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INTERACTION OF TRANSVERSE OSCILLATIONS IN A PLASMA

A. P. KAZANTSEV and I. A. GILINSKII

Institute of Radiophysics and Electronics, Siberian Branch, Academy of Sciences, U.S.S.R.

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The role of a nonlinear effect associated with the influence of the magnetic field of transverse waves in a plasma is discussed. It is shown that the waves are modulated if the frequency difference of two transverse waves is equal to the plasma frequency (resonance interaction). In a nonresonance interaction only a weak frequency shift occurs. The adiabatic invariants for the problem are found.

IN the usual analysis of small oscillations of a plasma the effect of the self magnetic field on the plasma is neglected because this effect is of order nv/c (n is the refractive index and v is some characteristic velocity) as compared with the effect of the electric field. It is of interest, however, to examine this effect because it can lead to an additional interaction between the waves.* It is clear that this effect will be of greatest importance in a magnetoactive plasma when the refractive index n becomes large.

We shall limit ourselves to the case of wave propagation along a uniform magnetic field **H** (0, 0, H). In this case the transverse waves interact via the excitation and absorption of longitudinal oscillations.

1. BASIC EQUATIONS

The equations that describe the plasma (hydrodynamic approximation) and the electromagnetic field are

$$du/dt + eE/m = -(e/2mc)(v\partial A^*/\partial z + v^*\partial A/\partial z), \quad (1)$$

$$\partial E/\partial z + 4\pi e n_0 \rho = 0, \qquad d\rho/dt + (1+\rho)\,\partial u/\partial z = 0, \quad (2)$$

$$\frac{d}{dt}\left(v - \frac{e}{c}A\right) = i\frac{eH}{mc}v, \quad (3)$$

$$\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} - \frac{\partial^2 A}{\partial z^2} = -\frac{4\pi}{c}en_0(1+\rho)v. \quad (4)$$

Here $d/dt = \partial/\partial t + u\partial/\partial t$; u and E are the electron velocity and the electric field along the z axis, and $\rho = (n - n_0)/n_0$ is the relative variation in electron density. The ions are fixed and characterized by a density n_0.

The transverse electromagnetic field is described by the vector potential $\mathbf{A}(A_x, A_y, 0)$ and $A = A_x + iA_y$, $v = v_x + iv_y$, where v_x and v_y are the transverse electron velocities. The electrons are assumed to be at zero temperature and dissipative processes are neglected.

Hereinafter we assume that no longitudinal self oscillations are excited. Taking the right-hand side of (1) to be small, we neglect quadratic terms in u and ρ.* Then, Eqs. (1) — (4) can be written in dimensionless form

$$\frac{\partial^2\rho}{\partial t^2} + \rho = \varepsilon^2\frac{\partial}{\partial z}\left(v\frac{\partial A^*}{\partial z} + v^*\frac{\partial A}{\partial z}\right), \quad (5)$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial u}{\partial z} = 0, \qquad \frac{d}{dt}(v - A) = i\omega_H v, \quad (6)$$

$$\partial^2 A/\partial t^2 - \partial^2 A/\partial z^2 + (1+\rho)v = 0. \quad (7)$$

Here, time, length, and longitudinal velocity are expressed in units of $1/\omega_0 = (m/4\pi e^2 n_0)^{1/2}$, c/ω_0, and c respectively. The unit of vector potential is some characteristic initial amplitude A_0 so that the dimensionless quantity A is of order unity. The transverse velocity is measured in units of $v_0 = eA_0/mc$; $\epsilon = |v_0|c\sqrt{2}$ is a small parameter and $\omega_H = eH/mc\omega_0$ is the dimensionless Larmor frequency.

2. APPROXIMATE SOLUTION

We seek an approximate solution of Eqs. (5) — (7) in the form of a superposition of plane waves with amplitudes that vary slowly in time:

$$A(z,t) = \sum_\omega A_\omega(t)\,e^{i\varphi_\omega},$$

$$v(z,t) = \sum_\omega \frac{\omega}{\omega - \omega_H}(A_\omega + B_\omega)\,e^{i\varphi_\omega},$$

$$\rho(z,t) = \sum_{\omega>\omega'} \alpha_{\omega\omega'}(t)\,e^{i(\varphi_\omega - \varphi_{\omega'})} + \text{c.c.} \quad (8)$$

*Wave interactions due to dissipation processes have been investigated in detail in a number of papers (cf. the review by Ginzburg and Gurevich[1]).

*The interaction of longitudinal oscillations in a plasma has been investigated by Sturrock.[2]

where

$\varphi_\omega = \omega t + k_\omega z,\ \ k_\omega = \omega n_\omega$ and $n_\omega^2 = 1 - 1/\omega(\omega - \omega_H)$

is the refractive index, which is determined from the solution of the linearized problem.

In (8) we have neglected combination waves since the amplitudes of these waves are proportional to a power of the small parameter. By slowly varying amplitudes here we mean that $\dot{A}_\omega \ll \omega A_\omega$, etc ($\dot{A}_\omega$ is also proportional to a power of ϵ). The quantity B_ω is a correction ($B_\omega \ll A_\omega$) that takes account of the deviation of v from the solution of the corresponding linearized problem. It is also assumed that any frequency of interest is not too close to ω_H.

Substituting Eq. (8) in Eqs. (5) — (7) and equating coefficients for the same phase we obtain a system of equations for the amplitudes:

$$2i(\omega - \omega')\dot{\alpha}_{\omega\omega'} + [1 - (\omega - \omega')^2]\alpha_{\omega\omega'} = -\varepsilon^2\lambda_{\omega\omega'}A_\omega A^*_{\omega'}, \tag{9}$$

$$\mu(\omega)\dot{A}_\omega = i\sum_{\omega'}\alpha_{\omega\omega'}\frac{\lambda_{\omega\omega'}}{(k_\omega - k_{\omega'})^2}A_{\omega'}, \tag{10}$$

$$B_\omega = i\frac{\omega_H}{\omega}\left(\frac{\dot{A}_\omega}{\omega - \omega_H} + \sum_{\omega'}\frac{k_{\omega'}(\omega - \omega')\alpha_{\omega\omega'}A_{\omega'}}{(k_\omega - k_{\omega'})(\omega' - \omega_H)}\right), \tag{11}$$

where

$$\lambda_{\omega\omega'} = (k_\omega - k_{\omega'})\left(\frac{k_\omega\omega'}{\omega' - \omega_H} - \frac{k_{\omega'}\omega}{\omega - \omega_H}\right),$$

$$\mu(\omega) = 2\omega + \frac{\omega_H}{(\omega - \omega_H)^2}.$$

Equations (9) — (10) have the following integrals:

$$\mu(\omega)|A_\omega|^2 + \frac{2}{\varepsilon^2}\sum_{\omega'}\frac{\omega - \omega'}{(k_\omega - k_{\omega'})^2}|\alpha_{\omega\omega'}|^2 = \mathrm{const}. \tag{12}$$

In the summation in the last expression only the resonance and (near-resonance) terms $\alpha_{\omega\omega'}$ are important; these terms satisfy the condition

$$(\omega - \omega')^2 = 1. \tag{13}$$

The resonance terms can be of order ϵ; far from resonance $\alpha_{\omega\omega'}$ is smaller than ϵ^2. It follows from (12) that in general when different waves interact the energy of the high-frequency waves is reduced while the energy of the low-frequency waves is increased.

Summing over ω in (12) we find that the following quantity is conserved:

$$\sum_\omega \mu(\omega)|A_\omega|^2 = \mathrm{const}. \tag{14}$$

The integrals in (12) and (14) represent adiabatic invariants of the problem being considered. The quantity $\mu(\omega)|A_\omega|^2/8\pi$ is the energy density of the plane waves divided by the frequency. In a dispersive medium this energy density is [3]

$$\frac{\omega}{8\pi}|A_\omega|^2\left[n_\omega^2 + \frac{d}{d\omega}(\omega n_\omega^2)\right]. \tag{15}$$

The quantity in the summation sign in (12) is proportional to the energy density of the longitudinal oscillations interacting with a given wave, divided by the frequency.

We note that (14) may be given the following quantum-mechanical interpretation. If we introduce the notion of transverse quasiparticles (A) and longitudinal quasi-particles (α), then the Hamiltonian of the interaction between them is cubic in the amplitude (quadratic with respect to A). Thus the meaning of (14) is that the number of transverse quasi-particles is conserved.*

Further investigation of Eqs. (9) — (10) is difficult for the general case. For this reason we make separate analyses of the resonance interaction of two waves and the nonresonance interaction (in which case the quantity $|\omega - \omega'|$ does not approach unity at any frequency).

In the nonresonance case we can omit $\dot{\alpha}$ in (9) and the solution of (9) — (10) can be easily found in the form

$$A_\omega(t) = A_\omega(0)\,e^{i\Delta\omega t}, \tag{16}$$

where

$$\Delta\omega = -\frac{\varepsilon^2}{\mu(\omega)}\sum_{\omega'}\frac{\lambda^2_{\omega\omega'}|A_{\omega'}(0)|^2}{(k_\omega - k_{\omega'})^2[1 - (\omega - \omega')^2]}.$$

The applicability condition for the last relation ($\Delta\omega \ll \omega$) is the inequality $\epsilon(\omega_H - \omega)^{-1} \ll 1$.

3. INTERACTION OF TWO WAVES

We now consider the case of a resonance interaction between two extraordinary waves ($\omega_1 - \omega_2 = 1$) assuming that $\omega_1, \omega_2 < \omega_H$ and $\omega_H > 1$. For definiteness we denote quantities pertaining to the high-frequency wave by the subscript "1." We also introduce the notation

$$I_{1,2} = \mu_{1,2}(\omega)|A_{1,2}|^2, \qquad |\alpha|^2 = \tfrac{1}{2}\varepsilon^2 I_{10}(k_1 - k_2)^2\zeta.$$

If it is assumed that there are no longitudinal oscillations at the initial time [$\dot{\zeta}(0) = \zeta(0) = 0$], then (12) can be written in the form

$$I_1 + I_{10}\zeta = I_{10}, \qquad I_2 - I_{10}\zeta = I_{20}, \tag{17}$$

so that $0 \le \zeta \le 1$. Using (9) — (10) and (17) we obtain the following differential equation for ζ:

*The author is indebted to V. L. Pokrovskii for these observations.

$$d^2\zeta/d\tau^2 + \zeta(I_{20} - I_{10}) + \tfrac{3}{2} I_{10}\zeta^2 = \tfrac{1}{2} I_{20}, \quad (18)$$

where

$$\tau = \varepsilon t\,[2\lambda^2/\mu_1\mu_2\,(k_1 - k_2)^2]^{1/2}.$$

Integrating once in (18) and introducing the initial conditions we have

$$(d\zeta/d\tau)^2 + \zeta^2(I_{20} - I_{10}) + I_{10}\zeta^3 - I_{20} = 0 \quad (19)$$

The cubic term in the left side of (19) has the following roots

$$0;\quad 1;\quad -I_{20}/I_{10}.$$

Consequently, ζ varies periodically from 0 to 1 with period $2\tau_0$, where

$$\tau_0 = \int_0^1 \frac{d\zeta}{\sqrt{\zeta(1-\zeta)(I_{20} + I_{10}\zeta)}}. \quad (20)$$

This same period characterizes the oscillations of the energy in wave 1 (E_1) between zero and the initial value and the oscillations of the energy of wave 2 between E_{20} and $E_{20} + \omega_2 E_{10}/\omega_1$. When the frequency of the first wave is close to ω_H ($\omega_H - \omega_1 \ll 1$) its energy increases sharply, $I_{10} \gg I_{20}$ and the modulation period is determined approximately by relation

$$\tau_0 = I_{10}^{-1/2} \ln(4\, I_{10}/I_{20}).$$

However, the applicability condition for Eqs. (9) — (10) imposes definite limitations on how small $\omega_H - \omega$ can be. Specifically, (11) and the requirement $B_\omega \ll A_\omega$ mean that the inequality $\epsilon(\omega_H - \omega)^{-3/2} \ll 1$ must be satisfied.

Using (9) we can easily estimate the width of the resonance interaction region. The "detuning" frequency is given by the relation

$$|1 - \omega_1 + \omega_2| \lesssim \varepsilon.$$

The quadratic terms in u and ρ are of order ϵ^4 far from resonance; at resonance these terms are of order ϵ^2 and the nonlinear inertia term $u\,\partial u/\partial z$ can be of the same order of magnitude as the driving force $\epsilon^2(v\,\partial A^*/\partial z + \text{c.c.})$. It is easily shown, however, that this term does not contain harmonics that interact. Thus it is valid to neglect the quadratic terms in u and ρ.

All the foregoing considerations are obviously valid to an accuracy of order ϵ for the resonance interaction and ϵ^2 for the nonresonance case. There is no qualitative change in the results if the electron temperature is nonzero. In place of (13) we obtain a resonance condition of the form

$$(\omega_1 - \omega_2)^2 - (c_s/c)^2 (k_1 - k_2)^2 = 1,$$

where c_s is the electron thermal velocity.

If we take $\omega_H = 0$ in the above formulas the interaction of circularly polarized waves in an isotropic plasma can be treated.

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[2] P. A. Sturrock, Proc. Roy. Soc. (London) **A242**, 277 (1957).

[3] Landau and Lifshitz, Elektrodinamika sploshnykh sred (Electrodynamics of Continuous Media), Gostekhizdat, 1957.

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33