ELECTROMAGNETIC PROPERTIES OF A RELATIVISTIC PLASMA, III

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We consider the problem of reflection and absorption of electromagnetic waves striking the plane bounding an electric plasma at an oblique angle. The main difference between oblique incidence and the case of normal incidence previously considered is that longitudinal waves are excited inside the plasma at frequencies close to those for which the longitudinal dielectric constant of the plasma vanishes. In the particular case of nonrelativistic temperatures, the energy spent in exciting the longitudinal waves exceeds the energy lost as a result of collisions between the plasma particles, provided that the condition (44) is satisfied.

1. The theory of electromagnetic properties of a semi-infinite plasma has been the subject of many papers.* At the present time the question of reflection, refraction, and absorption of electromagnetic waves incident perpendicularly to the surface of a sharply delineated plasma can be regarded as relatively well investigated. To the contrary, the case of inclined incidence remains essentially uninvestigated.

We have undertaken to fill this gap in part. We consider here the problem of oblique incidence of electromagnetic waves on the surface of an isotropic plasma (without constant field) with arbitrary particle distribution (in particular, relativistic). Such an analysis will enable us to study not only the question of losses connected with the occurrence of a transverse field in the plasma (called below the transverse losses), but also the question of the excitation of longitudinal waves and losses connected with the occurrence of a longitudinal field (longitudinal losses).

When we consider the semi-infinite plasma, we assume specular or diffuse reflection of the particles from its surface.[1] Both conditions must be regarded as rather approximate. These boundary conditions, however, can explain the role of effects occurring as a result of the presence of the plasma boundary, both in the case of a plasma bounded by a solid surface and in the case of a plasma confined by a magnetic field (under the condition that the transition layer is sufficiently small).

2. To study the electromagnetic properties of an electron plasma (assuming that the ions form a homogeneous background), we use, as is customary, the kinetic equation with self-consistent field. Being interested in the region of larger frequencies (much higher than the collision frequency), we can write here the kinetic equation in the form

$$\frac{\partial \delta f}{\partial t} + \mathbf{v}\frac{\partial \delta f}{\partial \mathbf{r}} + e\mathbf{E}\frac{\partial f_0}{\partial \mathbf{p}} = -\nu\delta f. \qquad (1)$$

Here f_0 is the equilibrium electron distribution function, and δf is the nonequilibrium addition; ν is the collision frequency.

Assuming specular reflection of the electrons from the surface of the plasma, when

$$\delta f\,(v_z > 0,\ z = 0) = \rho\delta f\,(v_z < 0,\ z = 0) \quad (\rho = 1) \qquad (2)$$

(here $z = 0$ corresponds to the surface of a plasma filling the half-space $z \geq 0$), we obtain the following solution of the kinetic equation for the case of an incident electromagnetic wave with time and space dependence $\exp\{i\omega(-t + z\cos\theta/c + y\sin\theta/c)\}$, where θ is the angle of incidence:

$$\delta f = -\frac{e}{v_z} f_0' \int_z^\infty dz' \exp\left\{-\frac{z-z'}{v_z}\chi\right\}\mathbf{vE}\,(z'), \qquad v_z < 0,$$

$$\delta f = \frac{e}{v_z} f_0' \int_0^z dz' \exp\left\{-\frac{z-z'}{v_z}\chi\right\}\mathbf{vE}\,(z') + \frac{e}{v_z} f_0' \int_0^\infty dz' \exp\left\{-\frac{z+z'}{v_z}\chi\right\} \times \{E_x v_x + E_y v_y - E_z v_z\}, \qquad v_z > 0.$$

Here $\chi = \nu - i\omega(1 - v_y \sin\theta/c)$ and f_0' is the derivative of the equilibrium distribution function with respect to the energy.

*For lack of space we cannot discuss these papers in any detail. We refer the reader, for example, to the book,[1] which contains a relatively complete bibliography.

Expressions (3) are used to determine the current density

$$\mathbf{j} = e\int \mathbf{v}\delta f d\mathbf{p},$$

which is contained in the field equations. If we now eliminate from the field equations the magnetic induction, and extend the electric field to include all space (the tangential components E_x and E_y in even fashion, and the normal component E_z in odd fashion), then, as can be readily verified, we obtain for the Fourier components of the electric field

$$E_i(q) = \frac{1}{\sqrt{2\pi}}\int dz e^{-iqz} E_i(z)$$

the following equation

$$k^2 E_i(q) - k_i(\mathbf{kE}(q)) - (\omega^2/c^2)\,\varepsilon_{ij}(\omega, \mathbf{k})\,E_j(q) = -\sqrt{2/\pi}\,a_i;$$
$$\mathbf{k} = (0, \omega\sin\theta/c, q), \quad \mathbf{a} = (E_x'(0), E_y'(0) - i\omega\sin\theta E_z(0)/c, 0). \tag{4}$$

The dielectric permittivity tensor ϵ_{ij} has in our case of isotropic plasma the form

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \varepsilon^t(\omega, k)(\delta_{ij} - k^{-2}k_ik_j) + k^{-2}k_ik_j\varepsilon^l(\omega, k).$$

Here the transverse and longitudinal permittivities are determined by the usual formulas[1]

$$\varepsilon^l(\omega, k) = 1 + \frac{4\pi e^2}{\omega k^2}\int d\mathbf{p}\,\frac{(\mathbf{kv})^2 f_0'}{\omega + i\nu - \mathbf{kv}}, \tag{5*}$$

$$\varepsilon^t(\omega, k) = 1 + \frac{2\pi e^2}{\omega k^2}\int d\mathbf{p}\,\frac{[\mathbf{kv}]^2 f_0'}{\omega + i\nu - \mathbf{kv}}. \tag{6}$$

The equations written out solve the formal aspect of the theory of inclined incidence of radiation on a semi-infinite plasma when the electrons are specularly reflected from the surface. We now proceed to an analysis of the pertinent results.

3. We turn first to the relatively simple case of s-polarization, when the electric vector of the incident wave is perpendicular to the plane of incidence. In this case only the x component differs from zero, and is given by

$$E(z) = \frac{E_x'(0)}{\pi}\int_{-\infty}^{+\infty}\frac{dq e^{iqz}}{(\omega/c)^2\varepsilon^t(\omega, k) - k^2}. \tag{7}$$

The difference between this equation and Eq. (7) of an earlier paper by one of the authors,[2] is that the quantity q in the denominator of the integrand is replaced by k. This corresponds in fact to the result obtained by Reuter and Sondheimer.[3]

The complex coefficient of reflection, in the case of s-polarization, is

$$r_s = \frac{(c/4\pi)\cos\theta Z_s - 1}{(c/4\pi)\cos\theta Z_s + 1},$$

where $Z_s(\omega) = (4\pi i\omega/c^2)\,[E_x(0)/E_x'(0)]$ is the surface impedance, connected by the relation

$$Z_s(\omega) = -(4\pi i\omega/c^2)\lambda_s$$

with the effective complex depth of penetration

$$\lambda_s^{(3)} = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dq}{(\omega/c)^2\varepsilon^t(\omega, k) - k^2}. \tag{8}$$

Under conditions when the spatial dispersion is weak, we can represent the transverse permittivity in the form $(\omega \gg \nu)$:[1,2]

$$\varepsilon^t(\omega, k) = \varepsilon(\omega) - \alpha^t c^2k^2/\omega^2 = 1 - \omega_0^2/\omega^2 - \alpha^t c^2 k^2/\omega^2 + i\nu\omega_0^2/\omega^3;$$

$$\omega_0^2 = -\frac{4\pi e^2}{3}\int d\mathbf{p} v^2 f_0', \qquad \alpha^t = -\frac{4\pi e^2}{15}\int\frac{v^4 f_0'}{c^2\omega^2}\,d\mathbf{p}. \tag{9}$$

Thereupon we get for the effective depth of penetration

$$\lambda_s = \frac{ic}{\omega}\,[(1 + \alpha^t)(\varepsilon(\omega) - (1 + \alpha^t)\sin^2\theta)]^{-1/2}. \tag{10}$$

In the case of a nonrelativistic plasma, the contribution due to a^t is negligibly small. To the contrary, in the relativistic case this is no longer so. Actually, for example, in the ultrarelativistic case $a^t = \omega_0^2/5\omega^2$.

In order for expansion (9) to be valid, the conditions

$$|\varepsilon(\omega)| \ll mc^2/\varkappa T_e \qquad (\omega^2 \gg \omega_{Le}^2\varkappa T_e/mc^2) \tag{11}$$

must be satisfied in the case of nonrelativistic plasma, and

$$|\varepsilon(\omega)| \ll 1 \tag{12}$$

in the relativistic case.

Expression (10) is due to the contribution of the pole of the integrand of (8). Let us consider now the contribution due to the branch point of the dielectric constant. We have

$$\delta\lambda_s^{(3)} = -\frac{2i}{\pi}\frac{c}{\omega}\left(1 + i\frac{\nu}{\omega}\right)\int_1^{\infty}\frac{dx\,x\,\mathrm{Im}\,\varepsilon_+^t(\omega, x(\omega+i\nu)/c)}{\sqrt{x^2 - \sin^2\theta\,\omega^2/(\omega+i\nu)^2}}$$
$$\times\left\{\left[\mathrm{Re}\,\varepsilon_+^t\left(\omega, \frac{\omega+i\nu}{c}x\right) - (1 + i\nu/\omega)^2x^2\right]^2 + \left[\mathrm{Im}\,\varepsilon_+^t\left(\omega, \frac{\omega+i\nu}{c}x\right)\right]^2\right\}^{-1}. \tag{13}$$

The expressions for ϵ_+^t were derived earlier (see [2]) for the case of a relativistic Boltzmann particle distribution. In the nonrelativistic case, when the main contribution is made by the values

*(kv) = k·v; [kv] = k × v.

$x \sim c/v_T \sim \sqrt{mc^2/\kappa T_e}$, the expression for $\delta\lambda_s^{(3)}$ is independent, accurate to terms $\sim \kappa T_e/mc^2$, of the angle of incidence and agrees with the following expression[2]

$$\delta\lambda_s^{(3)} \approx -\frac{2i\omega^2}{\pi c^2}\int_0^\infty \frac{dk}{k^4}\,\mathrm{Im}\,\varepsilon^t(\omega, k) = 2i\sqrt{\frac{2}{\pi}}\frac{c}{\omega}\frac{\omega_{Le}^2}{\omega^2}\left(\frac{\kappa T_e}{mc^2}\right)^{3/2}. \tag{14}$$

Here we assume $\omega \gg \nu$ and $(\omega_{Le}^2/\omega^2)(\kappa T_e/mc^2) \ll 1$.

In the ultrarelativistic case, assuming $\omega \gg \nu$, we have

$$\delta\lambda_s^{(3)} = \frac{ic}{\omega}\frac{3}{2}\frac{\omega_{0p}^2}{\omega^2}\int_1^\infty \frac{dx}{\sqrt{x^2-\sin^2\theta}}(1-1/x^2)\left\{\left(1+\frac{\omega_{0p}^2}{\omega^2}\frac{3}{4x}\right.\right.$$
$$\left.\times\left[-\frac{2}{x}+(1-1/x^2)\ln\frac{x-1}{x+1}\right]-x^2\right)^2$$
$$\left.+\left(\frac{3\pi}{4x}\right)^2\frac{\omega_{0p}^2}{\omega^2}(1-1/x^2)^2\right\}^{-1}, \tag{15}$$

where $\omega_{0p}^2 = 4\pi e^2 N_e c^2/3\kappa T_e$.

In order to exhibit the dependence of the latter expression on the angle of incidence θ, we give the values of $\delta\lambda_s^{(3)}(\omega/ic)$ when $\omega = \omega_{0p}$. Thus, when θ = 0, 30, 60, and 90° we have accordingly the values 0.09, 0.10, 0.115, and 0.24.

The formulas obtained permit, in particular, a determination of the absorbing ability of the plasma, $A = 1 - |r|^2$. In the case of nonrelativistic temperature with $\epsilon_0 - \sin^2 < 0$ we have

$$A_{\mathrm{nr}}^{(s)} \approx \frac{2\cos\theta}{\sqrt{\sin^2\theta-\varepsilon_0}}\frac{\nu}{\omega} + 8\sqrt{\frac{2}{\pi}}(\sin^2\theta-\varepsilon_0)\left(\frac{\kappa T_e}{mc^2}\right)^{3/2}\cos\theta. \tag{16}$$

When $\theta = 0$, this formula coincides with equation (38) of [2].

The contribution to the absorbing ability, a contribution not connected with the plasma-particle collisions with each other, can also be determined directly by calculating the energy lost by the electrons on colliding with the plasma surface. In the case when $\epsilon_0 - \sin^2\theta > 0$ this contribution is now equal to

$$\frac{8\sqrt{2/\pi}\,(\omega_{Le}/\omega)^2(\kappa T_e/mc^2)^{3/2}\cos\theta\,(\varepsilon(\omega)-\sin^2\theta)}{(\sqrt{\varepsilon(\omega)-\sin^2\theta}+\cos\theta)^2}. \tag{17}$$

In the case of s-polarization which we are considering, it is easy to obtain the solution also by assuming diffuse reflection of the electrons from the plasma boundary, corresponding to $r = 0$ in Eq. (2). Now we obtain for the field the following expression

$$E_x(z) = \frac{E_x(0)}{2\pi i}\int_{-i\delta-\infty}^{-i\delta+\infty}\frac{dq}{q}e^{iqz}\exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{dq'}{q-q'}\right.$$
$$\left.\times\ln\left[1-\frac{\omega^2}{c^2q'^2}(\varepsilon^t(\omega,k)-\sin^2\theta)\right]\right\}, \tag{18}$$

and for the complex depth of penetration we have

$$\lambda_s^{(D)} = \left\{\frac{1}{\pi}\int_0^\infty dq\,\ln\left[1-\frac{\omega^2}{c^2q^2}(\varepsilon^t(\omega,k)-\sin^2\theta)\right]\right\}^{-1}. \tag{19}$$

Under conditions of weak spatial dispersion, the contribution of the zero of the dispersion equation yields

$$\lambda_s^{(D)} = \frac{ic}{\omega}\left[\frac{1+\alpha^t}{\varepsilon(\omega)-(1+\alpha^t)\sin^2\theta}\right]^{1/2}. \tag{20}$$

The branch point of the dielectric permittivity makes a contribution

$$\{\delta\lambda_s^{(D)}\}^{-1} = \frac{i\omega}{\pi c}\left(1+i\frac{\nu}{\omega}\right)^3$$
$$\times\int_0^1 da\int_1^\infty dx\,x\left[x^2-\left(\frac{\omega}{\omega+i\nu}\right)^2\sin^2\theta\right]^{1/2}\mathrm{Im}\,\varepsilon_+^t$$
$$\times\{[a\,\mathrm{Re}\,\varepsilon_+^t-(1+i\nu/\omega)^2x^2$$
$$+(1-a)\sin^2\theta]^2+(a\,\mathrm{Im}\,\varepsilon_+^t)^2\}. \tag{21}$$

In the nonrelativistic limit, the result for $\delta\lambda$ does not depend on the angle of incidence, accurate to $(\kappa T_e/mc^2)$. In the opposite, ultrarelativistic limit, $\delta\lambda$ depends continuously on θ.

4. We now proceed to the somewhat more cumbersome but more interesting case of p-polarization, when the electric vector of the incident wave lies in the plane of incidence. This case is of special interest because a longitudinal wave, which cannot be excited in the case of s-polarization, can now penetrate into the plasma.

The field produced in the plasma as a result of the incident plane electromagnetic wave can be described in this case by the formulas

$$E_y(z) = E_y^t(z) + E_y^l(z), \tag{22}$$

$$E_y^t(z) = \left\{E_y'(0) - i\frac{\omega}{c}\sin\theta E_z(0)\right\}$$
$$\times\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dq\,q^2e^{iqz}}{[q^2+(\omega/c)^2\sin^2\theta][(\omega/c)^2\varepsilon^t(\omega,k)-(\omega/c)^2\sin^2\theta-q^2]}, \tag{23}$$

$$E_y^l(z) = \left\{E_y'(0) - i\frac{\omega}{c}\sin\theta E_z(0)\right\}$$
$$\times\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dq\,\sin^2\theta\,e^{iqz}}{[q^2+(\omega/c)^2\sin^2\theta]\,\varepsilon^l(\omega,k)}. \tag{24}$$

The complex reflection coefficient is determined by the following formulas:

$$r_p = \frac{\cos\theta - Z_p(c/4\pi)}{\cos\theta + Z_p(c/4\pi)}, \tag{25}$$

$$Z_p(\omega,\theta) = \frac{4\pi i\omega}{c^2}\frac{E_y(0)}{E_y'(0)-i(\omega/c)\sin^2\theta E_z(0)} = -\frac{4\pi i\omega}{c^2}\lambda_p. \tag{26}$$

From (22) — (24) it is clear that the effective depth of penetration is made up, in additive fashion, of the transverse and longitudinal depths, given by the formulas

$$\lambda_p^t = -\frac{1}{\pi} \times \int_{-\infty}^{+\infty} \frac{dqq^2}{[q^2 + (\omega/c)^2 \sin^2\theta][(\omega/c)^2 \varepsilon^t(\omega, k) - (\omega/c)^2 \sin^2\theta - q^2]}, \tag{27}$$

$$\lambda_p^l = -\frac{\sin^2\theta}{\pi} \int_{-\infty}^{+\infty} \frac{dq}{[q^2 + (\omega/c)^2 \sin^2\theta]\, \varepsilon^l(\omega, k)}. \tag{28}$$

We note that in the calculation of these integrals, the contributions of the singularity of the integrands at the point $q^2 + (\omega/c)^2 \sin^2\theta = 0$ cancel each other out in the summary expression for λ_p. The reason for it is that the longitudinal and transverse permittivities are equal to each other when $k = 0$. Consequently we disregard from now on the contribution of this singularity.

Under conditions when the spatial dispersion is weak and we can use expression (9) for the transverse permittivity, the contribution to the right half of (27), due to the zero of the dispersion equation for the transverse oscillations, has the form

$$\lambda_p^t = \frac{ic}{\omega\varepsilon(\omega)} \left[\frac{\varepsilon - (1+\alpha^t)\sin^2\theta}{1+\alpha^t}\right]^{1/2} = \frac{ic}{\omega^2 - \omega_0^2 + \omega_0^2 i\nu/\omega} \times \left[\frac{\omega^2 \cos^2\theta - \omega_0^2 + \omega_0^2 i\nu/\omega - \omega^2\alpha^t \sin^2\theta}{1+\alpha^t}\right]^{1/2}. \tag{29}$$

To be able to use expansion (9) here, inequalities (11) and (12) must be satisfied.

Analogously, in the case of weak spatial dispersion we can write for the longitudinal permittivity the following expression:*

$$\varepsilon^l(\omega, k) = \varepsilon(\omega) - \frac{c^2k^2}{\omega^2}\alpha^l, \tag{30}$$

$$\alpha^l = -\frac{4\pi e^2}{5c^2\omega^2}\int d\mathbf{p} v^4 f_0'. \tag{31}$$

In particular, in the case of a Boltzmann particle-energy distribution we have

$$\alpha^l = \frac{4\pi e^2 N_e mc^4}{5(\varkappa T_e)^2 \omega^2 K_2(mc^2/\varkappa T_e)} \int_1^\infty \frac{dx}{x^3}(x^2-1)^{5/2} \exp\left(-\frac{mc^2}{\varkappa T_e}x\right). \tag{31a}$$

For the case of nonrelativistic ($mc^2 \gg \kappa T_e$) and in the case of ultrarelativistic temperatures ($mc^2 \ll \kappa T_e$) we have now, respectively,

$$\alpha_{nr}^l = 3\varkappa T_e \omega_{Le}^2 / mc^2\omega^2, \qquad \alpha_{ur}^l = 3\omega_{0p}^2/5\omega^2. \tag{31b}$$

The contribution to the right half of (28), due to the zero of Eq. (30), has the form

$$\lambda_p^l = \frac{ic}{\omega}\frac{\sqrt{\alpha^l}}{\varepsilon(\omega)} \frac{\sin^2\theta}{\sqrt{\varepsilon(\omega) - \alpha^l \sin^2\theta}}. \tag{32}$$

This result, obtained with the aid of the approximate expression (30), will not be valid if the condition $|\epsilon(\omega)| \ll 1$ is violated.

The longitudinal permittivity is also relatively simplified if $kv_T \gg \omega$. Now

$$\varepsilon^l = 1 + (kr_{scr})^{-2}, \tag{33}$$

$$r_{scr}^{-2} = -4\pi e^2 \int d\mathbf{p} f_0'. \tag{34}$$

In the case of a Boltzmann electron distribution* we have

$$r_{scr}^{-2} = 4\pi e^2 N_e/\varkappa T_e. \tag{34a}$$

Now the pole contribution to formula (28) has the form

$$\lambda_p^l = -\sin^2\theta r_{scr}. \tag{35}$$

Finally, as $kc \to \omega + i\nu$ we obtain in the case of an ultrarelativistic plasma

$$\lambda_p^l = i\frac{8}{3}\frac{\omega^2}{\omega_{0p}^2}\frac{c}{\omega}\frac{\sin^2\theta}{\cos\theta}\exp\left[-\frac{2}{3}\frac{\omega^2}{\omega_{0p}^2} - 2\right].$$

Inasmuch as in this case $\omega \gg \omega_{0p}$, we are dealing with an exponentially small quantity.

Let us turn now to an examination of the contributions to the left halves of (27) and (28), due to the branch point of the permittivity. The integrals along the edges of the cut yield accordingly

$$\delta\lambda_p^t = -\frac{2i}{\pi}\frac{c}{\omega}(1 + i\nu/\omega) \times \int_1^\infty \frac{dx}{x}\left[x^2 - \sin^2\theta\left(\frac{\omega}{\omega+i\nu}\right)^2\right]^{1/2} \mathrm{Im}\,\varepsilon_+^t\left(\omega, \frac{\omega+i\nu}{c}x\right) \times \left\{\left[\mathrm{Re}\,\varepsilon_+^t\left(\omega, \frac{\omega+i\nu}{c}x\right) - (1+i\nu/\omega)^2 x^2\right]^2 + \left[\mathrm{Im}\,\varepsilon_+^t\left(\omega, \frac{\omega+i\nu}{c}x\right)\right]^2\right\}^{-1}, \tag{36}$$

$$\delta\lambda_p^l = -\frac{2i}{\pi}\frac{\sin^2\theta}{(1+i\nu/\omega)}\frac{c}{\omega}\int_1^\infty dx\, \mathrm{Im}\,\varepsilon_+^l\left(\omega, \frac{\omega+i\nu}{c}x\right)\left|\varepsilon_+^l\left(\omega, \frac{\omega+i\nu}{c}x\right)\right|^{-2}\left[x^2 - \sin^2\theta\left(\frac{\omega}{\omega+i\nu}\right)^2\right]^{-1/2}. \tag{37}$$

Formula (36) assumes the form (14) in the nonrelativistic case, when the main contribution to the

*In the transparency region we must take account of the contribution of the Cerenkov effect to the imaginary part of ε^l:

$$\delta\varepsilon^l = -i\frac{4\pi^2 e^2\omega}{k^2}\int d\mathbf{p}\,\delta(\mathbf{kv} - \omega) f_0'.$$

*We note that allowance for the motion of the ions causes a term $4\pi e_i^2 N_i/\varkappa T_i$ to appear in formula (34a).

integral is made by large values of x. Analogously, the integrand in (37) can be assumed independent* of the angle of incidence

$$\delta\lambda_p^l \approx -\frac{2ic}{\pi\omega}\frac{\sin^2\theta}{1+i\nu/\omega} \times \int_1^\infty \frac{dx}{x^2}\,\mathrm{Im}\,\varepsilon_+^l\left(\omega, \frac{(\omega+i\nu)}{c}x\right) \Big/ \left|\varepsilon_+^l\left(\omega, \frac{(\omega+i\nu)}{c}x\right)\right|^{-2}. \quad (37a)$$

For the case of Maxwellian electron distribution

$$\delta\lambda_p^l = \frac{ic}{\omega}\sin^2\theta\sqrt{\frac{2}{\pi}}\left(\frac{\varkappa T_e}{mc^2}\right)^{1/2}\frac{\omega_{Le}^2}{\omega^2} \times \int_0^{\beta_{max}} d\beta\beta e^{-\beta^2/2}\left\{\left(1+\frac{\omega_{Le}^2}{\omega^2}\beta^2\left[1-\mathrm{Re}\,J_+(\beta)\right]\right)^2 + \frac{\pi}{2}\frac{\omega_{Le}^4}{\omega^4}\beta^2 e^{-\beta^2/2}\right\}^{-1},$$

$$\beta_{max}^{-1} = \sqrt{\frac{\varkappa T_e}{mc^2}}\left[(1+i\nu/\omega)^2 - \sin^2\theta\right)^{1/2},$$

$$J_+(\beta) = \beta e^{-\beta^2}\int_{+i\infty}^{\beta} e^{\tau^2/2}\,d\tau.$$

We can put with high degree of accuracy $\beta_{max} = \infty$. The numerical integration yields for $\omega = \omega_{Le}$

$$\delta\lambda_p^l \approx 1.7\frac{ic}{\omega_{Le}}\sin^2\theta\sqrt{\frac{2}{\pi}}\left(\frac{\varkappa T_e}{mc^2}\right)^{1/2}. \quad (38)$$

Below the Langmuir frequency we can use for estimates the formula

$$\delta\lambda_p^l \sim \frac{ic}{\omega}\sin^2\theta\sqrt{\frac{2}{\pi}}\left(\frac{\varkappa T_e}{mc^2}\right)^{1/2}\frac{\omega^2}{\omega_{Le}^2}\left\{\frac{1}{2}\ln\left(1+\frac{\omega_{Le}^2}{\omega^2}\right) - \frac{\omega_{Le}^2}{\omega^2+\omega_{Le}^2}\right\}.$$

In the opposite case of ultrarelativistic temperatures, using Eqs. (11) and (12) of [4], we obtain $(\omega \gg \nu)$

$$\delta\lambda_p^t = -\frac{ic}{\omega}\frac{3}{2}\frac{\omega_{0p}^2}{\omega^2}\int_1^\infty \frac{dx}{x^2}\sqrt{x^2-\sin^2\theta}\left(1-\frac{1}{x^2}\right)\left\{\left(1+\frac{\omega_{0p}^2}{\omega^2}\frac{3}{4x} \times \left[-\frac{2}{x}+\left(1-\frac{1}{x^2}\right)\ln\frac{x-1}{x+1}\right] - x^2\right)^2 + \left(\frac{3\pi}{4x}\right)^2\frac{\omega_{0p}^2}{\omega^2}\left(1-\frac{1}{x^2}\right)^2\right\}^{-1}, \quad (39)$$

$$\delta\lambda_p^l = -\frac{ic}{\omega}3\frac{\omega_{0p}^2}{\omega^2}\sin^2\theta\int_1^\infty\frac{dx}{x^4(x^2-\sin^2\theta)^{1/2}} \times \left\{\left(1+3\frac{\omega_{0p}^2}{\omega^2}\frac{1}{x^2}\left[1+\frac{1}{2x}\ln\frac{x-1}{x+1}\right]\right)^2 + \left(\frac{3\pi}{2}\frac{\omega_{0p}^2}{\omega^2}\right)^2\frac{1}{x^6}\right\}^{-1}. \quad (40)$$

In order to display the dependence of these expressions on the angle of incidence θ, we give the following numerical values for $\omega_{0p}^2 = \omega^2$. Thus, for angles $\theta = 0, 30, 60, 90°$ we have for $\delta\lambda^t(\omega_{0p}/ic)$ the values 0.09, 0.088, 0.077, and 0.07 respectively. Analogously we have for $\delta\lambda^l(\omega_{0p}/ic)$ the values $(0, 0.12, 0.15, 0.16) \times \sin^2\theta$.

*The expression for ε_+^l for the case of Boltzmann electron distribution is given in the cited book[1] (see also the Appendix).

The results obtained can be used to determine the absorbing ability of a plasma, connected in the case of p-polarization with the effective depth of penetration by the following relation

$$A^{(p)} = \frac{4\cos\theta(\omega/c)\,\mathrm{Im}\,\lambda}{[\cos\theta + (\omega/c)\,\mathrm{Im}\,\lambda]^2 + [(\omega/c)\mathrm{Re}\lambda]^2}. \quad (41)$$

At large values of $\epsilon(\omega)$, when, as is well known, the Leontovich boundary conditions are applicable, an analysis of our formulas is of little interest, for in this case we can use directly the results obtained in the analysis of normal incidence (see [2]). We therefore concentrate our attention from now on on the case $|\epsilon(\omega)| \lesssim 1$.

For angles not too close to $\pi/2$, we can neglect the dissipative terms in the denominator of (41). Therefore to determine the effect of various factors on the dissipation of electromagnetic waves it is sufficient to compare the corresponding dissipative contributions to Im λ, contained in the numerator of (41). Thus, in the relativistic case when $\omega \approx \omega_{0p}$, as follows from the estimates given above, the longitudinal and transverse losses are comparable in magnitude when θ is not too small. A comparison of (38) with (14) shows that in the nonrelativistic case, when $\omega \sim \omega_{Le}$, the longitudinal losses connected with the collisions of the particles against the surface of the plasma are $(mc^2/\kappa T_e)$ times greater (at not too small angles θ) than the transverse losses in the case of specular reflection of electrons from the surface.

It must be noted that the indicated longitudinal losses are equal in order of magnitude to the transverse losses in the case of diffuse reflection of the electrons from the plasma boundary.[2] To clarify the role of such losses, we must compare expression (38) with the imaginary parts of formulas (29) and (32) under conditions when the plasma is not transparent, $\epsilon'(\omega) - \alpha^l\sin^2\theta < 0$. If $\omega_0^2 - \omega^2 \sim \omega_0^2$ in this case, then the losses connected with collisions between particles will be negligibly small if the inequality

$$25N_eL^2 \ll T_e^4\sin^4\theta \quad (42)$$

is satisfied, where T_e is in degrees Kelvin, N_e is the number of electrons per cubic centimeter, and L is the Coulomb logarithm. We note that this is similar to the analogous inequality obtained for the case of diffuse reflection of electrons.[2]

Under these conditions we obtain the following expression for the absorbing ability

$$A^{(p)} \cong \Big\{4\sqrt{\frac{2}{\pi}}\,1.7\cos\theta\sin^2\theta\left(\frac{\varkappa T_e}{mc^2}\right)^{1/2}\varepsilon'^2 + \cos\theta\,\frac{\nu}{\omega}\,2\,\frac{\omega_{Le}^2}{\omega^2}\,\frac{2\sin^2\theta-\varepsilon'}{\sqrt{\sin^2\theta-\varepsilon'}}\Big\} \times (1-\varepsilon')^{-1}(\sin^2\theta-\varepsilon'\cos^2\theta)^{-1}.$$

Further, in the region where the plasma is opaque to the transverse waves $[\epsilon'(\omega) - \sin^2\theta < 0]$ and transparent to the longitudinal waves $[\epsilon'(\omega) - \alpha^l \sin^2\theta > 0]$, the imaginary part of (32) makes the following contribution to the numerator of (41), along with $\operatorname{Im}\lambda_p^t$ and $\operatorname{Im}\delta\lambda_p^l$, at not too small values of ϵ' ($|\epsilon'| \gg |\epsilon''|$):

$$\operatorname{Im}\lambda_p^l = \frac{c}{\omega}\frac{\sqrt{\alpha^l}}{\varepsilon'(\omega)}\frac{\sin^2\theta}{\sqrt{\varepsilon'(\omega)-\alpha^l\sin^2\theta}}. \qquad (43)$$

This contribution is not at all connected with the dissipation of electromagnetic waves, and is brought about by the excitation of longitudinal waves in the plasma when a transverse wave is obliquely incident from the vacuum on the surface of the plasma.

In either an ultrarelativistic or a nonrelativistic plasma, the energy converted into longitudinal waves is much greater than the energy lost to collisions between the plasma particles and the surface, provided $\epsilon'(\omega) \ll 1$. To the contrary, when $\epsilon'(\omega) \sim 1$, the corresponding energies become commensurate, and therefore the conditions under which the energy lost to excitation of longitudinal waves is greater than the energy lost by collision between particles will be determined by inequality (42) for the case of a nonrelativistic plasma.

In the case $\alpha^l < \epsilon'(\omega) \ll 1$, the corresponding condition has the form

$$25 N_e L^2 \ll T_e^4 \sin^2\theta\,(1-\omega_{Le}^2/\omega^2). \qquad (44)$$

The region is narrower here than in the case of (42), owing to the increase in the dissipative losses due to collisions of the particles inside the plasma. This increase is due to an increase in depth of penetration of the transverse field into the plasma with decreasing $\epsilon'(\omega)$, and this naturally increases the fraction of the energy lost by the field in the plasma. We note that formula (32), and consequently also formula (43), were obtained under the condition $|\epsilon'(\omega)| \ll 1$, and therefore we must in fact use formula (44).

Under conditions when the inequality (44) is applicable, we have for the absorbing ability of the plasma the following expression:

$$A^{(p)} = \frac{4\cos\theta\sin^2\theta\sqrt{\alpha^l\varepsilon'^3(\omega)}}{[\varepsilon'^{3/2}\cos\theta+\sqrt{\alpha^l}\sin^2\theta]^2 + (-1+\sin^2\theta/\varepsilon')\varepsilon'^2}. \qquad (45)$$

If, in addition, $(\epsilon')^3 \gg \alpha^l$, then

$$A^{(p)} = \frac{4\sqrt{\alpha^l\varepsilon'(\omega)}}{1-\varepsilon'(\omega)}\,\frac{\cos\theta\sin^2\theta}{\sin^2\theta-\varepsilon(\omega)\cos^2\theta}. \qquad (46)$$

Finally, in the case when the plasma is transparent also to the transverse waves, $\epsilon'(\omega) > (1+\alpha^t) \times \sin^2\theta$, the main fraction of the energy is transferred to the transverse wave in the plasma. The waves formed are absorbed in the plasma, and their energy is converted into heat. As is well known, the corresponding heat is determined by the imaginary part of the dielectric constant, which is brought about by collisions of plasma particles with each other; in the case of longitudinal waves it is also due to the possibility of Cerenkov radiation. For the heat released by absorption of the transverse waves per unit volume at a depth z, we obtain

$$\frac{Q^t}{V} = \frac{\omega}{8\pi}\left(\frac{\nu}{\omega}\frac{\omega_0^2}{\omega^2}\right)|1+r_p|^2\,|H_{xi}(0)|^2 \times \exp\left\{-\frac{z\nu}{c}\frac{\omega_0^2/\omega^2}{[\varepsilon'(\omega)-\sin^2\theta\,(1+\alpha^t)]^{1/2}}\right\}, \qquad (47)$$

where r_p is given by (25).

Analogously we obtain for longitudinal waves (when $kv_T/\omega \ll 1$)

$$\frac{Q^l}{V} = \frac{\omega}{8\pi}\varepsilon^{l''}|1+r_p|^2\,|H_{xi}(0)|^2 \times \exp\left\{-\frac{z\omega}{c\sqrt{\alpha^l}}\frac{\varepsilon^{l''}}{[\varepsilon'-\alpha^l\sin^2\theta]^{1/2}}\right\},$$

$$\varepsilon^{l''} = \frac{\nu_{\text{eff}}\omega_{Le}^2}{\omega^3} + \sqrt{\frac{\pi}{2}}\frac{\omega\omega_{Le}^2}{k^3(\varkappa T_e/m)^{3/2}}\exp\left(-\frac{\omega^2 m}{2k^2\varkappa T_e}\right). \qquad (48)$$

When $1 \gg kr_D \gg 1/\sqrt{\ln \kappa T_e/e^2N_e^{1/3}}$, the collisions can be neglected, i.e., under these conditions the main contribution is produced by Cerenkov absorption.

We note, finally, that in the opacity region the Cerenkov absorption mechanism makes no contribution to the dissipation.

APPENDIX

ASYMPTOTIC BEHAVIOR OF THE FIELD AT LARGE z

Let us examine briefly the question of the asymptotic value of the field at large distances from the plasma surface. We can say that the asymptotic expression of interest to us is determined by the singularities of the Fourier transforms of the field, which we have obtained above. The presence of poles, obviously, leads here to solutions that depend exponentially on the coordinates. In particular, in the opacity region the field decreases exponentially, and in the transparency region we can speak of waves.

Under conditions when the spatial dispersion of the dielectric permittivity is small, we have for this wave part of the field

$$E_y^t(z) = -\left\{E_y'(0) - i\frac{\omega}{c}\sin\theta\, E_z(0)\right\}\frac{ic}{\omega c(\omega)} \times\left[\frac{\varepsilon(\omega)-\sin^2\theta\,(1+\alpha^t)}{1+\alpha^t}\right]^{1/2} \times \exp\left\{iz\frac{\omega}{c}\left[\frac{\varepsilon(\omega)-\sin^2\theta\,(1+\alpha^t)}{1+\alpha^t}\right]^{1/2}\right\}, \quad (A.1)$$

$$E_y^l(z) = -\left\{E_y'(0) - i\frac{\omega}{c}\sin\theta\, E_z(0)\right\}\frac{ic}{\omega}\frac{\sin^2\theta\,\sqrt{\alpha^l}}{\varepsilon(\omega)\,[\varepsilon(\omega)-\alpha^l\sin^2\theta]^{1/2}} \times \exp\left\{iz\frac{\omega}{c\sqrt{\alpha^l}}\sqrt{\varepsilon(\omega)-\alpha^l\sin^2\theta}\right\}. \quad (A.2)$$

At frequencies greater than the plasma frequency, we obtain for an ultrarelativistic plasma the following contribution of the pole to the longitudinal field

$$E_y^l(z) = -\left\{E_y'(0) - i\frac{\omega}{c}\sin\theta\, E_z(0)\right\}\frac{ic}{\omega}\frac{8}{3}\frac{\omega^2}{\omega_{0p}^2} \times\frac{\sin^2\theta}{\cos\theta}\exp\left(-\frac{2\omega^2}{3\omega_{0p}^2}-2\right)\exp\left\{iz\frac{\omega}{c}\cos\theta\right\}. \quad (A.3)$$

It is clear that the amplitude of the field decreases exponentially with increasing frequency.

We note that formulas (A.1) and (A.2) can be obtained from the differential equations of the field, which apply when the permittivities are given by formulas (9) and (30), and if in addition we use the boundary condition $E_Z(0) = 0$.

The branch point of the permittivity causes the field to have a coordinate dependence that cannot be identified with a wave (see [2,5]). The corresponding contribution to the longitudinal and transverse fields has the following form

$$E_y^t(z) = \left\{E_y'(0) - i\frac{\omega}{c}\sin\theta E_z(0)\right\}\frac{2ic}{\pi\omega}(1+i\nu/\omega) \times\int_1^\infty\frac{dx}{x}\left[x^2-\sin^2\theta\left(\frac{\omega}{\omega+i\nu}\right)^2\right]^{1/2} \times\exp\left\{iz\frac{\omega+i\nu}{c}\left[x^2-\sin^2\theta\left(\frac{\omega}{\omega+i\nu}\right)^2\right]^{1/2}\right\} \times\operatorname{Im}\varepsilon_+^t\left(\omega,\frac{\omega+i\nu}{c}x\right)\left\{\left[\operatorname{Re}\varepsilon_+^t\left(\omega,\frac{\omega+i\nu}{c}x\right) -\left(1+i\frac{\nu}{\omega}\right)^2x^2\right]^2 +\left[\operatorname{Im}\varepsilon_+^t\left(\omega,\frac{\omega+i\nu}{c}x\right)\right]^2\right\}^{-1}, \quad (A.4)$$

$$E_y^l(z) = \left\{E_y'(0) - i\frac{\omega}{c}\sin\theta E_z(0)\right\}\frac{2ic}{\pi\omega}\left(1-i\frac{\nu}{\omega}\right) \times\int_1^\infty\frac{dx}{x}\left[x^2-\sin^2\theta\left(\frac{\omega}{\omega+i\nu}\right)^2\right]^{-1/2} \times\exp\left\{iz\frac{\omega+i\nu}{c}\left[x^2-\sin^2\theta\left(\frac{\omega}{\omega+i\nu}\right)^2\right]^{1/2}\right\} \times\operatorname{Im}\varepsilon_+^l\left(\omega,\frac{\omega+i\nu}{c}x\right)\Big/\left|\varepsilon_+^l\left(\omega,\frac{\omega+i\nu}{c}x\right)\right|^2; \quad (A.5)$$

$$\operatorname{Re}\varepsilon_+^l\left(\omega,\frac{\omega+i\nu}{c}x\right) = 1-\frac{2\pi e^2N_ec^2}{\varkappa T_e\omega^2\left(1+i\frac{\nu}{\omega}\right)}\left[K_2\left(\frac{mc^2}{\varkappa T_e}\right)\right]^{-1}\left(\int_{-\infty}^{-1}+\int_1^\infty\right)\frac{dx'}{x'^3} \times P\frac{1}{x'-x}\left\{\frac{1}{1-x'^{-2}}+2\left(\frac{mc^2}{\varkappa T_e}\right)^{-1}\frac{1}{\sqrt{1-x'^{-2}}} +2\left(\frac{mc^2}{\varkappa T_e}\right)^{-2}\right\}\exp\left\{-\frac{mc^2}{\varkappa T_e}\frac{1}{\sqrt{1-x'^{-2}}}\right\}, \quad (A.6)$$

$$\operatorname{Im}\varepsilon_+^l\left(\omega,\frac{\omega+i\nu}{c}x\right) = \frac{2\pi^2e^2N_ec^2}{\varkappa T_e\omega^2\left(1+i\frac{\nu}{\omega}\right)}\left[K_2\left(\frac{mc^2}{\varkappa T_e}\right)\right]^{-1}\frac{1}{x'^3} \times\left\{\frac{1}{1-x'^{-2}}+2\left(\frac{mc^2}{\varkappa T_e}\right)^{-1}\frac{1}{\sqrt{1-x'^{-2}}} +2\left(\frac{mc^2}{\varkappa T_e}\right)^{-2}\right\}\exp\left\{-\frac{mc^2}{\varkappa T_e}\frac{1}{\sqrt{1-x'^{-2}}}\right\}, \quad (A.7)$$

and $\operatorname{Re}\epsilon_+^t$ and $\operatorname{Im}\epsilon_+^t$ are defined in [2].

In general, the resultant expressions are complicated. In the ultrarelativistic limit we have for the transverse field the following contribution to the asymptotic expression corresponding to formula (A.4):

$$E_y^t(z) \approx -\left\{E_y'(0) - i\frac{\omega}{c}\sin\theta E_z(0)\right\}\frac{3}{2}\left[1-\left(\frac{\omega}{\omega+i\nu}\right)^2\sin^2\theta\right] \times\frac{\omega_{0p}^2}{(\omega+i\nu)^2}\left(\frac{c}{\omega}\right)^3\left\{-2i\frac{\nu}{\omega}+\left(\frac{\nu}{\omega}\right)^2-\frac{3}{2}\frac{\omega_{0p}^2}{\omega^2\left(1+i\frac{\nu}{\omega}\right)}\right\}^{-2}\frac{1}{z^2} \times\exp\left\{i\frac{z}{c}\sqrt{(\omega+i\nu)^2-\omega^2\sin^2\theta}\right\}. \quad (A.8)$$

In the case of a nonrelativistic plasma, the asymptotic behavior of the field can be determined by the saddle-point method, as was done by Landau[6] and Shafranov.[7] In this case the limits of the integrals in (A.4) and (A.5) are 0 and ∞, while the integrands are assumed to be independent of the angle of incidence θ, in view of the large value of the ratio $mc^2/\kappa T_e$. It should be noted however, that relativistic effects come into play when z is sufficiently large, and the asymptotic expression obtained in this manner does not apply. The corresponding distances are determined from the conditions $z \gtrsim (mc^2/\kappa T_e)(c/\omega)$.

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