PLASMA IN A SELF-CONSISTENT MAGNETIC FIELD

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Submitted to JETP editor March 7, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 528-533 (August, 1961)

The stationary state of a multi-component plasma with axial symmetry in an external axial magnetic field is determined in the presence of an azimuthal plasma current by neglecting particle collisions and taking into account the proper magnetic field of the plasma. A Maxwellian velocity distribution is assumed with different longitudinal and transverse temperatures. Finally, the plasma is assumed to be neutral at each point.

In 1950 Tamm [1] considered certain general relations for a plasma located in constant electric and magnetic fields by starting from a particular form of the particle distribution function. In a number of works, [2-5] more detailed plasma configurations were considered with neglect of collisions; in particular, [3-5] the case of cylindrical symmetry for the azimuthal magnetic field H_φ and longitudinal plasma current J_z were considered. It was shown that stationary plasma states can exist in the self-consistent self-magnetic field.

In the research that is described (also in the absence of collisions), a second case is considered in which the magnetic field H_z is longitudinal and the field J_φ is azimuthal, and all the quantities are independent of t, z, φ. The particle distribution densities, the proper magnetic fields and the currents of the multi-component plasma are found, and the external magnetic fields are determined for which stationary states exist.

The problem reduces to a solution of the set of n kinetic equations and two Maxwell equations:

$$v_r \frac{\partial f_i}{\partial r} + \left(\frac{q_i}{m_i c} v_\varphi \frac{1}{r}\frac{\partial}{\partial r}(rA) - \frac{q_i}{m_i}\frac{\partial \varphi}{\partial r} + \frac{v_\varphi^2}{r}\right)\frac{\partial f_i}{\partial v_r} - \left(\frac{q_i}{m_i c} v_r \frac{1}{r}\frac{\partial}{\partial r}(rA) + \frac{v_\varphi v_r}{r}\right)\frac{\partial f_i}{\partial v_\varphi} = 0; \quad (1)$$

$$-\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA)\right] = \frac{4\pi}{c}\sum_i q_i \int v_\varphi f_i \, d\mathbf{v}, \quad (2)$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) = 4\pi \sum_i q_i \int f_i \, d\mathbf{v}. \quad (3)$$

Here q_i, m_i and f_i are respectively the charge, mass and distribution function of particles of the i-th type (i = 1, 2, ..., n); φ and A are the scalar and vector potentials, respectively.

The general solution of an equation of the form (1) is an arbitrary function of the first integrals of the equations of the characteristics:

$$C_{1i} = \tfrac{1}{2} m_i (v_r^2 + v_\varphi^2) + q_i \varphi, \qquad C_{2i} = r v_\varphi + (q_i / m_i c) A r.$$

As a boundary condition, we require that the distribution function $f_i(r, \mathbf{v})$ have the form

$$f_i(r_0, \mathbf{v}) = \frac{n_{0i}}{\pi^{3/2}} \frac{m_i^{3/2}}{2T_\perp^i \sqrt{2T_\parallel^i}} \times \exp\left\{-\frac{m_i}{2T_\perp^i}[v_r^2 + (v_\varphi - v_{\varphi 0}^i)^2] - \frac{m_i}{2T_\parallel^i} v_z^2\right\}$$

for any fixed value of $r = r_0$. Here n_{0i} and $v_{\varphi 0}^i$ are respectively the density and velocity of the directed motion of particles of the i-th type at the point r_0; $T_\perp^i$ and $T_\parallel^i$ are the transverse and longitudinal temperatures of the particles, expressed in energy units. Solving the problem for the chosen boundary condition, we find the distribution function

$$f_i(r, \mathbf{v}) = f_{0i} \exp\left(-\alpha_\perp^i C_{1i} - \beta_i m_i C_{2i} - \tfrac{1}{2}\alpha_\parallel^i m_i v_z^2\right);$$

$$f_{0i} = \frac{n_{0i}}{2^{3/2}\pi^{3/2}} m_i^{3/2} \alpha_\perp^i (\alpha_\parallel^i)^{1/2} \times \exp\left[-\alpha_\perp^i\left(\frac{m_i}{2} v_{\varphi 0}^{i2} - q_i \varphi(r_0) + \frac{q_i}{c} v_{\varphi 0}^i A(r_0)\right)\right],$$

$$\beta_i = -v_{\varphi 0}^i / T_\perp^i r_0, \qquad \alpha_\perp^i = 1/T_\perp^i, \qquad \alpha_\parallel^i = 1/T_\parallel^i.$$

The particle density n_i and the density of the azimuthal current of particles $P_{\varphi i}$ will be equal to

$$n_i = \int_{-\infty}^{\infty} f_i \, d\mathbf{v} = f_{0i}\left(\frac{2\pi}{m_i}\right)^{3/2} \frac{1}{\alpha_\perp^i \sqrt{\alpha_\parallel^i}} \times \exp\left(\frac{m_i \beta_i^2 r^2}{2\alpha_\perp^i} - \alpha_\perp^i q_i \varphi - \frac{\beta_i q_i}{c} A r\right),$$

$$P_{\varphi i} = \int_{-\infty}^{\infty} v_\varphi f_i \, d\mathbf{v} = -\frac{\beta_i r}{\alpha_\perp^i} n_i = \bar{v}_{\varphi i} n_i,$$

where $\bar{v}_{\varphi i} = -\beta_i r / \alpha_\perp^i = \omega_i r$ is the mean azimuthal component of the velocity. Correspondingly we have for the mean squares of the velocity components;

$$\overline{v_{ri}^2} = 1 / m_i \alpha_\perp^i, \quad \overline{v_{\varphi i}^2} = 1 / m_i \alpha_\perp^i + \beta_i^2 r^2 / \alpha_\perp^{i2}, \quad \overline{v_{zi}^2} = 1 / m_i \alpha_\parallel^i.$$

We shall consider such states when there is only a magnetic field. In absence of external sources of the field E, it follows from the vanishing of the volume charge $\Sigma q_i n_i = 0$ that

$$-\beta_i q_i/c = a = \text{const}, \qquad m_i\beta_i^2/2\alpha_\perp^i = b = \text{const},$$
$$\sum_i q_i f_{0i}(2\pi/m_i)^{3/2}\left[\alpha_\perp^i \sqrt{\alpha_\parallel^i}\right]^{-1} = 0, \qquad (4)$$

and there remains only the single Maxwell equation for the magnetic potential A:

$$-\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA)\right]$$
$$= \frac{4\pi}{c}\sum_i q_i f_{0i}\left(\frac{2\pi}{m_i}\right)^{3/2}\frac{1}{\alpha_\perp^i\sqrt{\alpha_\parallel^i}}\omega_i r \exp(br^2 + arA).$$

Transforming to the new function $\psi = br^2 + arA$ and introducing the variable $\rho = Lr^2$, where L is defined by the relation

$$L^2 = \frac{\pi}{c} a \sum_i q_i f_{0i}(2\pi/m_i)^{3/2}\omega_i / \alpha_\perp^i\sqrt{\alpha_\parallel^i},$$

we obtain the equation*

$$d^2\psi/d\rho^2 + e^\psi = 0,$$

the general solution of which has the form

$$e^\psi = 2C_1 \exp\{\sqrt{C_1}(\rho + C_2)\}\left[1 + \exp\{\sqrt{C_1}(\rho + C_2)\}\right]^{-2},$$

where C_1 and C_2 are arbitrary constants. Returning to the original variable, we find the following expression for the particle density n_i, the current J_i and the magnetic field H:

$$n_i = f_{0i}\left(\frac{2\pi}{m_i}\right)^{3/2}\frac{1}{\alpha_\perp^i\sqrt{\alpha_\parallel^i}} 2C_1 \frac{\gamma\exp\{\sqrt{C_1}Lr^2\}}{[1+\gamma\exp\{\sqrt{C_1}Lr^2\}]},$$
$$J_i = q_i\omega_i r n_i,$$
$$H = \frac{1}{r}\frac{\partial}{\partial r}(rA) = \frac{2L\sqrt{C_1}}{a}\frac{1-\gamma\exp\{\sqrt{C_1}Lr^2\}}{1+\gamma\exp\{\sqrt{C_1}Lr^2\}} - 2\frac{b}{a}, \qquad (5)$$

where $\gamma = \exp(C_2\sqrt{C_1})$.

In these expressions the meaning of the coefficients γ and $L\sqrt{C_1}$ is not clear. To determine the coefficients we make use of the boundary conditions for the magnetic field. We denote the external magnetic field by H_∞; taking into account the relations

$$I = \sum_i I_i = \int_0^\infty \sum_i J_i\, dr = \frac{1}{2\pi}\sum_i q_i\omega_i N_i, \qquad N_i = \int_0^{2\pi}\int_0^\infty n_i r\, dr\, d\varphi,$$

where I_i and N_i are respectively the total azimuthal plasma current and the total number of particles of the i-th type per unit length of the plasma cylinder, we can write down the conditions for the magnetic field H at zero and at infinity in the form

$$H(0) = H_\infty + \frac{4\pi}{c}I, \qquad H(\infty) = H_\infty.$$

This yields

$$\gamma = -1 + \frac{4L\sqrt{C_1}}{a}\Big/\frac{4\pi}{c}I.$$

For the density of particles at the center we have

$$n_{0i} = f_{0i}\left(\frac{2\pi}{m_i}\right)^{3/2}\frac{2C_1}{\alpha_\perp^i\sqrt{\alpha_\parallel^i}}\frac{4\pi}{c}I\left(-\frac{4\pi}{c}I + \frac{4L\sqrt{C_1}}{a}\right)\left(\frac{4L\sqrt{C_1}}{a}\right)^{-2}.$$

Then, by simple algebraic transformations, we get for γ and L:

$$\gamma = \frac{8\pi}{(4\pi I/c)^2}\sum_i n_{0i} m_i \overline{v_{ri}^2},$$
$$L\sqrt{C_1} = (1+\gamma)(4\pi I/c)^2/8\sum_i N_i m_i \overline{v_{ri}^2}.$$

Equations (5) now finally take the following form:

$$n_i = n_{0i}(1+\gamma)^2$$
$$\exp\{(1+\gamma)\lambda(r^2)\}[1 + \gamma\exp\{(1+\gamma)\lambda(r^2)\}]^{-2},$$
$$J_i = q_i v_{\varphi i} n_i, \quad H = \frac{2\pi}{c}I(1+\gamma)\frac{1-\gamma\exp\{(1+\gamma)\lambda(r^2)\}}{1+\gamma\exp\{(1+\gamma)\lambda(r^2)\}} - H^*,$$
$$\lambda(r^2) = r^2(4\pi I/c)^2/8\sum_i N_i m_i \overline{v_{ri}^2},$$
$$H^* = 2b/a = m_i\omega_i c/q_i. \qquad (6)$$

For analysis of the physical results, it is useful to obtain the set of equations:

$$H(0) = 2\pi I/c - 2\pi I\gamma/c - H^*,$$
$$H(\infty) = -2\pi I/c - 2\pi I\gamma/c - H^*.$$

From the equation

$$-\frac{\partial H_p}{\partial r} = \frac{4\pi}{c}\sum J_i$$

we have for the self field of the plasma

$$H_p = \frac{4\pi}{c}I\frac{1+\gamma}{1+\gamma\exp\{(1+\gamma)\lambda(r^2)\}}.$$

Equation (6) for the magnetic field can also be written in the form

$$H = \frac{4\pi}{c}I\frac{1+\gamma}{1+\gamma\exp\{(1+\gamma)\lambda(r^2)\}}$$
$$-\frac{2\pi}{c}I(1+\gamma) - H^* = H_p + H_\infty.$$

If Eq. (1) is multiplied by v_r and integrated over the velocity, we then get

$$m_i\overline{v_{ri}^2}\frac{\partial n_i}{\partial r} - \frac{q_i}{c}H\overline{v_{\varphi i}}n_i - \frac{1}{r}\overline{v_{\varphi i}}^2 m_i n_i = 0.$$

Summing this equation for all types of particles, making use of the expression for the current, and

*An equation of the same form has been obtained by F. M. Nekrasov in the plane case (private communication).

integrating over the coordinates, we obtain

$$\sum_i n_i m_i \overline{v_{ri}^2} + (H + H^*)^2 / 8\pi = (H_\infty + H^*)^2 / 8\pi. \quad (7)$$

DISCUSSION OF RESULTS

We see that thus in the absence of collisions there exists a stationary state of a multi-component plasma with an external longitudinal magnetic field. The presence of an azimuthal plasma current I is characteristic of this state. The configurations of the plasma, current and magnetic field depend on the quantity γ. Actually, we see from (6) that the extrema for the density of particles will be at $r = 0$ for $\gamma \geq 1$, and at $r = 0$ and

$$r = \left[\frac{-8\pi \ln \gamma}{\pi (1+\gamma)(4\pi I / c)^2} \sum_i N_i m_i \overline{v_{ri}^2}\right]^{1/2}$$

for $\gamma < 1$.

The characteristic form of the curves for n_i and H are plotted in Figs. 1 and 2 for the two cases $\gamma < 1$ and $\gamma \geq 1$. In the case when the kinetic energy of the particles at the center is small in comparison with the energy of the plasma self field H_p on the axis of the cylinder, the density of the plasma has a "well" at the center and the magnetic field H in this region can even take a direction opposite to the direction of the external field H_∞. For $\gamma \geq 1$,

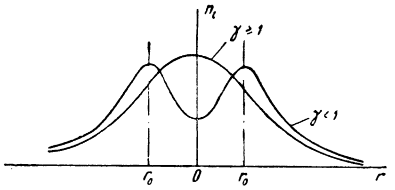

FIG. 1. Dependence of the concentration of particles of the i-th type on the radius for $\gamma \geq 1$ and $\gamma < 1$.

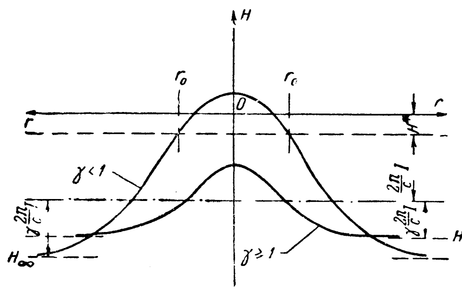

FIG. 2. Dependence of the magnetic field H on the radius for two cases $\gamma \geq 1$ and $\gamma < 1$. As $r \to \infty$, the field approaches the constant value H_∞.

the field H is always identical in direction with the external field, and the plasma has a maximum density at the center. The difference in the fields at the center and at infinity amounts to $4\pi I/c$. Thus the plasma possesses a diamagnetism, which is especially sharply pronounced for $\gamma < 1$.

It is seen from Eq. (6) that as the current I approaches zero the behavior of the field H depends on the quantity $\Sigma n_{0i} m_i \overline{v_{ri}^2}$. If the latter is constant or approaches zero more slowly than the current I ($\gamma \to \infty$), then $H \to H_\infty \to \infty$. In this case the self field of the plasma is almost nil. The particles are in a strong external magnetic field and interact weakly with one another; the decay curve (with distance) of the particle concentration is arbitrary; we have in the exponent

$$\pi r^2 \sum_i n_{0i} m_i \overline{v_{ri}^2} \Big/ \left(\sum_i N_i m_i \overline{v_{ri}^2}\right).$$

In the case $\gamma \to 1$ or $\gamma \to 0$, the field H approaches zero, but in this case the plasma configuration is spread out in space. In the last discussions, a constant number of particles N_i of each type has been assumed.

Equation (7) for the kinetic and magnetic pressures differs from the corresponding equation in the plane case only by the fact that the quantity $H + H^*$ appears in place of the field H. The quantity H^* has the physical meaning of the field which is necessary in order that the particle with mass m_i and charge q_i rotate with the cyclotron frequency ω_i about the axis of the cylinder, i.e., this is, as it were, that part of the field which is required to balance the centrifugal rotation forces of the plasma as a whole.

The first two equations of (4) are equivalent to the following conditions:

$$\overline{v_{\varphi i}}^2 q_i / m_i \overline{v_{ri}} = \text{const} \cdot r, \qquad \overline{v_{\varphi i}}^2 / \overline{v_{ri}^2} = \text{const} \cdot r^2$$

or

$$m_i \overline{v_{\varphi i}} / q_i = \text{const} \cdot r, \qquad m_i^2 \overline{v_{ri}^2} / q_i^2 = \text{const},$$

i.e., each type of particle rotates with a mean angular velocity that is independent of the radius; for particles with charges differing in sign, the directions of the angular velocities are different. The stationary states of the type under consideration exist only under the condition that the transverse temperatures of the particles of different types are inversely proportional to their masses. The azimuthal currents are principally connected with the light particles. Thus, in the case of an electron-ion plasma, the current will be created mainly by the electrons.

It can be shown that for the same absolute value of charge q_i, for instantaneous Larmor radii of particles of arbitrary type $r_{li} = m_i c v_\perp / eH$, the condition $\overline{r_{li}^2} = \text{const}$ is satisfied at each point.

Thus the results obtained make it possible to explain the behavior of the plasma upon application of a magnetic field, and to estimate the effect of diamagnetism. The application of an axial magnetic field to a plasma with an isotropic temperature distribution produces an ordered motion of the particles in the azimuthal direction, which leads to the appearance of a temperature anisotropy, since a part of the energy of random motion is transformed into ordered motion in the direction perpendicular to the field. The anisotropy produced can be estimated from a knowledge of the value of the total azimuthal current I_i and the total number of particles N_i of each type.

The states of the plasma considered admit of an arbitrary anisotropy in the temperature. However, it must be expected that its value will be limited by the criterion of stability.

In conclusion, the authors express their gratitude to R. A. Demirkhanov for constant interest in the work, and to I. I. Gutkin and A. I. Morozov for discussions.

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Translated by R. T. Beyer
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