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THEORY OF SIMPLE FINITE-AMPLITUDE MAGNETOHYDRODYNAMIC WAVES IN A DISSIPATIVE MEDIUM

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The propagation of magnetic-sound waves of finite amplitude is analyzed by taking into account the dissipation of energy in the medium. The analysis method is based on a simplification of the initial magnetic hydrodynamic equations, which are valid for small initial perturbations and small energy dissipation. The concept of simple waves is extended to the case of dissipative media. The formation and "smearing out" of wave fronts are studied for different types of wave configurations. The spatial scales of the phenomena are determined.

1. INTRODUCTION

SIMPLE waves are known to play an important role in magnetohydrodynamics. Among the three types of simple waves existing in magnetohydrodynamics, greatest interest is attached to magnetic-sound waves. These are plane waves and if their velocity and magnetic-field vectors are specified in the xy plane at the initial instant, they remain in the same plane in the future. The theory of simple waves has been studied by several authors,[1-5] who integrated the system of hydrodynamic equations in the absence of energy dissipation and who investigated certain features in the propagation of fast and slow magnetic-sound waves. The dissipative terms of the equations of magnetohydrodynamics come into play only in connection with special problems in the structure of stationary shock waves. The most complete analysis of this type was made by Sirotina and Syrovat-skii.[6]

It is of interest, however, to analyze questions in the formation and "spreading" of discontinuities. This can be done only by examining "non-stationary" solutions, with account of energy-dissipation effects. Such an analysis has not yet been made for magnetohydrodynamics.

In the present paper we use an approximate method[7] to obtain solutions of the magnetohydrodynamic equations in the form of simple waves, with account of the dissipative terms of the equations. The method is based on the fact that the non-linearity of the medium and the energy dissipation in the medium are small. The solutions, which are carried to the second approximation, enable us to trace the spatial scales of the distortion of the simple waves for arbitrary orientation of the magnetic-field intensity vector **H**, and to study the mechanism of formation and "spreading" of the shock waves as well as to investigate their fronts. The relations obtained are applicable to the investigation of particular special limiting cases, the transition to which is very simple; the results agree with the data obtained previously by other methods.

2. FORMULATION OF THE PROBLEM AND DERIVATION OF THE APPROXIMATE EQUATIONS

Proceeding to an examination of the propagation of waves of finite amplitude in the half-plane xy, it is necessary in general to specify at the initial point ($x = 0$, $y = 0$) small perturbations of the velocities v_x and v_y, of the density ρ, of the pressure P, and of the magnetic field intensity h_y. Corresponding to these perturbations are accelerated and retarded magnetic-sound waves, which propagate without practically interacting with each other, owing to the difference in the phase velocities. In fact, in the first approximation (for infinitesimally small perturbations of v_x, v_y, ρ, P, and h_y) the following equation holds for the rate of propagation of the accelerated and retarded magnetic-sound waves $u_{1,2}$:[8]

$$u_{1,2}=\frac{1}{2}\left\{\left[u_0^2+\frac{H^2}{4\pi\rho_0}+\frac{H_xu_0}{\sqrt{\pi\rho_0}}\right]^{1/2}\pm\left[u_0^2+\frac{H^2}{4\pi\rho_0}-\frac{H_xu_0}{\sqrt{\pi\rho_0}}\right]^{1/2}\right\}. \quad (1)$$

Here $u_0 = \sqrt{\gamma P_0/\rho_0}$ is the velocity of sound, P_0 is the pressure in the unperturbed medium, ρ_0 is the density of the unperturbed medium, and $\gamma = c_p/c_v$

is the ratio of the specific heats at constant volume.

Thus, the accelerated and retarded magnetic-sound waves can always be regarded separately, except in the particular case $H_y = 0$ and $H_x^2 \approx 4\pi\rho_0 u_0^2$, so nonlinear interactions between waves can manifest themselves only in the second order of smallness compared with nonlinear self-action of the waves.

The problem consists of finding the solutions of the system of magnetohydrodynamic equations in the form of simple waves propagating in a direction $x > 0$, when small perturbations of the velocity, density, pressure, and magnetic-field intensity are specified in some definite manner at the initial point, and the radiation conditions are satisfied at infinity. It is actually sufficient, for example, to specify at the point $x = 0$, $y = 0$ the x-component of the velocity, v_x, and assume v_y, ρ, h_y, and P to be specified in accordance with the formulas for infinitesimally small perturbations.

Thus, considering small velocities v_x and v_y, small deviations of the density ρ' and small deviations of the magnetic field intensity h_y from their equilibrium values ρ_0 and h_0, and specifying the equation of state in the form

$$P = P_0 + u_0^2\rho' + p, \tag{2}$$

we must assume that $v_x, v_y \ll u_{1,2}$, $\rho' \ll \rho_0$, $h_y \ll H_0$ and $p \ll P$, or, introducing the small parameter μ,

$$\frac{v_x}{u_{1,2}},\ \frac{v_y}{u_{1,2}},\ \frac{\rho'}{\rho_0},\ \frac{h_y}{H_0} \sim \mu, \qquad p \sim \mu^2, \tag{3}$$

i.e., v_x, v_y, ρ' and h_y are of first order of smallness and p is of second order.

The bulk and shear viscosities η and ζ, the heat conduction κ, and the magnetic viscosity $\beta = c_0^2/4\pi\sigma$ (c_0 is the velocity of light and σ is the electric conductivity of the medium) are also assumed small quantities of the first order of smallness, i.e.,

$$\eta,\ \zeta,\ \varkappa,\ \beta \sim \mu. \tag{4}$$

In the case of infinitesimally small perturbations, i.e., in the first approximation, where the equations have no dissipative or nonlinear terms, the following solution holds true

$$v_x,\ v_y,\ \rho',\ h_y,\ P = F\,(t - x/u_{1,2}), \tag{5}$$

where F is an arbitrary function of its argument.

It is natural to assume that in the general case, under the assumptions (2) — (4) made above, the solution of the magnetohydrodynamic equations has essentially the form (5), but the form of the function F changes slowly with distance, i.e.,

$$v_x,\ v_y,\ \rho',\ h_y,\ P = F\,(\mu x,\ t - x/u_{1,2}). \tag{6}$$

By introducing the new variables $x' = \mu x$ and $\tau = t - x/u_{1,2}$ into the initial system of equations, we neglect everywhere the small terms of order μ^3 and higher. After simple but cumbersome transformations, the initial system reduces to the following four equations:

$$\frac{\partial\rho'}{\partial x} - \frac{1}{u_{1,2}}\left(1 - \frac{v_x}{u_{1,2}}\right)\frac{\partial\rho'}{\partial\tau} + \frac{\rho_0}{u_0^2}\left(1 - \gamma\frac{u_0^2\rho'}{u_{1,2}^2\rho_0} + \frac{u_0^2\rho'}{u_{1,2}^2\rho_0} - \frac{v_x}{u_{1,2}}\right)\frac{\partial v_x}{\partial\tau} + \frac{H_y\rho_0}{H_x u_0^2}\left(1 - \frac{v_x}{u_{1,2}}\right)\frac{\partial v_y}{\partial\tau} - \frac{h_y}{4\pi u_0^2 u_{1,2}}\frac{\partial h_y}{\partial\tau} - \frac{1}{u_0^2 u_{1,2}^2}\left(\frac{4}{3}\eta + \zeta\right)\frac{\partial^2 v_x}{\partial\tau^2} - \frac{1}{\rho_0 u_{1,2}^3}\frac{\gamma - 1}{\gamma}\frac{\varkappa}{c_v}\frac{\partial^2\rho'}{\partial\tau^2} - \frac{H_y}{u_0^2 u_{1,2}^2 H_x}\eta\frac{\partial^2 v_y}{\partial\tau^2} = 0, \tag{7}$$

$$\frac{\partial v_x}{\partial x} - \frac{1}{u_{1,2}}\left(1 + \frac{\rho'}{\rho_0}\right)\frac{\partial v_x}{\partial\tau} + \frac{1}{\rho_0}\left(1 - \frac{v_x}{u_{1,2}}\right)\frac{\partial\rho'}{\partial\tau} = 0, \tag{8}$$

$$\frac{\partial h_y}{\partial x} - \frac{1}{u_{1,2}}\left(1 - \frac{\rho'}{\rho_0}\right)\frac{\partial h_y}{\partial\tau} - \frac{4\pi\rho_0}{H_x}\left(1 - \frac{v_x}{u_{1,2}}\right)\frac{\partial v_y}{\partial\tau} + \frac{4\pi}{u_{1,2}^2 H_x}\eta\frac{\partial^2 v_y}{\partial\tau^2} = 0, \tag{9}$$

$$\frac{\partial v_y}{\partial x} - \frac{1}{u_{1,2}}\frac{\partial v_y}{\partial\tau} - \frac{1}{H_x}\left(1 - \frac{v_x}{u_{1,2}}\right)\frac{\partial h_y}{\partial\tau} + \frac{H_y}{H_x\rho_0}\left(1 - \frac{v_x}{u_{1,2}}\right)\frac{\partial\rho'}{\partial\tau} + \frac{H_y}{H_x u_{1,2}}\left(\frac{h_y}{H_y} - \frac{\rho'}{\rho_0}\right)\frac{\partial v_x}{\partial\tau} + \frac{\beta}{u_{1,2}^2 H_x}\frac{\partial^2 h_y}{\partial\tau^2} = 0. \tag{10}$$

We have left the x unprimed throughout Eqs. (7) — (10).

It is well known that in the case of simple velocity waves the density, the pressure, and the magnetic-field intensity are functions of one and the same combination of the independent variables x and t. In the first approximation it follows from (7) — (10) that one can assume a combination of independent variables $(t - x/u_{1,2})$ and express v_x, v_y, ρ' and h_y (and consequently also P) simply in terms of each other:

$$\rho' = \frac{\rho_0}{u_{1,2}}v_x, \qquad h_y = -\frac{4\pi\rho_0 u_{1,2}}{H_x}v_y,$$
$$\rho' = -\frac{H_y\rho_0}{u_{1,2}H_x(1 - u_0^2/u_{1,2}^2)}v_y. \tag{11}$$

In the second approximation, however, if we keep v_x, v_y, ρ' and h_y dependent on the same combination $(t - x/u_{1,2})$, we can no longer express the velocity, density, magnetic field intensity, and pressure in terms of each other by the simple relations (11). It is natural to assume that these relations must be supplemented by small second-order terms. These are the second-order quadratic terms, and the terms due to energy dissipation, which are proportional to the derivatives with respect to τ with coefficients of order μ. Although v_x, v_y, ρ', and h_y may not have at the initial point a geometrically similar distribution with respect to τ, upon propagation of the initial perturbation into the region $x > 0$ the wave distributions of the velocity, of the density, of the pressure, and of the magnetic-field intensity should all vary in similar fashion.

The foregoing arguments are confirmed by an analysis of the structure of Eqs. (7) — (10), which were derived without any special assumptions regarding the connections between v_x, v_y, ρ' and h_y.

Finally, the simplest relationships (11) should in the second approximation be replaced by the following equations, which are compatible with each other (apart from arbitrary constant coefficients yet to be determined):

$$\rho' = \frac{\rho_0}{u_{1,2}} v_x + \frac{\rho_0}{u_{1,2}} \beta_1 v_x^2 + \gamma_1 \frac{\partial v_x}{\partial \tau}, \tag{12a}$$

$$v_x = \frac{u_{1,2}}{\rho_0} \rho' - \frac{u_{1,2}^2}{\rho_0^2} \beta_1 \rho'^2 - \frac{u_{1,2}^2}{\rho_0^2} \gamma_1 \frac{\partial \rho'}{\partial \tau}, \tag{12b}$$

$$h_y = -\frac{4\pi\rho_0 u_{1,2}}{H_x} v_y + \frac{4\pi\rho_0 u_{1,2} H_y}{H_x^2 (1 - u_0^2 / u_{1,2}^2)} \beta_2 v_y^2 - \frac{4\pi u_{1,2}^2}{H_x} \gamma_2 \frac{\partial v_y}{\partial \tau}, \tag{12c}$$

$$v_y = -\frac{H_x}{4\pi\rho_0 u_{1,2}} h_y + \frac{H_x H_y}{(4\pi\rho_0 u_{1,2})^2 (1 - u_0^2 / u_{1,2}^2)} \beta_2 h_y^2 + \frac{H_x}{4\pi\rho_0^2} \gamma_2 \frac{\partial h_y}{\partial \tau}, \tag{12d}$$

$$\rho' = -\frac{H_y \rho_0}{u_{1,2} H_x (1 - u_0^2 / u_{1,2}^2)} v_y + \frac{H_y^2 \rho_0}{u_{1,2} H_x^2 (1 - u_0^2 / u_{1,2}^2)^2} \beta_3 v_y^2 - \frac{H_y}{H_x (1 - u_0^2 / u_{1,2}^2)} \gamma_3 \frac{\partial v_y}{\partial \tau}, \tag{12e}$$

$$v_y = -\frac{u_{1,2} H_x}{\rho_0 H_y} \left(1 - \frac{u_0^2}{u_{1,2}^2}\right) \rho' + \frac{u_{1,2}^2 H_x}{\rho_0^2 H_y} \left(1 - \frac{u_0^2}{u_{1,2}^2}\right) \beta_3 \rho'^2 + \frac{u_{1,2}^2 H_x}{\rho_0^2 H_y} \left(1 - \frac{u_0^2}{u_{1,2}^2}\right) \gamma_3 \frac{\partial \rho'}{\partial \tau}. \tag{12f}$$

Here β_i and γ_i are arbitrary constant coefficients, with γ_1, γ_2, and γ_3 all proportional to μ.

From the fact that v_x, v_y, ρ' and h_y are characteristics of a single wave process it follows that their x-variations should be described by identical equations. After substituting (12b) and (12f) into (7), (12a) into (8), (12d) into (9), as well as (12c) and (12e) in (10), and after replacing the second-order terms with the aid of relations (11), we can indeed reduce Eqs. (7) — (10) to an identical form. Simultaneously, by equating pairwise the coefficients of the nonlinear terms and the coefficients of the second derivatives we determine automatically and uniquely the values of β_i and γ_i.

It is now sufficient to solve one of the transformed equations, say the equation for the x component of the velocity, which has the form

$$\partial v_x / \partial x - \alpha v_x \partial v_x / \partial \tau = \delta \partial^2 v_x / \partial \tau^2, \tag{13}$$

$$\alpha = \frac{1}{2u_{1,2}^2} \left\{ (\gamma + 1) + \frac{(2 - \gamma)(u_{1,2}^2 - u_0^2)^2}{(u_{1,2}^2 - u_0^2)^2 + H_y^2 u_0^2 / 4\pi\rho_0} \right\}, \tag{14}$$

$$\delta = \left\{ (u_{1,2}^2 - u_0^2)^2 (\eta + \beta\rho_0) - (u_{1,2}^2 - u_0^2) \frac{H_y^2}{4\pi\rho_0} \eta + \frac{H_y^2}{4\pi\rho_0} \left[u_0^2 \frac{\gamma - 1}{\gamma} \frac{\varkappa}{c_v} + u_{1,2}^2 \left(\frac{4}{3} \eta + \zeta \right) \right] \right\} \times \left\{ 2\rho_0 u_{1,2} \left[(u_{1,2}^2 - u_0^2)^2 + \frac{H_y^2}{4\pi\rho_0} u_0^2 \right] \right\}^{-1}. \tag{15}$$

Assuming v_x to be some definite function of t at the initial point, we can readily determine the corresponding solution of (13), since the substitution

$$v_x = \frac{2\delta}{\alpha W} \frac{\partial W}{\partial \tau} \tag{16}$$

reduces this equation to the usual form of heat-conduction equation. In the next section we shall consider the solutions of equation (13) at different boundary conditions.

However, before we proceed to the analysis of specific physical processes, it is appropriate to make the following remark concerning the preceding derivation. By supplementing relations (11) with derivatives with respect to τ we essentially deviate somewhat from the analysis of simple waves in the strict sense of this word, for we assume along with the dependence on the arbitrary combination ($\tau = t - x/u_{1,2}$) also a dependence on the derivative with respect to this combination of independent variables, albeit with a coefficient of order μ, made up of dissipated coefficients. This deviation must actually be regarded as a generalization of the concept of simple waves to include the case of dissipative media.

3. INVESTIGATION OF THE PROPAGATION OF MAGNETIC-SOUND WAVES AND OF THE STRUCTURE OF SHOCK WAVES

The case of sinusoidal boundary conditions ($v_x = v_{0x} \sin \omega t$) was considered by the authors in detail in a solution of the acoustic problem.[7] It is expedient here to consider only very briefly the final results of an analogous analysis for magnetic-sound waves for the purpose of comparing them with results obtained in ordinary hydrodynamics. We shall determine in passing the magnetohydrodynamic analogues of the Mach and

Reynolds numbers, which we need for the subsequent analysis.

The entire region of propagation of magnetic-sound waves $(x > 0)$ can be subdivided into three sections. As the wave propagates towards the $x > 0$ direction the nonlinear effects bring about a distortion of the wave profile, so that at a point x_1, defined by the relation $kx_1 = 1/M$, a quasi-discontinuity is formed. Here $k = \omega/u_{1,2}$ is the wave number and $M = \frac{1}{2} f(u_{1,2}, u_0) v_{0x}/u_{1,2}$ is the magnetohydrodynamic analogue of the Mach number.* The function $f(u_{1,2}, u_0)$ is the expression in the curly brackets of (14), and is readily seen to be positive for all values of H and for any orientation of this field; its numerical value lies between $\gamma + 1$ and 3.

A comparison of x_1 with the analogous parameter of ordinary hydrodynamics, x_1^*, shows that

$$x_1 = x_1^* \frac{\gamma + 1}{f(u_{1,2}, u_0)} \left(\frac{u_{1,2}}{u_0}\right)^2 \qquad (17)$$

for equal initial perturbations v_{0x} and v_0. At the same time, by specifying a definite magnetic-field intensity vector, we can directly evaluate u_1 and u_2 graphically, as was done for example by Syrovat-skii,[9] and determine the characteristic points x_1 corresponding to these velocities simply in the form $x_1 \approx x_1^*(u_{1,2}/u_0)^2$.

The occurrence of the "discontinuity" is accompanied by a strong energy dissipation, which causes the quasi-discontinuous wave to be transformed in the second region $(x > x_1)$ into an exponentially damped harmonic wave of frequency ω. This process can be regarded as completed at the point x_2, defined by the relation $kx_2 = 4\text{Re}/M$, where $\text{Re} = \alpha v_{0x}/2\omega\delta$ is the magnetohydrodynamic analogue of the Reynolds number, and takes into account the simultaneous influence of the bulk and shear viscosities, the heat conduction, and the magnetic viscosity of the medium. It is important to note that neither the characteristic points x_2 nor the amplitude of the signal at these points depend on the amplitude at the input of the system, v_{0x}. We can derive for the points x_1 a formula analogous to (17), but with cubic dependence on the velocity ratio.

In the third region $(x > x_2)$ the propagation process can be described by the linear equations of magnetohydrodynamics, since the waves are already so weak that there are no nonlinear effects, and no reconversion of the sinusoidal wave into a shock wave takes place.

Without reporting the data obtained on the structure of the front of the shock wave, which are quite analogous to the earlier[7] solutions of the acoustic problem, we proceed directly to a determination of the solutions of greatest interest to magnetohydrodynamics.

Assume that at the initial point we are given the x component of velocity $v_x = v_{0x} \tanh(\tau/\tau_0)$, where $\tau_0 \gg (\alpha v_{0x}/2\delta)^{-1}$ and τ ranges from $-\infty$ to $+\infty$. Then the function W, which satisfies an equation similar to the heat-conduction equation, can be written in the form

$$W = \frac{1}{2\sqrt{\pi\delta x}} \int_{-\infty}^{\infty} \exp\left\{\frac{\tau_0}{\tau'} \int_0^{y/\tau_0} \text{th}\, z\, dz - \frac{(\tau - y)^2}{4\delta x}\right\} dy, \quad \tau' \equiv \frac{2\delta}{\alpha v_{0x}}. \qquad (18)^*$$

When $\tau_0/\tau' \gg 1$, i.e., at large magnetohydrodynamic Reynolds numbers, the integral (18) is calculated by the saddle-point method, so that from the relation defining the saddle point y_0 and from the value of v_x at this point [given by Eq. (16)] we can generally speaking find a solution in the form

$$\frac{\tau}{\tau_0} = \text{Arth}\,\Phi - \frac{\alpha v_{0x} x}{\tau_0}\Phi, \quad \Phi = \text{th}\frac{\tau}{\tau_0}. \qquad (19)‡$$

A graphic analysis of the solution obtained (Fig. 1) demonstrates quite clearly how the profile of the initial perturbation is distorted as the wave propagates. The degree of distortion of the initial perturbation is determined here by the value of the slope $X = \alpha v_{0x} x/\tau_0$, which increases in direct proportion to the distance covered by the wave from the entrance of the system. The "discontinuity" point corresponds to the distance $x_1 = \tau_0/\alpha v_{0x}$, and when $x > x_1$ the function Φ becomes multiple-valued, which is a physical absurdity. In this case, however, the solution (19) is itself not valid; the principal value of the in-

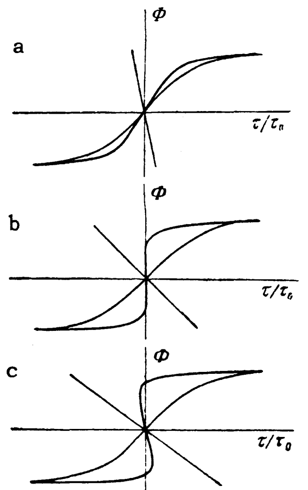

FIG. 1. Plot of the solutions of (19). $\tau/\tau_0 = f(\Phi)$ (heavy curve) is the sum of tanh Φ and a straight line with slope X: a – $|X| < 1$, b – $|X| = 1$, c – $|X| > 1$.

*In analogy with the acoustic analogue of the Mach number $M_{ac} = (\gamma + 1) v_0/2u_0$.

*th = tanh.
‡Arth = $\tanh^{-1}$.

tegral must now be calculated as the sum of its values over the small portions in the vicinity of the points y_1 and y_2, so that after calculating W_1 and W_2 and after substituting $W = W_1 + W_2$ in (16), the analysis of the form of v_x after formation of the "discontinuity" must be based on the expression

$$v_x = v_{0x}\frac{\operatorname{th}(y_1/\tau_0) + \operatorname{th}(y_2/\tau_0)\,e^{-Y}}{1+e^{-Y}}, \tag{20}$$

where

$$|Y| = \left|\frac{\tau_0}{\tau'}\left[\int\limits_0^{y_2/\tau_0}\operatorname{th} z\,dz - \int\limits_0^{y_1/\tau_0}\operatorname{th} z\,dz\right] + \frac{(\tau-y_2)^2-(\tau-y_1)^2}{4\delta x}\right|.$$

The value of $|Y|$ is easy to calculate if it is recognized that $(\tau - y_{1,2}) = \pm\alpha v_{0x}x$ at the saddle point, and if the integrals of the hyperbolic tangents are expanded in powers of τ and only the first approximation is used. Then $v_x = v_{0x}\tanh(\tau/\tau')$, where τ' is the duration of the shock-wave front, has the value 1/Re when reduced to dimensionless form. Thus, the width $L_f = u_{1,2}\,\tau_f$ of the shock-wave front established near the point x_1 remains stationary (unlike a sinusoidal front) and is given by the formula

$$L_f = u_{1,2}\tau' = 2\frac{u_{1,2}}{v_{0x}}\Big\{(u_{1,2}^2-u_0^2)^2[\eta+\beta\rho_0] - (u_{1,2}^2-u_0^2)\frac{H_y^2}{4\pi\rho_0}\eta + \frac{H_y^2}{4\pi\rho_0}\Big[u_0^2\frac{\gamma-1}{\gamma}\frac{\varkappa}{c_v} + u_{1,2}^2\Big(\frac{4}{3}\eta+\zeta\Big)\Big]\Big\} \times\Big\{\rho_0 u_{1,2}\Big[(\gamma+1)\frac{H_y^2}{4\pi\rho_0}u_0^2 + 3(u_{1,2}^2-u_0^2)^2\Big]\Big\}^{-1}. \tag{21}$$

Relation (21) goes in the limit into various particular magnetohydrodynamic discontinuities (parallel shock wave, perpendicular shock wave, singular oblique wave, etc.), which need not be analyzed here, since transitions of this type have been studied in detail by Sirotina and Syrovat-skii.[6] We need only point out here that the expressions determining the width of the front of the shock waves, obtained in [6] and in the present paper by different methods, coincide.

Particular interest is attached to an examination of the following boundary-value problems:

$$v_x = \begin{cases}-v_{0x}, & -\infty\leqslant\tau\leqslant 0,\\ +v_{0x}, & 0\leqslant\tau\leqslant+\infty,\end{cases} \tag{22}$$

i.e., when the x component of the velocity is specified at the initial point in the form of a discontinuous function with $L_f = u_{1,2}\tau_f = 0$. It is natural to expect the dissipative processes to predominate here from the very outset, so that the spatial scales of the established stationary width of the front will differ from those in the two preceding problems.

Thus, let the velocity v_x at the initial point be defined by (22). Then

$$W = \frac{1}{2\sqrt{\pi\delta x}}\left[\int\limits_{-\infty}^{0}\exp\left\{-\frac{y}{\tau'} - \frac{(\tau-y)^2}{4\delta x}\right\}dy + \int\limits_0^{\infty}\exp\left\{\frac{y}{\tau'} - \frac{(\tau-y)^2}{4\delta x}\right\}dy\right] \tag{23}$$

or, after reducing the exponential expressions to quadratic form, we obtain ($\tau' \equiv 2\delta/\alpha v_{0x}$)

$$W = \frac{1}{\sqrt{\pi}}\exp\left\{\frac{\alpha^2 v_{0x}^2 x}{4\delta}\right\}\left[e^{-\tau/\tau'}\int\limits_{Z_-}^{\infty}e^{-z^2}dz - e^{\tau/\tau'}\int\limits_{Z_+}^{-\infty}e^{-z^2}dz\right],$$

$$Z_{\mp} \equiv \frac{\tau \mp \alpha v_{0x}x}{2\sqrt{\delta x}}. \tag{24}$$

The analysis of the spatial variation of v_x, carried out in accordance with (16), is now more difficult. First, for x sufficiently large to satisfy the condition $x \geq \tau/\alpha v_{0x}$ in the region $\tau \geq 2\sqrt{\delta x}$, the x component of the velocity is simply given by

$$v_x = v_{0x}\operatorname{th}(\tau/\tau'). \tag{25}$$

The quantity τ', determines as before the duration of the front of the shock wave and its dimensionless value is 1/Re. Thus, at sufficiently large distances from the entrance to the system, the width of the shock-wave front displays no dependence on x and remains stationary.

Knowing the stationary duration of the front of the shock wave $\tau = \tau_{st}$, we can indicate a value x^1 beyond which $\tau = \tau_{st}$, namely:

$$x^1 = 2\delta/(\alpha v_{0x})^2. \tag{26}$$

We determine by the same token the interval $[0, x^1]$ within which the stationary width of the front of the shock wave is established.

It remains only to determine the variation of τ_f in the indicated interval. This variation can readily be ascertained by starting from the following considerations. When $x < \tau/\alpha v_{0x}$ and $|\tau|$ is sufficiently large, one of the integrals in (24) can be neglected as having equal limits, so that we obtain $v_x = |v_{0x}|$. Assuming this approximation to be possible when the lower limits of the integrals in (24) are greater than unity in absolute value, we can express τ as a function of x:

$$\tau = 2\sqrt{\delta x} + \alpha v_{0x}x. \tag{27}$$

However, even at the limiting point (26), the second term of formula (27) is $\sqrt{2}$ times less the first one, so that the law of establishment of the stationary width of the shock-wave front can be simply determined to be

$$L_f = u_{1,2}\tau_f = 2u_{1,2}\sqrt{\delta x}, \quad (28)$$

i.e., the growth in the width of the front of the shock wave is proportional to the square root of the distance from the entrance to the system covered by the wave before L_f reaches a stationary value.

Finally, it is interesting to consider the propagation of an arbitrary unit pulse, which for simplicity can be assumed to be triangular (Fig. 2a), in a certain interval $[0, \beta]$ such that the initial perturbation of the x component of the velocity remains of order μ while the area of the triangle P_{0x} remains constant, i.e.,

$$v_x = \begin{cases} 2P_{0x}\beta^{-1}(1-\tau/\beta), & 0 \leqslant \tau \leqslant \beta, \\ 0 & \tau < 0,\ \tau > \beta, \end{cases}$$

$$P_{0x} = \int_0^\beta v_{0x}\left(1 - \frac{\tau}{\beta}\right) d\tau. \quad (29)$$

An exact analytic expression for v_x, obtained in accordance with (16) after calculating W and the logarithmic derivative is, however, rather cumbersome. It is therefore advantageous, without writing out the expression, to plot the variation of v_x with τ for different distances from the entrance to the system, obtained on the basis of the exact solution.

First, for sufficiently small distances $x \leq x^1 = \delta\beta^2/2\,(\alpha P_{0x})^2$, the distortion of the pulse is such that its profile can be broken up into four characteristic regions (Fig. 2b), so the maximum of the pulse amplitude shifts to the right of the point $\tau = 0$ and there are simultaneously formed on its edges gently sloping portions (regions I and IV), on which the profile of the pulse is described by the following equations:*

$$v_x = \frac{4}{\sqrt{2\pi}}\frac{P_{0x}}{\beta^2}\sqrt{\delta x}\exp\left\{-\frac{\tau^2}{4\delta x}\right\} \text{ for } \tau < -2\sqrt{\delta x}, \quad (30)$$

$$v_x = \frac{4}{\sqrt{2\pi}}\frac{P_{0x}}{\beta^2}\sqrt{\delta x}\exp\left\{-\frac{(\beta-\tau)^2}{4\delta x}\right\} \text{ for } \tau > \beta. \quad (31)$$

The boundaries of the first and fourth regions are defined as $\tau_I = -4\sqrt{\delta x}$ and $\tau_{IV} = \beta + 2\sqrt{\delta x}$, respectively. The second region, corresponding to the values of τ within the interval $\pm 2\sqrt{\delta x}$ from the point $\tau = 0$, is the region of the leading front of the pulse. When $x = x^1$ this region reaches a value 1/Re (in dimensionless units). In the third region the deviation of the pulse from its initial configuration is still insignificant when $x \leq x^1$.

*The magnetohydrodynamic Reynolds number is assumed to be much greater than unity, Re ≫ 1.

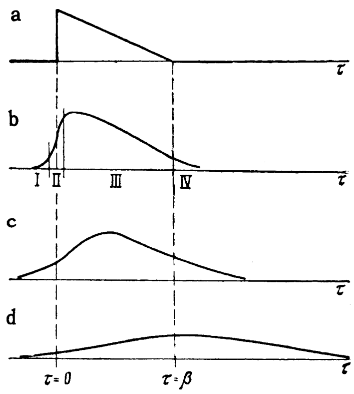

FIG. 2. Triangular unit pulse at different distances from the entrance to the system. a – $x = 0$, b – $x \leq x^1 = \delta\beta^2/2\,(\alpha P_{0x})^2$, c – $x^1 < x < x_2 = 4\,\mathrm{Re}/2\alpha P_{0x}\beta^{-2}$, d – $x \geq x_2$.

Further propagation of the pulse causes the maximum of v_x to shift in the direction $\tau \to \beta$ and broadens the regions I and IV and the region of the leading front at the expense of the third region (Fig. 2c), which is reduced. As the pulse attenuates, its form tends to be symmetrical, so that the duration of the front does not remain stationary, but increases as $\tau_f = 2\sqrt{\delta x}$, so long as the concept of τ_f retains in general a physical meaning.

At sufficiently large x, namely when $x \geq x_2 \approx \mathrm{Re}/2\alpha P_{0x}\beta^{-2}$ the following relation holds for all four regions (Fig. 2d):

$$v_x = \frac{2P_{0x}}{\beta}\exp\{-(\beta-\tau)^2/4\delta x\}/\sqrt{\mathrm{Re}}\sqrt{2\alpha P_{0x}x/\beta^2}. \quad (32)$$

The parameter x_2 is quite analogous here to the corresponding parameter in the boundary problem $v_x = v_{0x}\sin\omega t$, while the parameter x^1 corresponds to the analogous parameter of the boundary-value problem (22).

The last thing to be emphasized here is that the pulse configuration of Fig. 2b corresponds to any single pulse at a definite distance from the entrance to the system.

4. CONCLUSION

The system of magnetohydrodynamic equations, in the case of small nonlinearity and low energy dissipation in the medium, reduces to a system of like equations similar to (13). In the case of dissipative media, the simple waves are regarded here as waves in which the density, velocity, pressure, and magnetic-field intensity depend not simply on a definite combination of independent variables ($\tau = t - x/u_{1,2}$), but also on the derivative with respect to this combination. An investigation of an equation such as (13) for one of the velocity components of the magnetic-sound waves v_x or v_y, for the magnetic field intensity h_y, or for the density ρ' enables us to study the laws of propagation of waves with different initial forms.

Such an analysis of waves of various configurations has shown, first, that in the absence of a discontinuity at the initial point $(x = 0,\ y = 0)$, a discontinuity can be formed under certain conditions at a distance x_1, proportional to $1/M$, from the entrance of the system. Second, a discontinuity specified at the initial point becomes "spread out" as $\tau_f = 2\sqrt{\delta x}$, and attains at a distance $x^1 = 2\delta/(\alpha v_{0x})^2$ a dimensionless width $1/Re$. No further "smearing" of the front takes place only in the case of specially chosen functions (second and third boundary-value problems), for which $1/Re$ is thus found to be the stationary width of the front. Third, at a distance $x_2 \sim Re/M$, which is independent of the value of the initial perturbation, the amplitude of the wave is also independent of the amplitude at the entrance to the system, and wave propagation for $x > x_2$ can be described by the linear equations of magnetohydrodynamics.

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97