CAUSE OF VANISHING OF THE RENORMALIZED CHARGE IN THE LEE MODEL

D. A. KIRZHNITS

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor January 30, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) **41**, 417-422 (August, 1961)

With the example of a model which is a relativistic generalization of the models of Ruijgrok-Van Hove and of Lee, it is shown that the difficulties in the latter model arise from the violation of the Bloch consistency condition, and not from violation of crossing symmetry. In the Lee model a covariant S matrix exists only for vanishing renormalized charge. It is found that the Bloch condition in its usual form is too severe. A physical consistency condition is formulated which contains only renormalized quantities.

1. In recent years much attention has been given to the difficulties inherent in the well known Lee model (we refer to the vanishing of the renormalized charge, the existence of nonphysical states, and so on). The interest in these problems is primarily due to the fact that there are definite arguments in favor of the existence of analogous difficulties also in an actual field theory with a point interaction.

There are, however, serious grounds for thinking that the difficulties of the Lee model are specific to it and not directly related to possible difficulties of an actual field theory. This is indicated, for example, by the fact that these difficulties disappear if we go over from the Lee model to the more realistic model proposed by Ruijgrok and Van Hove (cf. [1]). To settle the question finally it is desirable to discover the reasons for the vanishing of the charge in the Lee model and to find out whether these reasons are effective in an actual field theory. The present paper is devoted to the analysis of this question.*

In connection with what has been said we must note an assumption due to Mandelstam,[2] that the difficulties in question are closely connected with the violation of the crossing symmetry (c.s.) of the theory.† If this assumption should turn out to be entirely correct, it would follow that the causes of the vanishing of the charge are specific features of the Lee model and that there are no such difficulties in an actual field theory. The c.s. is indeed strongly violated in the Lee model (the V particle can only emit a meson, and the N particle can only absorb one), whereas in an actual field theory there is complete symmetry between emission and absorption.

There is, however, no direct proof of Mandelstam's assumption. Essentially the only argument in its favor is the situation in simple models of field theory—the Lee model and the statistical scalar model.* If however, we go over to more complex models, we can verify (cf. Sec. 2) that this assumption is by no means always true and does not reflect the true causes of the appearance of the difficulties. It is convenient to use for this purpose a relativistic model considered by Smolyanskii and the writer[1] (hereafter referred to as I), which is a generalization of the models of Lee and Ruijgrok and Van Hove. It then turns out that the theory is free from difficulties even when there are some violations of c.s.†

It is shown in Sec. 3 that the immediate cause of the appearance of the difficulties is not the violation of c.s., but the violation of the Bloch consistency condition. In the light of this last condition the entire set of available facts receives a simple explanation. In particular, the vanishing of the charge in the Lee model is simply due to the fact that for other values of the charge the

*In this paper we are concerned exclusively with relativistic theories (in particular, with the relativistic Lee model). The discussion of nonrelativistic models is of little interest in itself, especially since the difficulties are due precisely to the relativistic range of momenta.

†By c. s. we mean the symmetry of the theory with respect to emission and absorption of particles, i. e., with respect to the interchanges $\varphi_+ \rightleftarrows \varphi_-$, and so on, where the indices + and − correspond to the creation and annihilation parts of the operator φ.

*Ter-Martirosyan[3] has shown that for these models this assumption is valid even outside the framework of the Hamiltonian method.

†On the other hand, a model is known (cf.[4]) in which the renormalized charge vanishes although there is c. s. This property is absent, however, in the relativistic generalization of the model.

correctly stated problem has no solutions at all.

In this connection it is important to note that for a renormalized theory, in which there is no mutually unambiguous correspondence between the bare and renormalized charges, the Bloch condition in its usual form can be excessively severe. This condition must be expressed not in terms of the bare quantities, but in terms of the renormalized quantities. It may be that the problem has no solution in any finite order of perturbation theory, but at the same time an exact solution exists. Precisely this situation exists in the model considered in I (cf. Sec. 4).

2. At first glance the situation existing in the model considered in I corresponds to the Mandelstam hypothesis. In fact, setting $g_N = 0$, we arrive at the Lee model, in which the difficulties under discussion are inherent; at the same time there is violation of c.s. as regards the θ particles. If, however, we set $g_V = g_N$, then the c.s. of the theory is restored, and, as is shown in I, the theory is free from difficulties.

On a more detailed examination, however, facts appear that are in contradiction with the hypothesis under discussion. Namely, it turns out that the theory is free from difficulties even when there are some violations of c.s.

First, in our model c.s. is radically violated with regard to the heavy particles. The usual c.s. condition, which requires interchange between emission of a particle and absorption of the antiparticle cannot be satisfied because of the absence of antiparticles. Even if one formulates a weakened c.s. condition, assuming simply interchange of emission of a particle and its absorption (this is possible if we proceed by the law of conservation of heavy particles, cf. [5]), this condition also is violated in view of the fact that the process $\theta \rightleftarrows V + N$ is forbidden.

Furthermore, in the case of bare charges g_V and g_N which are not equal to each other and to zero the theory also does not possess the property of c.s., and at the same time is free from difficulties. It is true that in this case we always have $g_{0V} = g_{0N}$, so that the renormalized theory is crossing-symmetrical. Therefore it could be thought that in the Mandelstam hypothesis one must be speaking of just the c.s. of the renormalized theory. In this form, however, the hypothesis loses its force. In fact, after charge renormalization the Lee model also acquires the property of c.s., because $g_0 = 0$.

Finally, we must recall the model with complex charge, considered at the end of I, which is also unsymmetrical with respect to interchange of creation and annihilation operators of the θ particles and has a nonvanishing renormalized charge.

It follows from what has been said that violation of c.s. is not the direct cause of the appearance of the difficulties. It is natural to think that these arise when there is at the same time a violation of some more fundamental requirement, which must be obeyed by every internally consistent field theory. It turns out that this requirement is the Bloch consistency condition.

3. A theory in which c.s. is violated is essentially a nonlocal theory. In fact, the Hamiltonian of the model under consideration (for notations see I)

$$H(x) = \bar{\psi}(x)(\sigma_+\varphi_+(x) + \sigma_-\varphi_-(x))\psi(x) \tag{1}$$

can be written in the typically nonlocal form

$$H(x) = \bar{\psi}(x)\int d\xi F(x-\xi)\varphi(\xi)\psi(x).$$

The Fourier transform of the form-factor F is of the form

$$F(k) = \sigma_+\theta_+(k) + \sigma_-\theta_-(k),$$

where $\varphi = \varphi_+ + \varphi_-$, and $\theta_\pm$ are step functions which accomplish the projection onto the positive and negative frequency ranges.

If the interaction Hamiltonian is nonlocal, the question arises sharply as to whether the consistency conditions are satisfied, i.e., as to whether the S matrix exists as a definite function of the spacelike surface σ. In the case in which the Hamiltonian does not depend explicitly on this surface the condition has the well known form

$$[H(x),\ H(x')] = 0, \tag{2}$$

where the points x and x′ lie on σ, i.e., are separated by a spacelike interval.

We shall show that in the model under consideration the vanishing of the renormalized charge occurs when and only when the condition (2) is violated (for a somewhat sharper form of this condition see Sec. 4). It is easy to understand this statement if we take account of the fact that a theory with the charge equal to zero satisfies Eq. (2) identically. Therefore under the indicated conditions no other solutions are possible.

First we note that in the model in question the fermions have no antiparticles. Therefore the creation and annihilation operators taken separately (and not their sum, as for Dirac particles) anticommute outside the light cone. The corresponding anticommutator

$$\{\psi(x)\bar{\psi}(x')\} \sim \delta(\mathbf{u}\xi_0 - \xi u_0),$$

where $\xi = x - x'$, vanishes for $\xi^2 > \xi_0^2$, in virtue of $u_0^2 > u^2$. It is clear that the corresponding violation of c.s. does not prevent the condition (2) from holding, in complete agreement with what was said in Sec. 2 and with the assertion just made.

Using these anticommutation rules for ψ and substituting Eq. (1) in Eq. (2), we get

$$[H(x)H(x')] = is_+(x)\,s_-(x')\,\Delta_+(x - x') + is_-(x)\,s_+(x')\,\Delta_-(x - x').$$

Here $s_\pm(x) = \bar{\psi}(x)\sigma_\pm\psi(x)$, and $\Delta_\pm$ are the well known commutator functions. Replacing them by the functions $\Delta = \Delta_+ + \Delta_-$, $\Delta_1 = i(\Delta_+ - \Delta_-)$, only the first of which vanishes outside the light cone, we arrive at a condition equivalent to Eq. (2)

$$s_+(x)\,s_-(x') - s_-(x)\,s_+(x') = 0. \qquad (3)$$

It is well known that this condition holds for a theory with crossing symmetry because $s_+ = s_-$, but this condition is less severe than the condition of c.s.

In particular, for a model with complex charge, where

$$s_+ = g(\bar{\psi}\tau_1\psi), \qquad s_- = g^*(\bar{\psi}\tau_1\psi),$$

the condition (3) is satisfied identically, and this is also in agreement with the present assertion.

The third case of violation of c.s. mentioned in Section 2—that in which the charges g_V and g_N are unequal—will be discussed in the next section.

If there is strong enough violation of c.s., the condition (3) may not be satisfied. In this case, according to our assertion, the theory must contain difficulties. This situation occurs in the Lee model,* where

$$s_+ = g(\bar{\psi}_N\psi_V), \qquad s_- = g(\bar{\psi}_V\psi_N).$$

The left member of Eq. (3) now has the form $g^2[A(x, x') - A(x', x)]$, where

$$A(x, x') = \bar{\psi}_N(x)\,\bar{\psi}_V(x')\,\psi_V(x)\,\psi_N(x')$$

and for it to vanish it is necessary that $g = 0$. From this it is clear that the renormalized charge must also be zero,

$$g_0 = 0. \qquad (4)$$

Thus the compatibility condition enables us to determine the value of the renormalized charge without making dynamical calculations. In the general case of charges g_V and g_N which are not zero this condition gives

$$g_V = g_N, \qquad (5')$$

from which it follows by considerations of symmetry that

$$g_{0V} = g_{0N}. \qquad (5'')$$

Precisely this relation is obtained by direct calculation (cf. I). If we violate the relation (5″), the corresponding bare charges turn out to be complex, and we have all of the difficulties that come from this.[7] This situation is also in agreement with the statement made earlier.

The only problem that remains unsettled is connected with the relation (5′), which at first glance contradicts the statement in question. In actual fact, as was noted in Sec. 2, the theory is free from difficulties even for charges g_V and g_N which are not equal to each other (and to zero). It turns out, however, that the condition (2) is in certain ways too severe.

4. Let us first look into the properties of the S matrix of the model under consideration in the case $g_V \neq g_N$. Because of the violation of the condition (2) the S matrix does not exist at all in the infinite-time representation of Tomonaga and Schwinger. We emphasize that for the time being the discussion is being conducted in the language of bare charges, which corresponds to the treatment of the unrenormalized theory.

When one uses the usual one-time formalism the violation of the compatibility condition manifests itself in the fact that violations of causality and of relativistic invariance arise at once. The point is that the expression for the S matrix involves retarded commutators of the type $\theta(x - x')\times[H(x)H(x')]$. Since this commutator does not vanish outside of the light cone because of the presence of the function Δ_1 (see above), and the function θ has an invariant meaning only inside the cone, a violation of the two conditions that have been indicated is inevitable. Thus the terms in the expansion of the S matrix can be divided into two classes. The first includes those that contain only the function Δ; terms of the second class contain also the function Δ_1 and thus contradict the conditions of causality and of relativistic invariance.

If, however, we analyze the structure of the terms of the second class, it turns out that their dependence on g_V and g_N is of a very specific type. In the lowest order of perturbation theory the matrix element in question depends on the combination $g_V^2 - g_N^2$ (for $g_V = g_N$ this element must vanish). If now we sum all of the reducible diagrams of higher orders that correspond to this

*The fact that the Bloch condition is not satisfied in the Lee model has been noted previously.[6]

matrix element, then after the renormalization is carried out the combination just mentioned goes over into the difference of the squares of the renormalized charges, $g_{0V}^2 - g_{0N}^2$. This fact is closely connected with the renormalizability of the model under consideration, which was already noted in the first papers on the Ruijgrok-Van Hove model (cf. [7]).

But the equation $g_{0V}^2 - g_{0N}^2 = 0$ holds for any value of g_V and g_N (cf. I). Therefore after renormalization the exact solution for the S matrix has an acceptable structure because of the actual vanishing of the terms of the second class. Returning to the infinite-time formalism, we can say that in this case the renormalized S matrix exists in spite of the violation of the condition (2).

It is important to emphasize that the vanishing of the terms of the second class occurs only in the exact solution; if we confine ourselves to terms of a finite order in g_V and g_N, the combination in question by no means vanishes. What has been said can be illustrated by the impossibility of the inverse expansion of the renormalized expression in terms of the bare constants, because of the nonanalytic behavior of the relation $g_{0V} = g_{0N} = (g_V g_N)^{1/2}$ (cf. I).

Thus the consistency condition in its usual form is in fact too severe. It requires the existence (in the one-time formalism, the causality and relativistic invariance) of each term of the expansion of the S matrix in terms of the bare coupling constants. This requirement is excessive, and, moreover, unphysical. The physical compatibility condition must be formulated in the language of the renormalized quantities only (see below).

Thus even if the original Hamiltonian does not satisfy the condition (2), there is a certain possibility for the theory to be "self-perfecting." For example, in the model considered just now the radiative corrections, which are different for each of the vertex parts, lead finally to the restoration of consistence.

This situation is possible owing to two circumstances. First, it is important that there is no reciprocally unique correspondence between the bare and renormalized charges. Whereas the region of variation of the bare charges is the plane (g_V, g_N), the region of variation of the renormalized charges (the so called normal zone) degenerates into the line $g_{0V} = g_{0N}$. In the Lee model there is an analogous degeneration of the line g into the point $g_0 = 0$. Therefore to a given value of the renormalized charge there corresponds a whole set of values of the bare charge. The degeneration of the normal zone, without which, by the way, the difficulties discussed in this paper are themselves impossible, is due to the great effect of virtual quanta of high energies.*

Second, it is important that the theory is renormalizable. This means that there exists a separation of the Lagrangian into free and interaction Lagrangians, different from the ordinary separation, and such that the bare quantities disappear from the theory; their place is taken by the renormalized charge and mass.[8] The corresponding S matrix is finite and contains only the renormalized quantities.

It is natural to take the physical consistency condition to be the condition of the existence of such a renormalized S matrix. This condition is hard to write in a closed mathematical form because of the presence of derivatives in the new interaction Lagrangian (cf. [8]). Nevertheless we can assert that all of the cases considered above, including the case $g_V = g_N$, are in accordance with this new condition. It is enough to note that this condition is equivalent to the requirement of causality and relativistic invariance, in the sense that has been indicated, for the S matrix written in the one-time formalism.

In conclusion we emphasize once more that in view of the complete equivalence of two theories that differ only in the values of the bare coupling constants but not in the values of the renormalized constants, the question raised at the end of Sec. 3 is completely disposed of.

5. The analysis made here thus shows that in the framework of the model considered the renormalized charge goes to zero only when the existence of a covariant S matrix is impossible with any other value of the charge. Of course we can still not conclude from this that there are no other causes for the vanishing of the renormalized charge.

In any case we can state that the cause that leads to the difficulties in the Lee model is without force in a real field theory (the corresponding Hamiltonian satisfies the consistency condition). Therefore the situation in the Lee model provides no additional arguments for solving the problem of the difficulties of a real field theory.

I express my deep gratitude to I. E. Tamm for his interest in this work and to E. S. Fradkin for numerous discussions.

*If the contribution of these quanta is small, the assertion of the preceding section (and also the Mandelstam hypothesis) loses its force. This applies in particular to the Lee model with a form-factor, and also to the model with a nonrelativistic dispersion law of the nucleons.[9]

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Translated by W. H. Furry
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