BEHAVIOR OF MULTIPLY CHARGED IONS IN A PLASMA

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It is shown that under certain conditions the direction of motion of multiply charged ions in a singly ionized plasma in a fixed electric field must be opposite to that of the singly charged ions. Under these conditions the velocity of the multiply charged ions can be of approximately the same magnitude as the directed electron velocity while their energy can be one to three orders of magnitude greater than the thermal energy of the singly charged ions and the electrons.

1. We consider an ion of charge Ze in a fully ionized plasma, consisting of electrons and singly charged ions, in a fixed electric field. The motion of the ions is obviously described by the equation

$$M d\mathbf{v}/dt = eZ\mathbf{E} - \mathbf{F}_e - \mathbf{F}_i. \tag{1}$$

Here, M is the ion mass, **v** is the ion velocity, $\mathbf{F}_e + \mathbf{F}_i$ is the friction force which arises by virtue of the interaction with the plasma electrons ($\mathbf{F}_e$) and ions ($\mathbf{F}_i$).

Because of the reciprocal nature of the interaction between the electrons and the ions, $\mathbf{F}_e$ can be written in the form

$$\mathbf{F}_e = m\nu_{ei}(N_e/N_Z)(\mathbf{v} - \mathbf{v}_{e0})$$

(cf. reference 1, Sec. 63). Here, m is the mass, N_e is the number density, $\mathbf{v}_{e0}$ is the mean directed velocity of the electrons, N_Z is the number density of ions with charge Ze, and ν_{ei} is the frequency of collisions between the electrons and these ions. Hereinafter it is convenient to consider Eq. (1) in a coordinate system that moves with the singly charged ions. The expression for $\mathbf{F}_e$ can then be written in the form

$$\mathbf{F}_e = m\nu_{e0}Z^2(\mathbf{v} - \mathbf{v}_0), \tag{2}$$

where $\mathbf{v}_0 = \mathbf{v}_{e0} - \mathbf{v}_{i0}$ is the difference between the mean directed velocities of the electrons and the singly charged ions, while ν_{e0} is the effective collision frequency for collisions between electrons and singly charged ions in the fixed electric field:

$$\nu_{e0} = \tfrac{4}{3}\sqrt{2\pi}e^4 N m^{-1/2}(kT_e)^{-3/2}\ln\Lambda. \tag{3}$$

Here, $N = N_e = N_i$ is the number density for the electrons or single charged ions in the plasma, T_e is the electron temperature, k is the Boltzmann constant and ln Λ is the Coulomb logarithm.

Since $\mathbf{v}_0 = e\mathbf{E}/m\nu_{e0}$, the expression for $\mathbf{F}_e$, Eq. (2), can be written

$$\mathbf{F}_e = eZ^2\mathbf{E} + m\nu_{e0}Z^2\mathbf{v}. \tag{2a}$$

We note that this expression applies only when

$$|\mathbf{v} - \mathbf{v}_0| \ll v_{Te}, \tag{2b}$$

where v_{Te} is the thermal velocity of the electrons. In other words, the essential requirement is that the mean directed velocity of the electrons must be smaller than their thermal velocity. Dreicer[2] (cf. also reference 3) has shown that this requirement is satisfied only when $E \ll E_c = 4\pi e^3 N \ln\Lambda/kT_e$. For the effects considered in the present paper, we shall be interested in relatively weak fields in which this condition is obviously satisfied [cf. Eqs. (5) and (10)].* Furthermore, it follows from Eq. (2b) that the velocity of the multiply charged ion must be smaller than the thermal velocity of the electrons, a condition which we naturally assume to be satisfied.

However, the analagous condition for the interaction with the plasma ions, $v < v_{Ti}$, cannot be satisfied. When $v > v_{Ti}$ the collision frequency goes as v^{-3} while the frictional force goes as v^{-2}. When this circumstance is taken into account the force $\mathbf{F}_i$ is given by the approximate expression:

$$\mathbf{F}_i = m\nu_{e0}Z^2\gamma\mathbf{v}/[1 + (v/v_{Ti})^3],$$
$$\gamma = (M_0 T_e^3/m T_{i0}^3)^{1/2}. \tag{4}$$

*We note that because of this condition, Eq. (2a), which gives the friction force, does not apply if the plasma density approaches zero (for a fixed value of E). Consequently, the features of the behavior of multiply charged ions noted below do not appear in this case; these ions then move in the direction of the field, as expected.

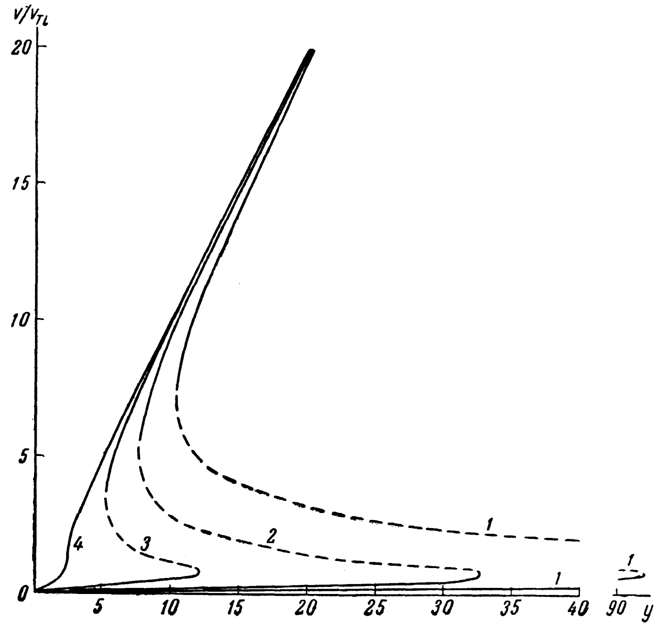

FIG. 1. The dependence of v/v_{Ti} on $y = e(Z-1) \times E/Zm\nu_{e0}v_{Ti} = (Z-1)v_0/Z\,v_{Ti}$ for different values of $\gamma = (M_0T_e^3/mT_{i0}^3)^{1/2}$; $\gamma = 172$, 60.8, 21.5, and 3 for curves 1, 2, 3, and 4, respectively.

Here, M_0, T_{i0} and v_{Ti} are respectively the mass, temperature, and thermal velocity of the plasma ions. The parameter γ is determined by the properties of the plasma in which the multiply charged ion is located. The value of γ is usually very large; for example, if $T_{i0} = T_e$, then in deuterium $\gamma \approx 60.8$ and in hydrogen $\gamma \approx 43$.

Equation (1) is now written in the form

$$M d\mathbf{v}/dt = -eZ(Z-1)\mathbf{E} - m\nu_{e0} Z^2 \mathbf{v}\,[1 + \gamma/(1 + v/v_{Ti})^3)]. \qquad (1a)$$

What is most obvious from the above is that a multiply charged ion moving with the singly charged ions in the plasma ($\mathbf{v} = 0$) is always acted on by a force in the direction of motion of the electrons. The origin of this force is easily understood if we consider that the force $e\mathbf{E}$ which the field exerts on the plasma ions (i.e., the singly charged ions) is exactly balanced by the friction caused by the interaction of these ions with the traveling electron stream. If a multiply charged ion is placed in this plasma, the force exerted on it by the field is larger by a factor of Z, while the friction force is larger by a factor of Z^2, so that the multiply charged ions are carried along by the electron stream.

Solving Eq. (1a) we can find the velocity of the multiply charged ion. The dependence of this velocity v on electric field, as determined from this equation for stationary conditions ($dv/dt = 0$), is shown in Fig. 1. It is obvious from the figure that in general the relation between v and E is not unique; in the region

$$3\gamma^{1/3} m\nu_{e0} v_{Ti} Z/2^{2/3} e(Z-1) \leqslant E \leqslant 2^{2/3} \gamma m\nu_{e0} v_{Ti} Z/3e(Z-1) \qquad (5)$$

a given value of field corresponds to two stable stationary values of the velocity rather than one, as is usually the case (the third stationary value of the velocity, shown in Fig. 1 by the dashed line, is unstable).

In the first stationary state, which corresponds to the lower curve, the velocity of the multiply charged ions is small: $v_1 \approx v_0(Z-1)/\gamma Z$. In this case the interaction with the plasma ions (the force $\mathbf{F}_i$) is decisive. On the other hand, in the second stationary state, corresponding to the upper curve, the velocity of the multiply charged ion is very large: $v_2 \approx v_0(Z-1)/Z$. Under these conditions the interaction with the plasma electrons assumes the dominant role because the ion interaction becomes unimportant at high velocities ($v \gg v_{Ti}$). The energy of the multiply charged ion is always large in the second stationary state:

$$\varepsilon_2 = \frac{Mv_2^2}{2} = \frac{M}{M_0}\left(\frac{v_0}{v_{Ti}}\right)^2 \left(\frac{Z-1}{Z}\right)^2 kT_{i0}.$$

This energy is many times (one to three orders of magnitude) greater than the thermal energy of the electrons or ions in the plasma. The characteristic time τ required for the ion to acquire an energy ϵ_2 is of order $M/m\nu_{e0}Z^2$.

As γ decreases the difference between the curves corresponding to the first and second states become smaller (cf. Fig. 1). These curves finally coalesce when $\gamma \leq 3$ [i.e., $T_e/T_{i0} \leq 2.1 \times (m/M_0)^{1/3}$] and in this case a given energy E corresponds to one stationary value of the velocity v.

With respect to a fixed observer (not with respect to a singly charged ion, as considered above) the velocity of the multiply charged ion is obviously $\mathbf{v} + \mathbf{v}_{i0}$, where $\mathbf{v}_{i0}$ is the mean velocity of the singly charged ions. Since the velocity $\mathbf{v}$ is always in the direction of motion of the electrons while the velocity $\mathbf{v}_{i0}$ is in the opposite direction, the multiply charged ion can move in either direction, depending on the ratio of the velocities v and v_{i0}. In particular, in a fully ionized equilibrium plasma (laminar) the velocity $\mathbf{v}_{i0}$ is very small (its magnitude is limited by the conservation of total momentum for the electrons and singly charged ions $v_{i0} = mv_{e0}/M_0$) so that v is almost always larger than v_{i0} (for $\gamma < M_0/m$). Thus, in an equilibrium plasma the multiply charged ion almost always moves in the same direction as the electrons. Under actual conditions, however, the directed velocity of the singly charged ions can be appreciably greater;[4] in any case, it can be greater than the first stationary velocity of the multiply charged ions.* On the other hand, the second stationary velocity is always greater than v_{i0}.

Thus, the elementary analysis given above indicates that two substantially different stationary states of multiply charged ions in a plasma can occur at a single value of the field. However, this analysis does not tell us which of these states is actually realized. Since any velocity is possible, by virtue of the existing particle velocity distribution, any of the ions will, in general, always be in one state (i.e., the velocities of these ions will be grouped about one of the stationary values) while the other ions will be in the other. Transfer of ions between states is also possible. How many ions will be in each of the states at total equilibrium? These questions can be answered only when the velocity distribution of the multiply charged ions is analyzed.

2. In a fully ionized plasma the kinetic equation for the velocity distribution $f(\mathbf{v}, t)$ of ions with charge Ze is of the form

$$\frac{\partial f}{\partial t}-\frac{eZ(Z-1)E}{M}\left(\cos\theta\frac{\partial f}{\partial v}-\frac{\sin\theta}{v}\frac{\partial f}{\partial\theta}\right)-\frac{1}{v^2}\frac{\partial}{\partial v}\left\{v^2\left[\frac{M}{M_0}\nu_i(v)\times G\left(\frac{v}{(2kT_{i0}/M_0)^{1/2}}\right)\left(\frac{kT_{i0}}{M}\frac{\partial f}{\partial v}+vf\right)+\frac{m}{M}\nu_{e0}Z^2\left(\frac{kT_e}{M}\frac{\partial f}{\partial v}+vf\right)\right]\right\}$$
$$-\frac{\nu_i(v)}{2\sin\theta}H\left(\frac{v}{(2kT_{i0}/M_0)^{1/2}}\right)\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)=0. \qquad (6)$$

Here, θ is the angle between $\mathbf{E}$ and $\mathbf{v}$, $\nu_i(v) = 4\pi e^4 NZ^2 \ln\Lambda/M^2v^3$ is the collision frequency for collisions between multiply charged ions and singly charged ions, and $G(x)$ and $H(x)$ are the functions introduced by Chandrasekhar:[5]

$$G(x)=\Phi(x)-2xe^{-x^2}/\sqrt{\pi},$$
$$H(x)=(1-\tfrac{1}{2}x^{-2})\Phi(x)+e^{-x^2}/\sqrt{\pi}x,$$

where $\Phi(x)$ is the probability integral. When $x \gg 1$ the functions $G(x)$ and $H(x)$ are close to unity; when $x \ll 1$ we have

$$G(x)=4x^3/3\sqrt{\pi},\quad H(x)=4x/3\sqrt{\pi}.$$

It is obvious from Eq. (6) that the multiply charged ions, which move with singly charged plasma ions, are subject to a force $\mathbf{F} = -eZ\times(Z-1)\mathbf{E}$ in the direction of motion of the electrons.*

For low velocities $v \lesssim (kT_{i0}/M)^{1/2}$ it is natural to seek a solution of (6), as usual, (cf. reference 7) in the form

$$f=f_0(v)+f_1(v)\cos\theta+\chi(v,\theta),\qquad \chi\ll f_0. \qquad (7)$$

The function f_1 is then easily found to be

$$\nu_i(v)H(x)f_1-\frac{M}{M_0v^2}\frac{\partial}{\partial v}\left\{v^2\nu_i(v)G(x)\left(\frac{kT_{i0}}{M}\frac{\partial f_1}{\partial v}+vf_1\right)\right\}=\frac{eEZ(Z-1)}{M}\frac{\partial f_0}{\partial v}. \qquad (7a)$$

In this case the function f_0 is Maxwellian with a characteristic temperature T_{i0}. It is obvious from Eqs. (7) and (7a) that at small velocities the distribution function is weakly dependent on the direction of the velocity.

On the other hand, at high velocities $v > (kT_{i0}/M)^{1/2}$ the distribution function is sensitive to the direction of the velocity. A method of solving an equation similar to Eq. (6) in the high-velocity region has been developed in reference 3; it is shown that at high velocities the distribution function decreases slowest in the direction of the acting force $\theta = 0$, i.e., in our case, in the direction opposite to the electric field. For velocities close to this direction ($\theta = 0$) the distribution function is of the form

$$f=(M/2\pi kT_{i0})^{3/2}N_1\times\exp\left\{-\int_{v_1}^{v}\frac{M\nu_i(v)Gv/M_0+m\nu_{e0}Z^2v/M-eZ(Z-1)E/M}{\nu_i(v)GkT_{i0}/M_0+\nu_{e0}Z^2kT_em/M^2}dv\right\}, \qquad (8)$$

where N_1 is the number density of the multiply charged ions in the first equilibrium state and v_1 is the first root of the equation

$$\frac{M}{M_0}\nu_i(v)G\left(\frac{v}{(2kT_{i0}/M_0)^{1/2}}\right)v+\frac{m}{M}\nu_{e0}Z^2v-\frac{eZ(Z-1)E}{M}=0. \qquad (9)$$

Equation (8) is obtained as a first approximation in an expansion in powers of the parameter $(kT_{i0}/Mv_c^2)^{1/2}$ in the exponential term; it is valid only when $v < v_c$ where the critical velocity v_c

*The relation $mv_{e0} + M_0v_{i0} = 0$ breaks down even if the plasma contains a large number of neutral particles or multiply charged ions, but all the more so in the presence of essentially nonequilibrium processes such as may cause a marked increase in the transfer of electron momentum to the walls of the chamber or to the inhomogeneities of the magnetic field.

*Equation (6) is written in a coordinate system that moves with the singly charged plasma ions. The collision integral for collisions between the multiply charged ions and the electrons and ions in the plasma is used in the differential form given by Landau[6] (terms that describe collisions with electrons are written under the assumption that the ion velocity v is smaller than the mean thermal velocity of the electrons). For simplicity it is assumed that the singly charged ions have a Maxwellian distribution. The directed velocity of the electrons is taken to be $v_0 = eE/m\nu_{e0}$. Interactions between the multiply charged ions themselves are neglected.

is the mean root of Eq. (9). It is apparent that a critical velocity v_c does not exist for every value of the field: E must be larger than E_{c1} and smaller than E_{c2} where

$$E_{c1} = \frac{Z}{Z-1}\left(\frac{3m}{2\pi M_0}\right)^{1/3} \frac{4\pi e^3 N \ln\Lambda}{kT_e} \approx 1.7\,\frac{Z}{Z-1}\left(\frac{M_p}{M_0}\right)^{1/3} \frac{N}{10^{14}}\frac{\ln\Lambda}{kT_e},$$

$$E_{c2} = 0.21\,\frac{Z}{Z-1}\frac{4\pi e^3 N \ln\Lambda}{kT_{i0}} \approx 5.5\,\frac{Z}{Z-1}\frac{N}{10^{14}}\frac{\ln\Lambda}{kT_{i0}} \qquad (10)$$

(in the numerical expression kT_e and kT_i are given in ev and E is in v/cm; M_p is the mass of the proton).

If $E < E_{c1}$ or $E > E_{c2}$, the distribution function decreases monotonically with increasing v; when $E_{c1} < E < E_{c2}$ the function exhibits a second maximum. Thus if the field E lies between E_{c1} and E_{c2} the second stationary state indicated in the preceding section is possible. As expected, the field values E_{c1} and E_{c2} coincide with the limiting values of the field indicated by Eq. (5) to within a numerical factor of order unity.

Further, it has been noted above that if $E_{c2} > E > E_{c1}$, then Eq. (6) gives the distribution function only when $v \le v_c$. The form of the distribution function for $v > v_c$ can be found easily by means of the preceding method but now the origin in Eq. (6) must be displaced to the point v_2, corresponding to the second equilibrium value of the velocity, i.e., the second maximum in the distribution function in Eq. (8). In this case the velocity v_2 is naturally determined by the same relation, Eq. (9). As before, when $v > v_c$ the distribution function exhibits a maximum in the direction $\theta = 0$. In the first approximation it is

$$f(v') = \left(\frac{M}{2\pi kT_e}\right)^{3/2} N_2$$

$$\times \exp\left\{-\int_{v_2}^{v_2 - v'} \frac{eZ(Z-1)E/M - m\nu_{e0}Z^2 v/M - M\nu_i(v)Gv/M_0}{\nu_{e0}Z^2 kT_e m/M^2 + \nu_i(v)\, GkT_{i0}/M_0}\, dv\right\}, \qquad (11)$$

where v' is the velocity in the new coordinate system, while N_2 is the density of the multiply charged ions under consideration in the second equilibrium state (it is obvious that $N_Z = N_1 + N_2$ where N_Z is the total density of the indicated ions).

When $v = v_c$ the values of the functions in Eqs. (8) and (11) must obviously be the same at equilibrium; this condition then determines the ratio between the number of multiply charged ions in the first and second states at equilibrium:

$$\frac{N_2}{N_1} \approx \exp\left\{-\int_{v_1}^{v_2} \frac{M\nu_i(v)\,Gv/M_0 + m\nu_{e0}Z^2 v/M - eZ(Z-1)E/M}{\nu_i(v)\,GkT_{i0}/M_0 + \nu_{e0}Z^2 kT_e m/M^2} dv\right\}. \qquad (12)$$

It is important that the ratio N_2/N_1 is usually very sensitive to changes in the field E; when E

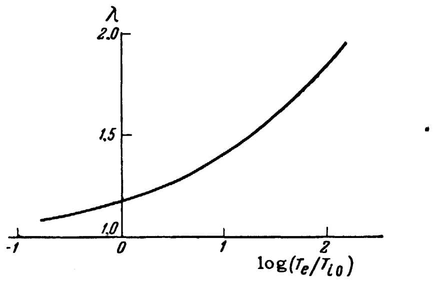

FIG. 2

is changed by the order of ten percent the ratio N_2/N_1 can change by a factor of ten. Consequently, it is natural to introduce a critical field E_c, defined by the condition $N_2(E_c) = N_1(E_c)$. When $E < E_c$ almost all the particles are in the first stationary state and when $E > E_c$ almost all the particles are in the second state. In this case the field E_c can be written in the form

$$E_c = \lambda\,(T_e/T_{i0})\,E_{c1},$$

where E_{c1} is determined by (10) and $\lambda(T_e/T_{i0})$ is the numerical factor given in Fig. 2. It is clear from the figure that λ is a rather weak function of the ratio T_e/T_{i0}.

Using the distribution function obtained for the multiply charged ions we can naturally compute the mean directed velocity v in the equilibrium state. The velocity v is given as a function of electric field in Fig. 3. It is obvious from the figure that near the value $E = E_c$ there is something like a transition from the first stationary state for the directed velocity, indicated in Sec. 1 (cf. Fig. 1), to the second, as expected.

We now estimate the characteristic time required for the establishment of the equilibrium

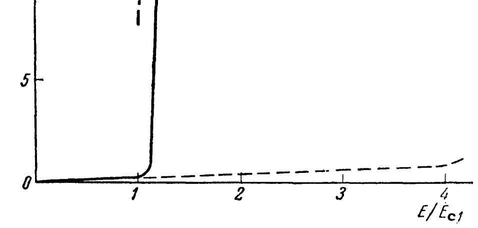

FIG. 3. The dependence of $v/(kT_{i0}/M_0)^{1/2}$ on E/E_{c_1} for $T_e = T_{i0}$, $M_0/m = 3.7 \times 10^3$ (deuterium).

state under consideration. We assume, for example, that at the instant the field is switched on all the particles have low velocities so that the first stationary state is established in a time $\tau \approx M/m\nu_{e0}Z^2$. This state is naturally unstable when $E > E_c$: some of the multiply charged ions are continuously transferred to the second state. The flow of ions from the first state can be determined without difficulty if we make use of the results of references 3 and 8:

$$S = -\frac{dN_1}{dt} = \frac{N_1}{\sqrt{2\pi}} Z^2 \nu_{e0} \left(\frac{m}{M_0}\right)^{1/2} \left(\frac{T_e}{T_{i0}}\right)^{3/2} \left(\frac{E}{E_{c2}}\right)^{3/4} \times \exp\left\{-1.2\frac{M}{M_0}\frac{E_{c2}}{E}\right\}.$$

Thus the characteristic time for the establishment of the equilibrium state is

$$\Delta t \sim \left((M_0/m)^{1/2}/\nu_{e0}Z^2\right)\exp\left\{1.2ME_{c2}/M_0E\right\}.$$

This time is quite large when $M \gg M_0$. In this case $\tau \ll \Delta t$ and the first state indicated above becomes "quasi-stationary" when $E > E_c$. The same situation obtains for the second state when $E < E_c$. In Fig. 3 both of these conditions are shown by dashed lines. However, if $M \approx M_0$, then Δt and τ are of the same order of magnitude. In this case only the equilibrium state shown by the solid line in Fig. 3 is meaningful.

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