SPATIAL DISPERSION IN A RELATIVISTIC PLASMA

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The dielectric constant of a relativistic plasma $\epsilon_{ij}(\omega, \mathbf{k})$ is considered with pair production taken into account. The effect of recoil on the Cerenkov absorption is also considered. The spectra for longitudinal and transverse plasma oscillations are analyzed at high densities and temperatures, in which case absorption due to pair production is possible in addition to Cerenkov absorption.

1. Spatial dispersion in an ultrarelativistic plasma has been considered by Silin[1] in the classical (nonquantum) limit. The classical analysis cannot be used at high plasma temperatures and densities. For example, at densities $N \sim 10^{32}$ cm^{-3} the natural frequency of the longitudinal oscillations is of the order of twice the mass of the electron (for $\hbar = c = 1$) and the processes of virtual and real pair production have an important effect on the dielectric constant.

However, even at low densities the relativistic quantum-mechanical calculation makes it possible to take account of recoil in the Landau damping of the longitudinal waves.[2] This damping is an inverse Cerenkov effect for the longitudinal waves, in which the wave is absorbed by free plasma electrons by virtue of the transfer of momentum to the medium (Fig. 1a). The quantum-mechanical effect of recoil on Cerenkov radiation of particles in a medium was first considered by Ginzburg.[3]

In addition to Cerenkov absorption (cf. Fig. 1a), at high densities it is possible to have absorption due to pair production (Fig. 1b); the latter process is allowed by the conservation laws because part of the momentum is taken up by the medium itself. This damping mode is possible for both the longitudinal and transverse electromagnetic waves (in appropriate regions of ω and $\mathbf{k}$). The process shown in Fig. 1a is also possible for the transverse waves.

The production of pairs in a medium has been treated by Saakyan,[4] who used a phenomenological quantum-electrodynamical description, but the spatial dispersion of the dielectric constant was not taken into account.*

*Furthermore, the expression used for $\varepsilon(\omega)$ in reference 4 is not applicable at high densities and temperatures.

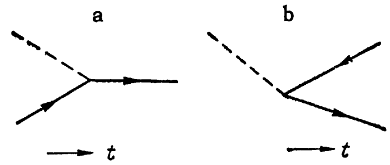

FIG. 1

2. When spatial dispersion is taken into account in an isotropic plasma, the dielectric constant becomes a three-dimensional tensor:*[6]

$$\varepsilon_{ij} = \delta_{ij} + 4\pi\omega^{-1}\sigma_{ij} = \varepsilon^t(\delta_{ij}\mathbf{k} - \mathbf{k}^{-2}k_ik_j) + \varepsilon^l\mathbf{k}^{-2}k_ik_j, \quad (1)$$

$$j_i = \sigma_{ij}(\omega, \mathbf{k})E_j, \quad \sigma_{ij}(\omega, \mathbf{k}) = \sigma^t(\delta_{ij} - \mathbf{k}^{-2}k_ik_l) + \sigma^l\mathbf{k}^{-2}k_ik_j, \quad (2)$$

where j_i is the four-dimensional Fourier component of the current, E_j is the Fourier component of the electric field, σ^l, σ^t, ϵ^l, and ϵ^t are respectively the longitudinal and transverse (in the three-dimensional sense) electrical conductivity and dielectric constant.

The four-dimensional representation can also be used:

$$j_\mu = \Pi_{\mu\nu}(\omega, \mathrm{k})A_\nu, \quad \Pi_{ik} = i\omega\sigma_{ik}, \quad \Pi_{i4} = \Pi_{4i} = -ik_i\sigma^l,$$
$$\Pi_{44} = -i(\mathrm{k}^2/\omega)\sigma^l, \quad (3)$$

where A_ν is the Fourier component of the potential. The relation between $\Pi_{\mu\nu}$ and σ_{ik} can be obtained easily from Eq. (1) if E_j is expressed in terms of the potentials and j_4 is found from the equation of continuity: $j_4 = -(1/i\omega)k_ij_i$. The relation in (3) can be substituted in Maxwell's equations:

$$(k_\lambda^2\delta_{\mu\nu} - k_\mu k_\nu - 4\pi\Pi_{\mu\nu})A_\nu = 4\pi j_\mu^0, \quad (4)$$

where j_μ^0 is the Fourier component of the external current.

*Hereinafter we use $\hbar = c = 1$; i, j = 1, 2, 3; μ, ν = 1, 2, 3, 4.

If the external source is given by a δ-function we obtain the following equation for the Green's function D:

$$(k_\lambda^2\delta_{\mu\nu} - k_\mu k_\nu - 4\pi\Pi_{\mu\nu}) D_{\nu\sigma} = 4\pi\delta_{\mu\sigma}. \quad (5)$$

The equations for the Green's function in quantum statistics has been investigated in detail by Fradkin.[7] The equations for the time Green's functions in quantum statistics have been considered by Kogan[8] and Bonch-Bruevich.[9]

3. To terms of order e^2, the causal polarization operator $\Pi^c_{\mu\nu}$ is made up additively of the corresponding operators for the electrons and the ions. As an example let us consider the electrons:

$$\Pi^c_{\mu\nu}(\omega, \mathbf{k}) = \frac{ie^2}{(2\pi)^4}\int \mathrm{Sp}\,\gamma_\mu G(\mathbf{p}+\mathbf{k}, \omega+\lambda)\gamma_\nu G(\mathbf{p}, \lambda)\, d\mathbf{p}d\lambda, \quad (6)$$

where the γ_μ are the Dirac matrices while $G(\mathbf{p}, \lambda)$ is the electron Green's function in the momentum representation:

$$G(\mathbf{p}, \lambda) = \int \mathrm{Tr}\,[\rho T(\hat{\psi}_1\hat{\bar{\psi}}_2)]\, e^{-i\mathbf{pr}+i\lambda t}\, d\mathbf{r}dt, \quad (7)$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $t = t_1 - t_2$, T is the sign of the time ordering, ρ is the density matrix ($\mathrm{Tr}\,\rho = 1$), $\hat{\psi}_1 \equiv \hat{\psi}(\mathbf{r}_1, t_1)$ and $\hat{\bar{\psi}}_2 = \hat{\bar{\psi}}(\mathbf{r}_2, t_2)$ are operators in the interaction representation:

$$\hat{\psi}(\mathbf{r}, t) = \exp[it(\hat{H}_0 - \mu\hat{N}_0)]\hat{\psi}(\mathbf{r})\exp[-it(\hat{H}_0 - \mu\hat{N}_0)],$$

μ is the chemical potential of the system, $\hat{H}_0$ is the Hamiltonian for the free Dirac particles, $\hat{N}_0$ is the operator for the conservation of the difference in the number of particles and antiparticles, Sp denotes summation over the spin indices, and Tr is the statistical average.

Expanding $\hat{\psi}$ in plane waves and substituting in Eq. (7) we have*

$$G(\mathbf{p}, \omega) = G^-(\mathbf{p}, \omega)(m - i\hat{p}^-)/2\varepsilon_p + G^+(\mathbf{p}, \omega)(m - i\hat{p}^+)/2\varepsilon_p, \quad (8)$$

$$\hat{p} = \gamma_\mu p_\mu, \quad p_\mu^- = \{\mathbf{p}, i\varepsilon_p\}, \quad p_\mu^+ = \{\mathbf{p}, -i\varepsilon_p\};$$

$$G^\pm(\mathbf{p}, \omega) = \frac{1}{i}\left\{\frac{n_p^\pm}{\varepsilon_p \mp \mu \pm \omega + i\delta} + \frac{1 - n_p^\pm}{\varepsilon_p \mp \mu \mp \omega - i\delta}\right\}, \quad (9)$$

where $\epsilon_p = \sqrt{p^2 + m^2}$ is the modulus of the energy while $n_p^\pm$ is respectively the mean number of positrons and electrons with energy ϵ_p for the density matrix ρ. The expression for $G(\mathbf{p}, \omega)$ in the nonrelativistic limit (without the positron part) has been given, for example, by Kogan.[8]

*The chemical potential appears with opposite sign in the expression for the mean number of positrons, since the number of positrons n_p^+ is

$$1 - n^-_{\varepsilon<0} = 1 - \{\exp(-\varepsilon - \mu)\beta + 1\}^{-1} = \{1 + \exp(\varepsilon + \mu)\beta\}^{-1}.$$

The further calculations are very simple. It is necessary to substitute (8) in (6), compute the traces of the γ-matrices by the conventional rule,[10] and integrate the resultant expression with respect to λ, taking account of the fact that the products of the terms $G^\pm$, which contain poles in different half-planes of the complex variable λ, make a contribution to the integral. We give the result for the longitudinal and transverse parts of the dielectric constant ϵ^l and ϵ^t:*

$$\varepsilon^{l,t} = 1 - \frac{4\pi e^2}{\omega^2}\int f(\varepsilon_p)\, d\mathbf{p}\left\{\frac{\varepsilon_p - \varepsilon_{p-k}}{(\varepsilon_p - \varepsilon_{p-k})^2 - \omega^2}\Lambda_-^{l,t} + \frac{\varepsilon_p + \varepsilon_{p-k}}{(\varepsilon_p + \varepsilon_{p-k})^2 - \omega^2}\Lambda_+^{l,t}\right\} + \delta\varepsilon_B^{l,t},$$

$$\Lambda_\pm^l = 1 \mp \frac{\varepsilon_p^2 + (\mathbf{pk}) - 2(\mathbf{pk})^2/\mathbf{k}^2}{\varepsilon_p\varepsilon_{p-k}},$$

$$\Lambda_\mp^t = 1 \mp \frac{m^2 - (\mathbf{pk}) + (\mathbf{pk})^2/\mathbf{k}^2}{\varepsilon_p\varepsilon_{p-k}}, \quad f(\varepsilon_p) = 2(2\pi)^{-3}(n_p^- + n_p^+),$$

$$\delta\varepsilon_B^t\omega^2 = (\omega^2 - k^2)\,\delta\varepsilon_B^l = \Pi_B = \frac{e^2}{\pi^2}\int \Lambda_+^t \frac{\varepsilon_p + \varepsilon_{p-k}}{(\varepsilon_p + \varepsilon_{p-k})^2 - \omega^2}\, d\mathbf{p}; \quad (10)$$

the quantity $\delta\epsilon^{l,t}$ is obtained from that part of the polarization operator which does not vanish when $n_p^+ = n_p^- = 0$; after the standard renormalization, this quantity gives the usual expression for the vacuum polarization† (cf. reference 10).

*The causal operator $\Pi^c_{\mu\nu}$ can only be used to find the real parts of ε^l and ε^t and the momentum integrals are to be understood in the sense of the principal value. The imaginary part of the causal operator will not correspond to the imaginary parts of ε^l and ε^t, which are determined by the retarded Green's function for the photon. The imaginary parts can be found by means of the Kramers-Kronig relations for σ^l and σ^t and the easily derived formula

$$\frac{1}{\pi}\int\frac{dx}{x - \omega}\int\frac{\varphi(y)}{f(y) - x}\, dy = -\pi\int dy\,\varphi(y)\,\delta(f(y) - \omega),$$

where the slash denotes an integral in the sense of the principal value. Thus, to obtain the imaginary parts the energy denominators must be replaced by $-i\pi\delta$-functions or, what is the same, for the complex ε^l and ε^t in Eq. (10) we are to understand ω in the sense of $\omega = \omega + i\delta$ with $\delta \to +0$.

†The relation between $\delta\varepsilon^t$ and $\delta\varepsilon^l$ follows from the four-dimensional tranversality of the vacuum polarization operator. The real parts of $\delta\varepsilon$ are small also away from the light cone ($\omega \approx k$), and will be neglected hereafter. However, the imaginary part of $\delta\varepsilon$ is very important for the description of pair production in the medium by a photon. In particular, if the imaginary part of $\delta\varepsilon$ is not taken into account a meaningless result is obtained; the amplitude of the wave increases because of pair production, i.e., the oscillations are excited rather than damped. The imaginary part of $\delta\varepsilon$ is usually not considered because it vanishes everywhere except far from the light cone $\omega^2 > 4m^2 + k^2$ (cf. below). The dispersion curves for the longitudinal and transverse waves can only fall in this region if the particle density of the medium is greater than 10^{32} cm^{-3} (cf. below).

In the nonrelativistic limit Eq. (10) coincides with the familiar expressions (cf. reference 6) obtained from the kinetic equation. If the positron contribution is neglected ($n_{\mathbf{p}}^{+} = 0$) in the non-quantum-mechanical ($\mathbf{k} \ll \mathbf{p}$) ultrarelativistic limit $p \gg m$, the equations in (10) lead to the results obtained by Silin[1] by means of the kinetic equation. Below we shall only discuss results not contained in reference 1.

It should be emphasized that the positron contribution cannot be neglected in the ultrarelativistic limit if the system has attained total equilibrium. In this case the chemical potential of the system μ must be found from the relation

$$\frac{2}{(2\pi)^3}\int (n_{\mathbf{p}}^{-} - n_{\mathbf{p}}^{+})\, dp = N, \qquad (11)$$

where N is the difference between the number of particles and antiparticles in the system.

4. Spatial dispersion can be neglected in the limit of small wave vectors $\mathbf{k}$, and the real part becomes*

$$\mathrm{Re}\,\varepsilon(\omega) = 1 - \frac{4\pi e^2}{\omega^2}\int f(\varepsilon_{\mathbf{p}})\frac{4\varepsilon_{\mathbf{p}}(1 - p^2/3\varepsilon_{\mathbf{p}}^2)\,dp}{4\varepsilon_{\mathbf{p}}^2 - \omega^2}. \qquad (12)$$

At relatively low frequencies $\omega \ll 2\epsilon_{\mathbf{p}}$ Eq. (12) yields the well-known expression

$$\mathrm{Re}\,\varepsilon(\omega) = 1 - \frac{\omega_0^2}{\omega^2}, \qquad \omega_0^2 = 4\pi e^2\int\frac{f(\varepsilon_{\mathbf{p}})}{\varepsilon_{\mathbf{p}}}\left(1 - \frac{1}{3}\frac{p^2}{\varepsilon_{\mathbf{p}}^2}\right)dp. \qquad (13)$$

When $n_{\mathbf{p}}^{+} = 0$ the expression for the natural plasma frequency ω_0 in ultrarelativistic ($p \gg m$) degenerate and nondegenerate gases coincides with the expression given by Silin.[1] If, however, a system at relativistic temperatures reaches total equilibrium, then at sufficiently high temperatures $\mu = 0$, $n_{\mathbf{p}}^{+} \approx n_{\mathbf{p}}^{-} = [\exp(p\beta) + 1]^{-1}$ and

$$\omega_0^2 = 2\zeta(2)\, e^2/3\pi\beta^2, \qquad (14)$$

where ζ is the Riemann zeta function.

At high frequencies $\omega \gg 2\epsilon_{\mathbf{p}}$

$$\mathrm{Re}\,\varepsilon(\omega) = 1 + Z/\omega^4,$$

$$Z = 16\pi e^2\int f(\varepsilon_{\mathbf{p}})\,\varepsilon_{\mathbf{p}}\left(1 - \frac{1}{3}\frac{p^2}{\varepsilon_{\mathbf{p}}^2}\right)dp. \qquad (15)$$

For nonrelativistic temperatures $Z = 16\pi e^2 mN$ whereas for ultrarelativistic temperatures, if the positron contribution is neglected ($n_{\mathbf{p}}^{+} = 0$), in a nondegenerate gas $Z = 32\pi e^2 N/\beta$ while in a degenerate gas $Z = 4\pi e^2 N p_0$ ($p_0 = 2\pi[3N/8\pi]^{1/3}$ is the limiting momentum at the Fermi surface). For total equilibrium and ultrarelativistic temperatures

$$Z = 120 e^2\zeta(4)/\pi\beta^4. \qquad (16)$$

5. To obtain the imaginary part of $\epsilon^{l,t}$ we must replace the energy denominators by δ functions which express the conservation of momentum and energy for the diagrams in Fig. 1. In momentum space $\mathbf{p}$ we introduce a cylindrical coordinate system with Z axis in the direction of the wave vector $\mathbf{k}$. With no loss of generality (formally, we replace $\mathbf{p}$, the variable of integration in Eq. (10), by $\mathbf{p} \pm \mathbf{k}/2$) we can set the initial momentum of the electron equal to $\mathbf{p} - \mathbf{k}$ so that $\mathbf{p}$ is finite and the energy conservation relation for Cerenkov dissipation* (Fig. 1a)

$$\omega = \varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{p}-\mathbf{k}} \qquad (17)$$

allows us to find the electron momentum component p_z as a function of k, ω and $\epsilon_{\perp} = \sqrt{p_{\perp}^2 + m^2}$ ($\mathbf{p}_{\perp}$ is the momentum perpendicular to $\mathbf{k}$).

Equation (17) has two roots

$$p_z^{\pm} = k/2 \pm (\omega/k)\,\varkappa,$$

$$\varkappa = [\varepsilon_{\perp}^2/(1 - \omega^2/k^2) + k^2/4]^{1/2}. \qquad (18)$$

Substituting Eq. (17) in $\epsilon_{\mathbf{p}}$ and $\epsilon_{\mathbf{p}-\mathbf{k}}$ we obtain different signs for the two possible values $\epsilon_{\mathbf{p}}^{\pm}$ and $\epsilon_{\mathbf{p}-\mathbf{k}}^{\pm}$ (ϵ^{+} denotes ϵ for $p_z = p_z^{+}$ while ϵ^{-} denotes ϵ for $p_z = p_z^{-}$). Equation (17) is satisfied only by p_z^{+} and

$$\varepsilon_{\mathbf{p}}^{+} = \omega/2 + \varkappa, \qquad \varepsilon_{\mathbf{p}-\mathbf{k}}^{+} = \varkappa - \omega/2. \qquad (19)$$

On the other hand, the values p_z^{-}, $\epsilon_{\mathbf{p}}^{-} = \kappa - \omega/2$, $\epsilon_{\mathbf{p}-\mathbf{k}}^{-} = \omega/2 + \kappa$ satisfy the conservation law $\omega = \epsilon_{\mathbf{p}-\mathbf{k}} - \epsilon_{\mathbf{p}}$ which corresponds to another possible absorption process in which the initial momentum is $\mathbf{p}$ and the final momentum is $\mathbf{p} + \mathbf{k}$. It thus follows that Cerenkov dissipation is possible only when $\kappa^2 > \omega^2/4$ or, what is the same thing, when $\omega/k \ll 1$. This result means that Cerenkov damping occurs only for phase velocities smaller than the velocity of light when recoil is taken into account.

The energy conservation relation for dissipation due to pair production $\omega = \epsilon_{\mathbf{p}} + \epsilon_{\mathbf{p}-\mathbf{k}}$ has the same solutions (18) for p_z as (17). Both values $p_z^{\pm}$ satisfy the conservation relation; the signs must be chosen in the following manner: $\epsilon_{\mathbf{p}}^{\pm} = \omega/2 \pm \kappa$. For $\epsilon_{\mathbf{p}}^{\pm}$ to be positive we require $\omega^2/4 > \kappa^2$; this is possible only when $\omega^2/k^2 > 1$. Thus dissipation due to pair production arises only when the phase velocity of the wave is greater than the velocity of light.

*Equation (12) coincides with the expression obtained by Fradkin.[7]

*To be specific we take $\omega > 0$.

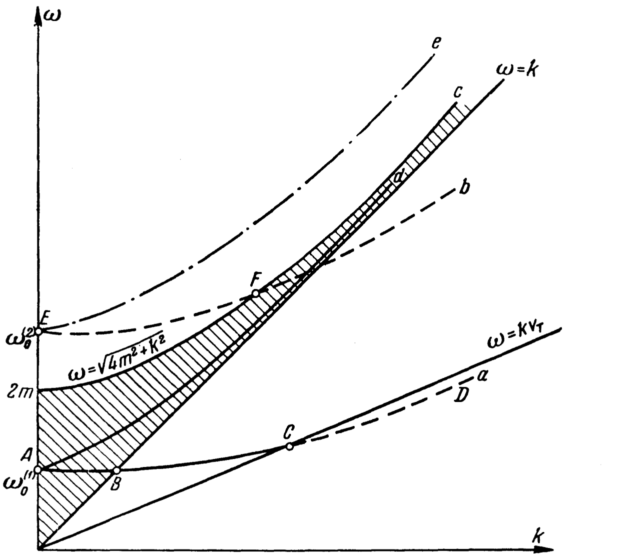

FIG. 2. a – dispersion curve for longitudinal waves for low densities, d – dispersion curve for transverse waves for low densities, b – dispersion curve for longitudinal waves for high densities, e – dispersion curve for transverse waves for high densities, BC – region of weak damping, CD – region of strong damping, EF – region of damping due to pair production.

However, there is an energy threshold for the latter process; this threshold follows from the requirement that the quantity under the radical in the expression for κ must be positive. Since the distribution contains particles with arbitrary momenta, in particular, momenta parallel to **k** ($p_\perp = 0$), it is sufficient to satisfy this condition for $\epsilon_\perp^2 = m^2$:

$$\omega^2 > 4m^2 + k^2. \qquad (20)$$

In the $\omega - k$ plane the line $\omega = k$ and the curve $\omega^2 = 4m^2 + k^2$ define the region in which neither of the dissipation mechanisms considered above is possible (the cross-hatched region in Fig. 2). This region is bordered on both sides by regions in which the absorption is relatively small.

6. The calculation of the imaginary parts of $\epsilon^{l,t}$ is simplified by the fact that the integration over p_z can be carried out in elementary fashion by means of δ-functions. Thus, for the imaginary part of ϵ^l, which describes the Cerenkov absorption of the longitudinal waves, we have from Eq. (10)

$$\operatorname{Im}\varepsilon^l_{\text{cer}} = \frac{8\pi^3 e^2}{k^3}\int_{\varkappa_0}^{\infty}\left(\varepsilon^2 - \frac{k^2}{4}\right)\left[f\left(\varepsilon - \frac{\omega}{2}\right) - f\left(\varepsilon + \frac{\omega}{2}\right)\right]d\varepsilon, \qquad (21)$$

where $\kappa_0 = \kappa\,|_{\epsilon_\perp = m}$. If the positrons are neglected, in a Boltzmann gas $f(\epsilon) = 2(2\pi)^{-3}e^{(\mu-\epsilon)\beta}$ and after an elementary integration we have

$$\operatorname{Im}\varepsilon^l_{\text{cer}} = \frac{4e^2}{k^3\beta^3}e^{\mu\beta}\,\text{sh}\,\frac{\omega\beta}{2}\left\{\frac{m^2}{1-\omega^2/k^2} + \frac{2}{\beta}\varkappa_0 + \frac{2}{\beta^2}\right\}e^{-\beta\varkappa_0}. \qquad (22)^*$$

*sh = sinh.

The integration is also elementary for a degenerate gas.

The imaginary part of ϵ^t, which describes Cerenkov absorption (Fig. 1a), is computed in similar fashion

$$\operatorname{Im}\varepsilon^t_{\text{cer}} = \frac{4\pi^3 e^2}{\omega^2 k}\left(1 - \frac{\omega^2}{k^2}\right)\int_{\varkappa_0}^{\infty}\left(\varepsilon^2 + \frac{k^2}{2} - \varkappa_0^2\right)\left[f\left(\varepsilon - \frac{\omega}{2}\right) - f\left(\varepsilon + \frac{\omega}{2}\right)\right]d\varepsilon. \qquad (23)$$

In particular, for a Boltzmann electron gas

$$\operatorname{Im}\varepsilon^t_{\text{cer}} = \frac{4e^2(1-\omega^2/k^2)}{\omega^2 k\,\beta^3}\left(1 + \beta\varkappa_0 + \frac{k^2\beta^2}{4}\right)e^{(\mu-\varkappa_0)\beta}\,\text{sh}\,\frac{\omega\beta}{2}. \qquad (24)$$

The damping due to pair production (Fig. 1b) is computed in similar fashion from Eq. (10):

$$\operatorname{Im}\varepsilon^l_{\text{pair}} = -\frac{8\pi^3 e^2}{k^3}\frac{\omega}{|\omega|}\int_0^{\varkappa_0}\left(\frac{k^2}{4} - \varepsilon^2\right)\left[f\left(\varepsilon + \frac{\omega}{2}\right) + f\left(\frac{\omega}{2} - \varepsilon\right)\right]d\varepsilon + \operatorname{Im}\delta\varepsilon^l_{\text{pair}}, \qquad (25)$$

$$\operatorname{Im}\varepsilon^t_{\text{pair}} = -\frac{4\pi^3 e^2}{\omega^2 k}\left(\frac{\omega^2}{k^2} - 1\right)\frac{\omega}{|\omega|}\int_0^{\varkappa_0}\left(\frac{k^2}{2} - \varkappa^2 + \varepsilon^2\right)\left[f\left(\frac{\omega}{2} + \varepsilon\right) + f\left(\frac{\omega}{2} - \varepsilon\right)\right]d\varepsilon + \operatorname{Im}\delta\varepsilon^t_{\text{pair}}, \qquad (26)$$

$$\operatorname{Im}\delta\varepsilon^l_{\text{pair}} = \frac{\omega^2}{\omega^2 - k^2}\operatorname{Im}\delta\varepsilon^t_{\text{pair}} = \frac{e^2\varkappa_0}{3k^3}\left(k^2 + \frac{2m^2}{\omega^2/k^2 - 1}\right). \qquad (27)$$

In particular, for a Boltzmann gas ($\operatorname{Im}\epsilon_{\text{pair}} = \operatorname{Im}\tilde{\epsilon}_{\text{pair}} + \operatorname{Im}\delta\epsilon_{\text{pair}}$)

$$\operatorname{Im}\tilde{\varepsilon}^l_{\text{pair}} = -\frac{4e^2}{k^3\beta^3}e^{-\beta(\omega/2-\mu)}\frac{\omega}{|\omega|}\left[\left(\frac{k^2\beta^2}{4} - 2 - \varkappa_0^2\beta^2\right)\text{sh}\,\varkappa_0\beta + 2\varkappa_0\beta\,\text{ch}\,\varkappa_0\beta\right], \qquad (28)^*$$

$$\operatorname{Im}\tilde{\varepsilon}^t_{\text{pair}} = -\frac{4e^2}{\omega^2 k\beta^2}\frac{\omega}{|\omega|}\left(\frac{\omega^2}{k^2} - 1\right)e^{-\beta(\omega/2-\mu)}\left[\left(1 + \frac{k^2\beta^2}{4}\right)\text{sh}\,\varkappa_0\beta - \varkappa_0\beta\,\text{ch}\,\varkappa_0\beta\right], \qquad (29)$$

7. We now consider the spectrum of electromagnetic oscillations and the damping. We start with the longitudinal waves, for which

$$\varepsilon^l(\omega, \mathbf{k}) = 0.$$

For weak damping $\tilde{\omega} = \omega - i\gamma$ ($\gamma \ll \omega$) (cf. reference 6)

$$\operatorname{Re}\varepsilon^l(\omega, \mathbf{k}) = 0, \qquad \gamma = \operatorname{Im}\varepsilon^l \Big/ \frac{\partial}{\partial\omega}\operatorname{Re}\varepsilon^l. \qquad (30)$$

At relatively low densities $\omega_0 \ll 2m$ there is a region of ω and **k** in which the spatial dispersion of the longitudinal waves can be considered weak ($k \ll \omega$) for any temperature $1/\beta$:

$$\operatorname{Re}\varepsilon^l(\omega, k) = \varepsilon(0) - \omega_0^2/\omega^2 - (k^2/\omega^2)\,u^l(\omega^2), \qquad (31)$$

$$\varepsilon(0) = 1 - \pi e^2\int d\mathbf{p}\,f(\varepsilon_{\mathbf{p}})\,\frac{1}{\varepsilon_{\mathbf{p}}^2}\left(1 - \frac{1}{3}\frac{\mathbf{p}^2}{\varepsilon_{\mathbf{p}}^2}\right), \qquad (32)$$

$$u^l(\omega^2) = \varepsilon(0) - 1 + \frac{4\pi e^2}{\omega^2}\int d\mathbf{p}\,f(\varepsilon_{\mathbf{p}})\,\frac{\mathbf{p}^2}{\varepsilon_{\mathbf{p}}^3}\left(1 - \frac{3}{5}\frac{\mathbf{p}^2}{\varepsilon_{\mathbf{p}}^2}\right), \qquad (33)$$

*ch = cosh.

while ω_0 is given by Eq. (13).

Together with Eqs. (22) and (31), the dispersion equations (30) lead to a spectrum of weakly damped longitudinal oscillations:

$$\omega^2 = \frac{1}{\varepsilon(0)}(\omega_0^2 + k^2 u^l (\omega_0^2/\varepsilon(0))), \tag{34}$$

$$\gamma = \sqrt{\frac{\pi}{8}} \frac{1}{k^3 d^3} \omega_0 \left(\frac{2}{\omega_0\beta} \operatorname{sh}\frac{\omega_0\beta}{2}\right) \frac{1}{1-\omega^2/k^2} e^{-\beta\varkappa_0}; \quad d^2 = (4\pi N e^2 \beta)^{-1}. \tag{35}$$

It should be noted that while Eq. (34) is valid for undamped oscillations ($k < \omega$) and ultrarelativistic temperatures,* the damping expression (35) applies only for nonrelativistic temperatures because the expansion in (31) does not apply in the region $k > \omega$ for $u^l \sim 1$ (u^l is of the order of the mean square thermal velocity of the particles). For this reason we have neglected terms of order $1/\beta m \ll 1$ in Eq. (22). When $\omega_0\beta \ll 1$, $k \gg \omega$ and $k \ll m$, Eq. (35) gives the familiar Landau damping.[2] As is well known, in this case the oscillations are weakly damped at wavelengths large compared with the Debye radius $dk \ll 1$ (dashed line in Fig. 2).

We now consider the extent to which damping is reduced at nonrelativistic temperatures when recoil is taken into account in absorption (the $k^2/4$ term in the expression for κ_0^2). When $k^2/4 \ll m^2$ there is an additional factor $\exp[-\beta(k^2/8m)\times \sqrt{1-\omega^2/k^2}$ which can reduce the damping at $k \gg \omega$ if $k^2 \gg 8m/\beta$. For wavelengths of the order of the Debye radius this gives $\beta^2 \gg 8/\omega_0^2$; if the gas is nondegenerate $N \ll 2m^{3/2}(2\pi\beta)^{-3/2}$ we find

$$\beta m \gg 8\pi/e^4 = 0.5\cdot 10^6, \tag{36}$$

where (36) must be satisfied with a margin of at least two orders of magnitude. In other words, the damping can decrease only at sufficiently low temperatures. It also follows from Fig. 2 that a region of undamped and weakly damped oscillations exists for $\omega \gg 2m$ when $k \sim m$. It is easy to verify that Eq. (30) always has real solutions in this region. We find $\epsilon^l|_{\omega=k}$ for this region. Integrating over the angles, we have from Eq. (10):

$$\operatorname{Re}\varepsilon^l(\omega, k)|_{\omega=k} = 1 - \omega_l^2/\omega^2,$$

$$\omega_l^2 = 4\pi e^2 \int f(\varepsilon_p)\left[\frac{2}{p}\ln\frac{\varepsilon_p + p}{m} - \frac{1}{\varepsilon_p}\right] dp. \tag{37}$$

At nonrelativistic temperatures $\omega_l^2 \approx \omega_0^2$ whereas for ultrarelativistic temperatures there is an additional factor of order $\ln(\alpha/m\beta)$, where $\alpha \sim 1$. Thus, it is convenient to seek a solution of Eq. (30) in the region $k \approx \omega \approx \omega_l$ taking $k = \omega_l + \Delta k$, $\omega = \omega_l + \Delta\omega$, and expanding in Δk and $\Delta\omega$:

$$\operatorname{Re}\left\{\Delta\omega \frac{\partial\varepsilon^l}{\partial\omega}\bigg|_{\omega=k=\omega_l} + \Delta k \frac{\partial\varepsilon^l}{\partial k}\bigg|_{\omega=k=\omega_l}\right\} = 0;$$

$$\operatorname{Re}\frac{\partial\varepsilon^l}{\partial\omega}\bigg|_{\omega=k=\omega_l} = \frac{8\pi e^2}{\omega_l^3}\int\frac{f(\varepsilon_p)}{\varepsilon_p}\left[\frac{2\varepsilon_p^2}{m^2} - \frac{\varepsilon_p}{p}\ln\frac{\varepsilon_p+p}{m}\right] dp, \tag{38}$$

$$\operatorname{Re}\frac{\partial\varepsilon^l}{\partial k}\bigg|_{\omega=k=\omega_l} = -\frac{8\pi e^2}{\omega_l^3}\int\frac{f(\varepsilon_p)}{\varepsilon_p}\left[1 + 2\frac{\varepsilon_p^2}{m^2} - \frac{3\varepsilon_p}{p}\ln\frac{\varepsilon_p+p}{m}\right] dp. \tag{39}$$

It follows that the slope of the dispersion curve with respect to the line $\omega = k$ (Fig. 2) decreases rapidly as the temperature increases

$$\Delta\omega/\Delta k = 1 - \tfrac{1}{3} m^2\beta^2 \ln(2C\sqrt{e}/m\beta), \tag{40}$$

where C is the Euler constant.

Equation (38) is used to determine the damping. Thus, for ultrarelativistic temperatures close to threshold ($\Delta k \ll \omega_l$)

$$\gamma = \omega_l \pi m^2\beta^2 \frac{\operatorname{sh}(\omega_l\beta/2)}{\omega_l\beta/2}\left[\frac{3\omega_l}{2\Delta k\ln\Lambda} + 2 + 2\left[\frac{3\omega_l}{2\Delta k\ln\Lambda} + \frac{\omega_l^2\beta^2}{4}\right]^{1/2}\right] \times \exp\left[-\left(\frac{3\omega_l}{2\Delta k\ln\Lambda} + \frac{\omega_l^2\beta^2}{4}\right)^{1/2}\right], \quad \Lambda = \frac{2C\sqrt{e}}{m\beta}. \tag{41}$$

At threshold ($\omega \approx k$) the damping is exponentially small at any temperature.

The damping is described by Eq. (41) for any density, in particular for $\omega_l \gg 2\epsilon_p$. At high densities ($\omega_l \gg 2m$) the gas is always ultrarelativistic.* To analyze the damping far from threshold we use the fact that the dispersion curve approaches $\omega = k$ in the ultrarelativistic limit. For arbitrary ω we write $k = \omega + \Delta k$. Using Eq. (39) with Eq. (30) we can express Δk as a function of ω

$$\frac{\Delta k}{\omega} = \frac{\omega^2 - \omega_l^2}{\omega_s^2}, \quad \omega_s^2 = 8\pi e^2\int\frac{f(\varepsilon_p)}{\varepsilon_p}\left(1 + 2\frac{\varepsilon_p^2}{m^2} - \frac{3\varepsilon_p}{p}\ln\frac{\varepsilon_p+p}{m}\right)dp. \tag{42}$$

Since ω_s^2 is of order $\omega_l^2/m^2\beta^2 \gg \omega_l^2$, Eq. (42) gives the dispersion curve up to frequencies of approximately $\omega_l/m\beta \gg \omega_l$. In the region $\omega_l \ll \omega \ll \omega_l/m\beta$ the damping is given by

$$\gamma = \frac{\pi}{48}\omega m^2\beta^2\frac{\operatorname{sh}(\omega\beta/2)}{\omega\beta/2}\left[\frac{m^2\beta^2\omega_s^2}{2\omega^2} + 2\left(\frac{m^2\beta^2\omega_s^2}{2\omega^2} + \frac{\omega^2\beta^2}{4}\right)^{1/2} + 2\right] \times \exp\left[-\left(\frac{m^2\beta^2\omega_s^2}{2\omega^2} + \frac{\omega^2\beta^2}{4}\right)^{1/2}\right] \tag{43}$$

and ceases to be exponentially small only when $\omega \sim \omega_l/m\beta$; however, it is still rather small: $\gamma/\omega \sim m^2\beta^2 \ll 1$.

When $k \gg \omega$ and $m\beta \ll 1$, in the classical limit the quantity $\operatorname{Im}\epsilon^l$ coincides with the results of reference 1. The oscillations are strongly damped in this case.

*In this case $\omega^2 = \omega_0^2/\varepsilon(0) + k^2[8/5 - 1/\varepsilon(0)]$.

*The limiting momentum for a degenerate gas p_0 is high when $\omega_l = 2m$: this momentum is given by $p_0^2/m^2 = 9\pi/2e^2 \gg 1$.

8. At high densities and temperatures the longitudinal oscillations can be damped as a result of pair production (Fig. 1b). The dispersion curve for the longitudinal oscillations must intersect the line $\omega^2 = 4m^2 + k^2$ (cf. Fig. 2) for this to occur. To analyze the pair-production damping we calculate Re ϵ^l for an ultrarelativistic Boltzmann electron gas by means of Eq. (10)*

$$\mathrm{Re}\,\varepsilon^l = 1 + \frac{4\pi Ne^2\beta}{k^2} + \frac{4\pi Ne^2\beta}{k^2(1-k^2/\omega^2)(\varepsilon_1-\varepsilon_2)} \times \left[\frac{m^2+\varepsilon_1(k^2-\omega^2)/2\omega}{\sqrt{\varepsilon_1^2-m^2}}\ln\frac{\varepsilon_1+\sqrt{\varepsilon_1^2-m^2}}{m} + \frac{m^2+\varepsilon_2(k^2-\omega^2)/2\omega}{\sqrt{\varepsilon_2^2-m^2}}\ln\frac{\varepsilon_2+\sqrt{\varepsilon_2^2-m^2}}{m}\right],$$

where

$$\varepsilon_{1,2} = \omega/2 \pm \varkappa^2, \qquad \omega\beta \ll 1, \qquad k\beta \ll 1, \qquad m\beta \ll 1. \quad (44)$$

It follows from Eq. (44) that on the curve $\omega^2 = 4m^2 + k^2$

$$\mathrm{Re}\,\varepsilon^l = 1 - \frac{4\pi Ne^2\beta}{k^2}\left[\frac{\sqrt{4m^2+k^2}}{k}\ln\frac{k+\sqrt{k^2+4m^2}}{2m} - 1\right]. \quad (45)$$

In particular, as $k \to 0$, Re $\epsilon^l \to 1 - \pi Ne^2\beta/3m$.

At high densities $N > 10^{33}$ cm^{-3}, Re ϵ^l can vanish. When k increases, the function $1 - \mathrm{Re}\,\epsilon^l$ decreases monotonically at threshold, corresponding to the intersection of the curve $\omega^2 = 4m^2 + k^2$ with the dispersion curves for all high densities N.

The behavior of the dispersion curve close to threshold can be described approximately by expanding the relation $\epsilon^l(\omega, k) = 0$ in powers of $\Delta\omega = \omega - \omega_0$ and $\Delta k = k - k_0$, where k_0 and ω_0 define the point at which the dispersion curve intersects the threshold $\omega_0^2 = 4m^2 + k_0^2$. By definition $\epsilon^l(\omega_0, k_0) = 0$ so that

$$\frac{\Delta\omega}{\Delta k} = -\left(\frac{\partial\varepsilon^l}{\partial k}\Big/\frac{\partial\varepsilon^l}{\partial\omega}\right)_{\omega_0, k_0}. \quad (46)$$

The derivatives which appear here can be computed from Eq. (44):

$$\left.\frac{\partial\varepsilon^l}{\partial k}\right|_{\omega_0, k_0} = 4\pi Ne^2\beta\left[\frac{3\sqrt{k_0^2+4m^2}}{k_0^4}\ln\frac{k_0+\sqrt{k_0^2+4m^2}}{2m} - \frac{1}{4k_0m^2} - \frac{3}{k_0^3}\right], \quad (47)$$

$$\left.\frac{\partial\varepsilon^l}{\partial\omega}\right|_{\omega_0, k_0} = -\frac{4\pi Ne^2\beta}{k_0^2}\left[\frac{1}{k_0}\ln\frac{k_0+\sqrt{k_0^2+4m^2}}{2m} - \frac{1}{4m^2}\sqrt{k_0^2+4m^2}\right]. \quad (48)$$

When $k_0 \ll m$ we have $\Delta\omega = (3k_0/40m)\Delta k$. When $k_0 \gg 2m$, we have $\Delta k = \Delta\omega$. The approximate behavior of the longitudinal dispersion curve at high densities and temperatures is shown in Fig. 2 (dashed line).

The damping factor close to the pair-production threshold can be found easily from Eqs. (27) and (30):*

$$\frac{\gamma^l_{\mathrm{pair}}}{2m} = \frac{1}{32}e^2\sqrt{\frac{\Delta\omega}{m}}, \qquad k_0 \ll m,$$

$$\frac{\gamma^l_{\mathrm{pair}}}{k_0} = \frac{2e^2}{3}\sqrt{\frac{\Delta\omega}{m}}\left(\frac{m}{k_0}\right)^{3/2}\frac{1}{\ln 0.37\,k_0/m}, \qquad k_0 \gg m. \quad (49)$$

We note that, in contrast with Cerenkov damping, the pair-production damping vanishes close to threshold not exponentially but as $\sqrt{\Delta\omega}$.

9. We now consider the transverse plasma oscillations. Separating the real and imaginary parts in the transverse dispersion equation

$$\omega^2\varepsilon^t = k^2, \quad (50)$$

we obtain for weak damping

$$\omega^2\,\mathrm{Re}\,\varepsilon^t = k^2, \qquad \gamma = \omega^2\,\mathrm{Im}\,\varepsilon^t \Big/ \frac{\partial}{\partial\omega}\omega^2\,\mathrm{Re}\,\varepsilon^t. \quad (51)$$

It follows from (51) that $k^2 \ll \omega^2$ corresponds to Re $\epsilon^t \ll 1$. In other words, the frequency ω must be close to the plasma frequency ω_0 [Eq. (13)] if the spatial dispersion is to be weak.

At frequencies far from the plasma frequency spatial dispersion is important only at relativistic temperatures. If we assume that Re ϵ^t is approximately unity so that $\omega \sim k$, we can write in accordance with Eq. (51) $\omega = k$ in Eq. (10), the expression for ϵ^t. Thus we find

$$\varepsilon^t(\omega, \omega) = 1 - \frac{\omega_t^2}{\omega^2}, \qquad \omega_t^2 = 4\pi e^2\int\frac{f(\varepsilon_p)}{\varepsilon_p}dp. \quad (52)$$

In the ultrarelativistic limit ω_t^2 is $3/2$ times larger than ω_0^2. At low densities ($\omega_0^2 \ll 4m^2$) the dispersion curve for the transverse oscillations lies inside the region in which there is neither Cerenkov nor pair-production damping (cf. Fig. 2).

Pair-production damping appears at high densities. To analyze the spectrum of transverse os-

*The real parts of ε^l and ε^t were computed exactly for a Boltzmann gas. Equation (44) is obtained as a limiting case of the resulting complicated expressions, which are not given here.

*The temperature effects contained in Eqs. (28) and (29) reduce the damping to some extent; however, this reduction is small for a Boltzmann gas because the corresponding terms contain the small factor $\exp(\mu\beta) \ll 1$. These terms, which are proportional to β^2, are omitted in Eq. (49). It is also easy to write general formulas for an ultrarelativistic Boltzmann gas by dividing Eq. (27) by $\partial\varepsilon/\partial\omega$ from Eq. (44). For a degenerate gas the contributions of Im $\varepsilon^l_{\mathrm{pair}}$ and Im $\delta\varepsilon^l_{\mathrm{pair}}$ are of the same order of magnitude and decrease together for $\mu > \omega/2 + \varkappa$, whereas for $\omega/2 > \mu + \varkappa$ we have Im $\varepsilon^l_{\mathrm{pair}}$ = Im $\delta\varepsilon^l_{\mathrm{pair}}$.

cillations we calculate Re ϵ^t by means of Eq. (10) for $m\beta \ll 1$, $\omega\beta \ll 1$, $k\beta \ll 1$:

$$\text{Re}\,\varepsilon^t = 1 - \omega_t^2/\omega^2 - \tfrac{1}{2}(\varepsilon^l - 1)(1 - k^2/\omega^2), \quad (53)$$

where Re ϵ^l is given by Eq. (44).

The dispersion curve for the transverse oscillations is obtained from Eq. (53):

$$\varepsilon^l = 3 - 2\omega_t^2/(\omega^2 - k^2). \quad (54)$$

It is apparent that Eq. (53) and the curve $\omega^2 = 4m^2 + k^2$ can intersect only for a limited range of densities

$$1 < \omega_t^2/4m^2 < 3/2. \quad (55)$$

If the second inequality in (55) is not satisfied the dispersion curve lies above $\omega^2 = 4m^2 + k^2$ everywhere, that is to say, it lies in the region in which the oscillations are damped by pair production. Then, for weak spatial dispersion ($k \ll \omega$), using Eqs. (28) and (51) we find

$$\gamma^t_{\text{pair}}/\omega = \tfrac{1}{12}e^2\sqrt{1 - 4m^2/\omega_0^2}\,(1 + 2m^2/\omega_0^2). \quad (56)$$

This result corresponds to weak damping.

An approximate analytical solution of Eq. (54) can be found at high frequencies. We seek a solution of Eq. (54) in the form $\omega^2 = \omega_s^2 + k^2$ where $\omega_s^2 > 4m^2$ and $k^2 \gg \omega_s^2$. Then

$$\varepsilon^l \approx 1 - (\omega_t^2/k^2)\ln(k^2/\omega_s^2)$$

and for values of k/ω_s which are not excessively large, the third term in Eq. (53) is approximately $(\omega_t^2/2k^4)\,\omega_s^2 \ln(k^2/\omega_s^2)$ i.e., one order higher than the second term. Writing $\epsilon^l \approx 1$ in Eq. (54) we obtain $\omega_s^2 = \omega_t^2$, i.e., for the accuracy required here we can use Eq. (52). In this case the damping due to pair production is

$$\frac{\gamma^t_{\text{pair}}}{\omega} = \frac{e^2}{12}\frac{\omega_t^2}{\omega^2}\sqrt{1 - \frac{4m^2}{\omega_t^2}}\left(1 + \frac{2m^2}{\omega_t^2}\right). \quad (57)$$

The temperature corrections $\delta\gamma^t_{\text{pair}}$ given by (29) are small for an ultrarelativistic Boltzmann gas. For $\mu\beta \ll 1$ these corrections are proportional to β^2; when $\omega\beta \gg 1$ they are exponentially small.

$$\frac{\delta\gamma^t_{\text{pair}}}{\omega} = -\frac{e^2}{12}\frac{\pi\omega_t^2}{\omega^2}\left(1 + \frac{2m^2}{\omega^2}\right)\sqrt{1 - \frac{4m^2}{\omega_t^2}}\,\omega_t^2\beta^2 \quad \text{for } \omega\beta \ll 1. \quad (58)$$

Equation (57) is valid for a degenerate gas when $\mu + \kappa < \omega/2$; when $\omega/2 + \kappa < \mu$ the damping vanishes.

10. We discuss further the possibility of realizing the inequality Re $\epsilon^t > 1$ for the transverse oscillations. We consider the limiting case Re $\epsilon^t \gg 1$ or, what is the same thing, $k^2 \gg \omega^2$. It follows from Eq. (10)* that for $k^2 \gg 4\epsilon^2$

$$\varepsilon^t = 1 + \frac{\Lambda}{\omega^2 k^2}\left(1 + 3\frac{\omega^2}{k^2}\right), \quad (59)$$

$$\Lambda = \frac{16\pi e^2}{3}\int \frac{p^2}{\varepsilon_p} f(\varepsilon_p)\,dp. \quad (60)$$

The condition Re $\epsilon^t \gg 1$ is then actually possible when $\Lambda \gg \omega^2, k^2$, i.e., at very high densities. Thus, for a nondegenerate ultrarelativistic electron gas $\Lambda = 32\pi e^2 N/\beta$ and for a degenerate gas, $\Lambda = 8\pi e^2 N p_0$ where p_0 is the limiting momentum at the Fermi surface.

In this case the transverse oscillations are damped because of the inverse Cerenkov effect. This damping is exponentially small ($k\beta \gg 1$) for a nondegenerate electron gas [cf. Eq. (24)]

$$\gamma_{\text{cer}}/\omega = \tfrac{1}{128}\pi\,(\text{Re}\,\varepsilon^t)\,k^3\beta^3 e^{-\beta k/2}. \quad (61)$$

A particle moving in a medium with $\epsilon^t > 1$ will experience Cerenkov losses; in the region in which the medium is transparent these losses are given by the well-known expression (cf. reference 6):

$$W^t = 2e^2\int_0^\infty \omega\,d\omega \int_0^\infty \frac{q^3\,dq}{q^2 + \omega^2/v^2}\,\delta\left[q^2 + \omega^2\left(\frac{1}{v^2} - \varepsilon^t\left(\omega, \sqrt{q^2 + \frac{\omega^2}{v^2}}\right)\right)\right], \quad (62)$$

where v is the particle velocity and W^t is the energy loss per unit path of the particle. Making the substitution $\epsilon^t = 1 + \Lambda/\omega^2 k^2$ throughout, we have

$$W^t = \frac{e^2}{2v^2\sqrt{\Lambda}}\int_0^{\omega_{max}} \omega\,d\omega\,(\omega_{max}^2 - \omega^2), \quad (63)$$

where $\omega_{max}^2 = v^2\sqrt{\Lambda}$. Thus, a wide spectrum is radiated (up to $\omega = \omega_{max}$). Finally, after integrating over frequency we have

$$W^t = \tfrac{1}{8}e^2\omega_{max}^2. \quad (64)$$

11. The static magnetic susceptibility of a relativistic electron gas has been investigated by Silin and Rukhadze.[11] The dielectric constant found in the present work can be utilized to find a more general expression that takes account of the positron contribution. We shall use for $\chi \ll 1$ the definition

$$\chi(0) \approx \frac{\omega^2}{k^2}(\varepsilon^t - \varepsilon^l) = \frac{\omega^2}{k^2}(\varepsilon^t - 1) \quad \text{for } \omega, k \to 0. \quad (65)$$

Using Eq. (10) we have

$$\chi = \frac{4}{3}\frac{e^2\pi}{(2\pi)^3}\int_0^\infty \frac{\overset{-}{n}_p + \overset{+}{n}_p}{\varepsilon_p}\,dp, \quad (66)$$

*When $\omega^2 \ll \Lambda/k^2 \ll k^2$ the second term in the round bracket can be neglected.

which differs from the result obtained in reference 11 only in that there is an additional term $n_{\mathbf{p}}^{+}$. In particular, at equilibrium with respect to pair production ($\mu \approx 0$)

$$\chi = (e^2/6\pi^2) \ln (2.9/m\beta). \qquad (67)$$

A similar modification, specifically, a term with $n_{\mathbf{p}}^{+}$, must be introduced into the expression for the shielding radius for a static field (Debye radius) found in reference 1:

$$r_{\mathrm{D}}^{-2} = -4\pi e^2 \int \frac{2}{(2\pi)^3} d\mathbf{p} \left(\frac{dn_{\mathbf{p}}^{-}}{d\varepsilon_{\mathbf{p}}} + \frac{dn_{\mathbf{p}}^{+}}{d\varepsilon_{\mathbf{p}}} \right). \qquad (68)$$

In particular, when $\mu = 0$

$$r_{\mathrm{D}}^{-2} = (8e^2/\pi\beta^2)\, \zeta(2). \qquad (69)$$

This dependence of the Debye radius on temperature has been indicated by Fradkin.[7]

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