\varepsilon) + L_2(-\varepsilon) + \ln\varepsilon\Big(\ln(1-\varepsilon^2) - \frac{M}{2}\frac{\varepsilon^2}{1-\varepsilon^2}\ln\varepsilon\Big) - \frac{M}{2}\frac{\pi^2}{4}\frac{1-\varepsilon}{1+\varepsilon}\Big]\Big\}, \quad (31)$$

$$R_1 = -2\alpha Z M\varepsilon^3\Big(\frac{1-\beta^2}{1-\varepsilon^2}\Big)^{1/2}\Big\{\beta\ln\varepsilon + \pi\alpha Z\Big[\frac{2\ln 2}{\varepsilon} - \frac{1+\varepsilon}{\varepsilon^2}\ln(1+\varepsilon)\Big]\Big\}, \quad (32)$$

$$Q_2 = 2M\gamma_i\Big\{\frac{\varepsilon^2}{\varepsilon^2+\mu_i^2}(1-\beta^2\varepsilon^2) + \alpha Z\Big[\frac{\varepsilon^2}{\varepsilon^2+\mu_i^2}(1-\beta^2\varepsilon^2)\frac{1}{\beta}\times\Big(\pi\frac{M}{2}\frac{\varepsilon}{1+\varepsilon} - 2\operatorname{arctg}\frac{1}{\mu_i}\Big) + \beta\varepsilon\Big((1-\varepsilon)\operatorname{arctg}\frac{\mu_i(1+\varepsilon)}{-\varepsilon+\mu_i^2}$$
$$-(1+\varepsilon)\operatorname{arctg}\frac{\mu_i(1-\varepsilon)}{\varepsilon+\mu_i^2}\Big)\Big]\Big\}. \quad (33)\dagger$$

*The single integral (26) is obtained from the double integral (18c) by integration by parts.

*$[\mathbf{kp}] = \mathbf{k}\times\mathbf{p}$.

†$\operatorname{arctg} = \tan^{-1}$.

$$R_2 = -2\alpha ZM\gamma_i\beta(1-\beta^2)^{1/2}\frac{\varepsilon^3}{(1-\varepsilon^2)^{1/2}}\ln\frac{\varepsilon^2+\mu_i^2}{1+\mu_i^2}; \quad (34)$$

$$\gamma_i \equiv \sum_{i=1}^{4} a_i - \frac{4\pi}{Z}\int_0^\infty d\mu_i\,\frac{\varphi(2p\mu_i)}{2p\mu_i^2} \quad \text{(cf. (14) and (11))},$$

$$\varepsilon = \frac{q}{2p} = \sin\frac{\vartheta}{2},\quad \beta = \frac{p}{E},\quad M = \frac{2\pi\xi}{1-e^{-2\pi\xi}},$$

$$\xi = \frac{\alpha ZE}{p} = \frac{\alpha Z}{\beta},\quad \mu_i = \frac{\lambda_i}{2p}.$$

In the extreme relativistic case ($\beta \to 1$, $p \to \infty$), the screening potential drops out as a result of (11a), and (31) – (34) assume the form

$$Q_1^{cr} = 1-\varepsilon^2+\alpha Z\pi\varepsilon(1-\varepsilon)+2(\alpha Z)^2\varepsilon\Big\{(1-\varepsilon)L_2(\varepsilon) - (1+\varepsilon)L_2(-\varepsilon)+\ln\varepsilon\Big[(1-\varepsilon)\ln(1-\varepsilon) - (1+\varepsilon)\ln(1+\varepsilon)+\frac{\varepsilon(2-\varepsilon^2)}{2(1-\varepsilon^2)}\ln\varepsilon\Big]-\frac{\pi^2}{24}\frac{\varepsilon(1+7\varepsilon)}{1+\varepsilon}\Big\}, \quad (35)$$

$$Q_2^{cr} = 2\gamma\Big\{\frac{\varepsilon^2}{\varepsilon^2+\mu^2}+\alpha Z\Big[\frac{\varepsilon^2}{\varepsilon^2+\mu^2}\Big(\pi\frac{3\varepsilon+2}{2(1+\varepsilon)}-2\,\mathrm{arctg}\,\frac{1}{\mu}\Big) + \varepsilon\Big((1-\varepsilon)\,\mathrm{arctg}\,\frac{\mu(1+\varepsilon)}{-\varepsilon+\mu^2}-(1+\varepsilon)\,\mathrm{arctg}\,\frac{\mu(1-\varepsilon)}{\varepsilon+\mu^2}\Big)\Big]\Big\}; \quad (36)$$

$$\gamma = -\frac{4\pi}{Z}\int_0^\infty d\mu\,\frac{\varphi(2p\mu)}{2p\mu^2},\quad R_1^{cr} = R_2^{cr} = 0. \quad (37)$$

In conclusion, the author thanks Prof. L. A. Sliv for continuous support during the work.

APPENDIX

Using the formula (see [3] and [8])

$$\frac{1}{2\pi^2}\int\frac{d^3s}{(\mathbf{q}_{ns}^2-\Lambda^2)[(\mathbf{q}_{sp}+\mathbf{r})^2-(r+i\eta)^2]} = \frac{i}{2|\mathbf{q}_{np}+\mathbf{r}|}\ln\frac{|\mathbf{q}_{np}+\mathbf{r}|+r+\Lambda+i\eta}{|\mathbf{q}_{np}+\mathbf{r}|+r+\Lambda+i\eta} = \int_\eta^\infty\frac{d\lambda}{(\mathbf{q}_{np}+\mathbf{r})^2-(r+\Lambda+i\lambda)^2}, \quad (A.1)$$

We transform the three-dimensional integrals in (18d), after which we get

$$H(\mathbf{q},\mathbf{k},\mathbf{p};\varepsilon_1,\varepsilon_2;\lambda) = \frac{1}{(2\pi i)^2}\oint^{(0^+,1^+)}\frac{dx_2}{x_2}\Big(\frac{-x_2}{1-x_2}\Big)^{i\xi_2}\oint^{(0^+,1^+)}\frac{dx_1}{x_1}\Big(\frac{-x_1}{1-x_1}\Big)^{i\xi_1} \times\int_{\varepsilon_2}^\infty d\varepsilon_2'\int_{\varepsilon_1}^\infty d\varepsilon_1'\frac{1}{(\mathbf{q}+\mathbf{p}x_1-\mathbf{k}x_2)^2-(kx_2+px_1+i\eta)^2};$$

$$\mathbf{q} = \mathbf{k}-\mathbf{p},\quad \eta = \varepsilon_1+\varepsilon_2'+\lambda. \quad (A.2)$$

By evaluating the integral with respect to x_1 by residues and putting $x = x_2$, we obtain (19).

[1] N. F. Mott, Proc. Roy. Soc. **A124**, 425 (1929); **A135**, 429 (1932).

[2] W. A. McKinlay and H. Feshbach, Phys. Rev. **74**, 1759 (1948).

[3] R. H. Dalitz, Proc. Roy. Soc. **A206**, 509 (1951).

[4] H. Mitter and P. Urban, Acta Phys. Austriaca, **7**, 311 (1953).

[5] C. Kacser, Nuovo cimento **13**, 303 (1959).

[6] Johnson, Weber, and Mullin, Phys. Rev. **121**, 933 (1961).

[7] A. I. Akhiezer and V. B. Berestetskii, Kvantovaya élektrodinamika (Quantum Electrodynamics), 2d Ed. Fizmatgiz, 1959.

[8] V. G. Gorshkov, JETP **40**, 1481 (1961), Soviet Phys. JETP **13**, 1037 (1961).

[9] G. Moliere, Z. Naturforsch. **2a**, 133 (1947).

[10] H. Uberall, Phys. Rev. **119**, 365 (1960).

[11] H. Bateman, Higher Transcendental Functions, **1**, McGraw-Hill, (1953) p. 31.

Translated by J. G. Adashko
168