COLLISION INTEGRAL FOR CHARGED PARTICLES

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Submitted to JETP editor January 4, 1961

J. Exptl. Theoret. Phys. (U.S.S.R.) 40, 1768-1774 (June, 1961)

The equation for the correlative distribution function with screening of the interaction of charged particles taken into account, which was obtained earlier by Klimontovich and Temko, is here solved. The correlative function is found. The collision integral is obtained for a system consisting of several types of nonrelativistic charged particles; this integral is suitable in particular for the description of states very different from the thermodynamic equilibrium state. It is shown that the screening of the Coulomb interaction is described by a complex permittivity tensor. This has made possible an extension to the case of relativistic distributions and the obtaining of the relativistic collision integral with the screening of the fields of the charged particles taken into account.

1. Klimontovich and Temko[1] have extended results obtained by Bogolyubov[2] to the quantum case and have shown that the collision integral $J_\alpha(\mathbf{p}_\alpha)$ for charged particles is determined by the formula

$$\frac{N_\alpha}{V} J_\alpha = \frac{i}{2\hbar}\int \frac{d\mathbf{k}}{(2\pi)^3}\Big\{h_\alpha\Big(\mathbf{k}, \mathbf{p}_\alpha + \frac{\hbar\mathbf{k}}{2}\Big) - h_\alpha\Big(\mathbf{k}, \mathbf{p}_\alpha - \frac{\hbar\mathbf{k}}{2}\Big) - h_\alpha\Big(-\mathbf{k}, \mathbf{p}_\alpha + \frac{\hbar\mathbf{k}}{2}\Big) + h_\alpha\Big(-\mathbf{k}, \mathbf{p}_\alpha - \frac{\hbar\mathbf{k}}{2}\Big)\Big\}, \quad (1.1)$$

where N_α/V is the number of particles of type α per unit volume, and the function $h_\alpha(\mathbf{k}, \mathbf{p}_\alpha)$ is connected with the correlative function $g_{\alpha\beta}(\mathbf{r}_\alpha - \mathbf{r}_\beta, \mathbf{p}_\alpha, \mathbf{p}_\beta)$ by the relation

$$h_\alpha(\mathbf{k}, \mathbf{p}_\alpha) = \sum_\beta \frac{N_\alpha}{V}\frac{N_\beta}{V} \nu_{\alpha\beta}(k) \int d\mathbf{p}_\beta G_{\alpha\beta}(\mathbf{k}, \mathbf{p}_\alpha, \mathbf{p}_\nu). \quad (1.2)$$

Here

$$G_{\alpha\beta}(\mathbf{k}, \mathbf{p}_\alpha, \mathbf{p}_\beta) = \int e^{-i\mathbf{kr}}\, d\mathbf{r}\, g_{\alpha\beta}(\mathbf{r}, \mathbf{p}_\alpha, \mathbf{p}_\beta),$$

$$\nu_{\alpha\beta}(k) = \int e^{i\mathbf{kr}}\, d\mathbf{r}\, \frac{e_\alpha e_\beta}{r}.$$

Furthermore, according to the work of Klimontovich and Temko[1] one has the following equation which determines the correlative function:

$$\frac{N_\alpha}{V}\frac{N_\beta}{V}\nu_{\alpha\beta}(k)\, G_{\alpha\beta}(\mathbf{k}, \mathbf{p}_\alpha, \mathbf{p}_\beta) = \frac{1}{\hbar}\Big\{i\pi\delta\Big(\frac{\mathbf{kp}_\alpha}{m_\alpha} - \frac{\mathbf{kp}_\beta}{m_\beta}\Big) + \mathrm{P}\frac{1}{\mathbf{kp}_\alpha/m_\alpha - \mathbf{kp}_\beta/m_\beta}\Big\}\Big\{\nu_{\alpha\alpha}(k)\,\nu_{\beta\beta}(k)\frac{N_\alpha}{V}\frac{N_\beta}{V}$$
$$\times [f_\alpha(\mathbf{p}_\alpha + \hbar\mathbf{k}/2) f_\beta(\mathbf{p}_\beta - \hbar\mathbf{k}/2) - f_\alpha(\mathbf{p}_\alpha - \hbar\mathbf{k}/2) f_\beta(\mathbf{p}_\beta + \hbar\mathbf{k}/2)]$$
$$+ (N_\alpha/V)\,\nu_{\alpha\alpha}(k)\,[f_\alpha(\mathbf{p}_\alpha + \hbar\mathbf{k}/2) - f_\alpha(\mathbf{p}_\alpha - \hbar\mathbf{k}/2)]\, h_\beta(-\mathbf{k}, \mathbf{p}_\beta)$$
$$- (N_\beta/V)\,\nu_{\beta\beta}(k)\,[f_\beta(\mathbf{p}_\beta + \hbar\mathbf{k}/2) - f_\beta(\mathbf{p}_\beta - \hbar\mathbf{k}/2)]\, h_\alpha(\mathbf{k}, \mathbf{p}_\alpha)\Big\}. \quad (1.3)$$

The symbol P means that here and in what follows the singular integrals are to be taken by using the principal value.

In the paper of Klimontovich and Temko[1] this equation was not solved, although it was shown that such an equation must lead to screening of the interaction of the particles at large distances. Screening of the Coulomb interaction in the quantum collision integral was obtained by Konstantinov and Perel'[3] in the case of states differing slightly from the state of thermodynamic equilibrium. Here, by solving Eq. (1.3), we shall obtain a collision integral that is valid for the description of states decidedly different from the equilibrium state. In the classical theory the analogous treatment for collisions of electrons with electrons has been carried out in papers by Balescu[4] and Lenard.[5]

2. We introduce functions of the complex variable ω,

$$H(\omega, \mathbf{k}, \pm) = \frac{1}{2\pi i}\sum_\alpha \int \frac{d\mathbf{p}_\alpha}{\omega - \mathbf{kp}_\alpha/m_\alpha} h_\alpha(\pm\mathbf{k}, \mathbf{p}_\alpha), \quad (2.1)$$

which have no singularities in either the upper or lower half-plane, but which have a discontinuity on passage across the real axis. On the real axis the limit H^+ of the function analytic in the upper half-plane and the limit H^- of the function analytic in the lower half-plane obey the Sokhotskii-Plemel' relations

$$H^\pm(\omega) = \frac{1}{2\pi i}\sum_\alpha \int d\mathbf{p}_\alpha h_\alpha(\mathbf{p}_\alpha)\Big\{P\frac{1}{\omega - \mathbf{kp}_\alpha/m_\alpha} \mp i\pi\delta\Big(\omega - \frac{\mathbf{kp}_\alpha}{m_\alpha}\Big)\Big\}. \quad (2.2)$$

From Eq. (1.3) we have

$$h_\alpha(\mathbf{k}, \mathbf{p}_\alpha) = \frac{2\pi i v_{\alpha\alpha}(k) N_\alpha / V}{\hbar \varepsilon^-(\mathbf{k}\mathbf{p}_\alpha/m_\alpha, \mathbf{k})} \Big[f_\alpha\Big(\mathbf{p}_\alpha + \frac{\hbar\mathbf{k}}{2}\Big) F^-\Big(\frac{\mathbf{k}\mathbf{p}_\alpha}{m_\alpha}, \mathbf{k}, -\Big) - f_\alpha\Big(\mathbf{p}_\alpha - \frac{\hbar\mathbf{k}}{2}\Big) F^-\Big(\frac{\mathbf{k}\mathbf{p}_\alpha}{m_\alpha}, \mathbf{k}, +\Big)\Big] + \frac{v_{\alpha\alpha}(k)}{\hbar}\frac{N_\alpha}{V}\Big[f_\alpha\Big(\mathbf{p}_\alpha + \frac{\hbar\mathbf{k}}{2}\Big) - f_\alpha\Big(\mathbf{p}_\alpha - \frac{\hbar\mathbf{k}}{2}\Big)\Big] 2\pi i \frac{H^-(\mathbf{k}\mathbf{p}_\alpha/m_\alpha, \mathbf{k}, -)}{\varepsilon^-(\mathbf{k}\mathbf{p}_\alpha/m_\alpha, \mathbf{k})}, \quad (2.3)$$

where ϵ^- and F^- are the limits on approaching the real axis from below of the functions

$$\varepsilon(\omega, \mathbf{k}) = 1 + \frac{2\pi i}{\hbar}[F(\omega, \mathbf{k}, +) - F(\omega, \mathbf{k}, -)], \quad (2.4)$$

$$F(\omega, \mathbf{k}, \pm) = \frac{1}{2\pi i}\sum_\alpha \int \frac{d\mathbf{p}_\alpha}{\omega - \mathbf{k}\mathbf{p}_\alpha/m_\alpha} v_{\alpha\alpha}(k) \frac{N_\alpha}{V} f_\alpha\Big(\mathbf{p}_\alpha \pm \frac{\hbar\mathbf{k}}{2}\Big). \quad (2.5)$$

It follows from Eqs. (1.3) and (2.3) that for the solution of the system of singular integral equations (1.3) it is sufficient to determine the functions $H(\omega, \mathbf{k}, \pm)$. The corresponding equations for the determination of these functions can be obtained in the following way. Let us multiply Eq. (2.3) by $\delta(\omega - \mathbf{k}\mathbf{p}_\alpha/m_\alpha)$ and integrate over $\mathbf{p}_\alpha$. On summing the result of the integration over α, we get

$$[H^-(\omega, \mathbf{k}, +) - H^+(\omega, \mathbf{k}, +)]\,\varepsilon^-(\omega, \mathbf{k}) - H^-(\omega, \mathbf{k}, -)\,[\varepsilon^-(\omega, \mathbf{k}) - \varepsilon^+(\omega, \mathbf{k})] = (2\pi i/\hbar)\,[F^+(\omega, \mathbf{k}, -)F^-(\omega, \mathbf{k}, +) - F^+(\omega, \mathbf{k}, +)F^-(\omega, \mathbf{k}, -)]. \quad (2.6)$$

We get a second equation by changing the signs of ω and $\mathbf{k}$ in Eq. (2.6):

$$[H^-(\omega, \mathbf{k}, -) - H^+(\omega, \mathbf{k}, -)]\,\varepsilon^+(\omega, \mathbf{k}) - H^+(\omega, \mathbf{k}, +)\,[\varepsilon^-(\omega, \mathbf{k}) - \varepsilon^+(\omega, \mathbf{k})] = (2\pi i/\hbar)[F^+(\omega, \mathbf{k}, -)F^-(\omega, \mathbf{k}, +) - F^+(\omega, \mathbf{k}, +)F^-(\omega, \mathbf{k}, -)]. \quad (2.7)$$

The system (2.6) and (2.7) enables us to determine the functions H.

To solve (2.6) and (2.7) we subtract one equation from the other. The result is the following relation:

$$[H^-(\omega, \mathbf{k}, +) - H^-(\omega, \mathbf{k}, -)]\,\varepsilon^-(\omega, \mathbf{k}) = [H^+(\omega, \mathbf{k}, +) - H^+(\omega, \mathbf{k}, -)]\,\varepsilon^+(\omega, \mathbf{k}). \quad (2.8)$$

The left member of this relation is analytic in the lower half-plane of the complex variable ω, and the right member is analytic in the upper half-plane. The analytic function with zero discontinuity on the line that separates the regions of analyticity is obviously analytic in the entire plane of the complex variable. The condition that the distribution functions go to zero at infinitely large momenta means that both the right and left members of Eq. (2.8) go to zero at infinity, and from this and the condition that

$$\varepsilon^\pm(\omega, \mathbf{k}) \neq 0 \quad (2.9)$$

we get

$$H(\omega, \mathbf{k}, +) = H(\omega, \mathbf{k}, -) \equiv H(\omega, \mathbf{k}). \quad (2.10)$$

The absence of zeroes of the functions $\epsilon^\pm(\omega, \mathbf{k})$ in the regions in which they are analytic has a simple physical meaning. The fact is that the function $\epsilon(\omega, \mathbf{k})$ is connected with the complex permittivity tensor $\epsilon_{ij}(\omega, \mathbf{k})$ of the plasma by the relation*

$$k^2\varepsilon(\omega, \mathbf{k}) = k_i k_j \varepsilon_{ij}(\omega, \mathbf{k}).$$

In this connection, the condition (2.9) corresponds to the absence of undamped and increasing self-consistent oscillations of the density (so called longitudinal plasma waves†) in the state in which the distribution of the particles is described by the functions $f_\alpha(\mathbf{p}_\alpha)$. In other words, the condition (2.9) is the condition for the stability of the system of charged particles against perturbations associated with changes of the charge density. In what follows it is assumed that this condition is satisfied.‡

Equation (2.10) enables us to write Eq. (2.6) in the following form

$$\frac{H^-(\omega, \mathbf{k})}{\varepsilon^-(\omega, \mathbf{k})} - \frac{H^+(\omega, \mathbf{k})}{\varepsilon^+(\omega, \mathbf{k})} = \frac{F^+(\omega, \mathbf{k}, -) + F^+(\omega, \mathbf{k}, +)}{2\varepsilon^+(\omega, \mathbf{k})} - \frac{F^-(\omega, \mathbf{k}, -) + F^-(\omega, \mathbf{k}, +)}{2\varepsilon^-(\omega, \mathbf{k})} - \frac{F^+(\omega, \mathbf{k}, -) + F^+(\omega, \mathbf{k}, +) - F^-(\omega, \mathbf{k}, -) - F^-(\omega, \mathbf{k}, +)}{2\varepsilon^+(\omega, \mathbf{k})\,\varepsilon^-(\omega, \mathbf{k})}. \quad (2.11)$$

This equation determines the discontinuity of the function H/ϵ on the real axis of the plane of the complex variable ω. As is well known,[6,7] the problem of the determination of an analytic function $\mathfrak{A}$ which goes to zero at infinity from its discontinuity a on a path L is solved [as can be seen without difficulty from the Sokhotskii-Plemel' relations of the type of Eq. (2.3)] by the formulas**

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*Regarding the complex permittivity tensor of a plasma see reference 9.

†The damped plasma oscillations which are often considered correspond to zeroes of the analytic continuation of the function $\varepsilon(\omega, \mathbf{k})$ to adjacent sheets of the complex variable ω.

‡We note that for the obtaining of the collision integral it is sufficient for the condition (2.9) to be satisfied on the path of integration of the formulas (2.1) and (2.5). The condition then corresponds to the absence of self-consistent oscillations capable of being absorbed and emitted by particles with the distributions f_α.

**Furthermore,

$$\mathfrak{A}^+(z) = \frac{1}{2}a(z) + \frac{1}{2\pi i}\int_L \frac{dz'}{z' - z} a(z'),$$

$$\mathfrak{A}^-(z) = \frac{1}{2}a(z) + \frac{1}{2\pi i}\int_L \frac{dz'}{z' - z} a(z')$$

$$\mathfrak{A}^+(z) - \mathfrak{A}^-(z) = a(z) \text{ on } L, \quad \mathfrak{A}(z) = \frac{1}{2\pi i}\int_L \frac{dz' a(z')}{z' - z}.$$

Therefore the solution of Eq. (2.11) can be written in the following form:

$$\frac{H(\omega, \mathbf{k})}{\varepsilon(\omega, \mathbf{k})} = -\frac{F(\omega, \mathbf{k}, +) + F(\omega, \mathbf{k}, -)}{2\varepsilon(\omega, \mathbf{k})} + \frac{1}{2\pi i}\int \frac{d\omega'}{\omega' - \omega}$$
$$\times \frac{F^+(\omega', \mathbf{k}, -) + F^+(\omega', \mathbf{k}, +) - F^-(\omega', \mathbf{k}, -) - F^-(\omega', \mathbf{k}, +)}{2\varepsilon^+(\omega', \mathbf{k})\,\varepsilon^-(\omega', \mathbf{k})}. \quad (2.12)$$

3. Equations (2.12), (2.3), and (1.3) enable us to write an explicit expression for the correlative function in a system of charged particles. We note that a knowledge of this function can be necessary, for example, for the determination of the energy of the system of particles in a nonequilibrium state. It is obvious that now we can also write an expression for the collision integral. We note that Eq. (2.11) suffices for this purpose, because Eq. (1.1) contains the difference $h_\alpha(\mathbf{k}, \mathbf{p}_\alpha) - h_\alpha(-\mathbf{k}, \mathbf{p}_\alpha)$. Substituting the expressions obtained in Eq. (1.1), we find

$$\frac{N_\alpha}{V} J_\alpha(\mathbf{p}_\alpha) = \sum_\beta \frac{N_\alpha}{V}\frac{N_\beta}{V}\int \frac{d\mathbf{p}'_\alpha}{(2\pi\hbar)^3}\, d\mathbf{p}'_\beta\, d\mathbf{p}_\beta\, w_{\alpha\beta}(\mathbf{p}_\alpha, \mathbf{p}'_\alpha)$$
$$\times \delta(\mathbf{p}'_\alpha + \mathbf{p}'_\beta - \mathbf{p}_\alpha - \mathbf{p}_\beta)\,\delta(p'^2_\alpha/2m_\alpha + p'^2_\beta/2m_\beta \quad (3.1)$$
$$- p^2_\alpha/2m_\alpha - p^2_\beta/2m_\beta)\,[f_\alpha(\mathbf{p}_\alpha) f_\beta(\mathbf{p}_\beta) - f_\alpha(\mathbf{p}'_\alpha) f_\beta(\mathbf{p}'_\beta)],$$

$$w_{\alpha\beta}(\mathbf{p}_\alpha, \mathbf{p}'_\alpha) = \frac{2\pi \nu^2_{\alpha\beta}\left(\left|\frac{\mathbf{p}_\alpha - \mathbf{p}'_\alpha}{\hbar}\right|\right)}{\hbar\,\varepsilon^+\left(\frac{p'^2_\alpha - p^2_\alpha}{2\hbar m_\alpha}, \frac{\mathbf{p}'_\alpha - \mathbf{p}_\alpha}{\hbar}\right)\varepsilon^-\left(\frac{p'^2_\alpha - p^2_\alpha}{2\hbar m_\alpha}, \frac{\mathbf{p}'_\alpha - \mathbf{p}_\alpha}{\hbar}\right)}. \quad (3.2)$$

We note that (3.2), like the original (1.3), has been obtained on the assumption that the interaction is weak. This means that for very small impact parameters Eq. (3.2) must not be used. In the case of states only slightly different from the state of thermodynamic equilibrium, for which we can use the linear approximation, Eq. (3.2) goes over into the formula of the paper of Konstantinov and Perel'.[3]

In the classical limit, which corresponds to sufficiently distant collisions, Eq. (3.1) takes the form

$$\frac{N_\alpha}{V} J_\alpha = \frac{\partial}{\partial p^i_\alpha}\sum_\beta \int d\mathbf{p}_\beta I^{ij}_{\alpha\beta}\left(\frac{\mathbf{p}_\alpha}{m_\alpha}, \frac{\mathbf{p}_\beta}{m_\beta}\right)\left[\frac{\partial f_\alpha}{\partial p^j_\alpha} f_\beta - f_\alpha \frac{\partial f_\beta}{\partial p^j_\beta}\right]\frac{N_\alpha}{V}\frac{N_\beta}{V}, \quad (3.3)$$

$$I^{ij}_{\alpha\beta}(\mathbf{v}_\alpha, \mathbf{v}_\beta) = \int \frac{d\mathbf{k}}{(2\pi)^3}\frac{k^i k^j \pi \nu^2_{\alpha\beta}(k)\,\delta(\mathbf{k}\mathbf{v}_\alpha - \mathbf{k}\mathbf{v}_\beta)}{\varepsilon^+_{\mathrm{cl}}(\mathbf{k}\mathbf{v}_\alpha, \mathbf{k})\,\varepsilon^-_{\mathrm{cl}}(\mathbf{k}\mathbf{v}_\alpha, \mathbf{k})}, \quad (3.4)$$

where ϵ_{cl} is given as a function of the complex variable ω by the formula

$$\varepsilon_{\mathrm{cl}}(\omega, \mathbf{k}) = 1 + \sum_\alpha \frac{4\pi e^2_\alpha}{k^2}\frac{N_\alpha}{V}\int \frac{d\mathbf{p}_\alpha}{\omega - \mathbf{k}\mathbf{p}_\alpha/m_\alpha}\left(\mathbf{k}\frac{\partial f_\alpha}{\partial \mathbf{p}_\alpha}\right). \quad (3.5)$$

In the special case of electron-electron collisions the formula (3.4) corresponds to the formula obtained by Balescu and by Lenard.[4,5] We note that in Eq. (3.4) the integral diverges at large values of k, which correspond to small impact parameters; this is due to the fact that here the classical approximation cannot be applied, as it is in the passage from Eq. (3.1) to Eq. (3.3). The necessity of cutting off the integral in Eq. (3.4) can be connected with the lack of validity of perturbation theory, which with the Boltzmann distribution begins to fail at the impact parameter $\rho_{\mathrm{min}} \sim e^2/\kappa T$.

Finally, if we neglect the difference between ϵ_{cl} and unity, the expression (3.3) goes over into the collision integral for charged particles in the form that was given by Landau.[8] Here also at large impact parameters one must resort to cutting off the integral, which converges automatically in our treatment. The convergence is due to the consistent inclusion of effects of polarization of the medium as described by the permittivity.

4. The results of the preceding section regarding the collision integral mean that in calculating collision probabilities we must use instead of the Coulomb field the expression for the field of a particle in a plasma, with the complex permittivity taken into account. This is particularly clear from Eq. (3.2). Here the value of the complex permittivity corresponds to the first approximation of perturbation theory, and in the nonquantum case, to which we confine ourselves from now on, is given by[9]

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \delta_{ij} + \sum_\alpha \frac{4\pi e^2_\alpha}{\omega}\frac{N_\alpha}{V}\int \frac{d\mathbf{p}_\alpha}{\omega - \mathbf{k}\mathbf{v}_\alpha} v^i_\alpha \frac{\partial f_\alpha}{\partial p^l_\alpha}$$
$$\times\left(\delta_{jl}\left[1 - \frac{\mathbf{k}\mathbf{v}_\alpha}{\omega}\right] + \frac{k_l v^j_\alpha}{\omega}\right). \quad (4.1)$$

Here, as before, we are dealing with analytic functions that have cuts along the real axis.

Only the longitudinal interaction plays any part in the nonrelativistic approximation, and therefore in the formulas written above it is the quantity $k_i k_j \epsilon_{ij}$ that appears. In the nonrelativistic case this is no longer true. We shall now proceed to the consideration of this case. Here we shall not deal with the equation for the correlative functions, but shall at once take into account the polarization of the medium, and use the permittivity for the determination of the field in the plasma.

It is clear that for what follows we must define the probability of collision between two particles. For this we need to know in the nonquantum limit the quantity

$$\lim_{\hbar\to 0} w_{\alpha\beta}(\mathbf{p}_\alpha, \mathbf{p}_\alpha + \hbar\mathbf{k})\,\hbar/2,$$

where $w_{\alpha\beta}$ is the probability of a collision of particles α and β with change of the momentum of particle α from $\mathbf{p}_\alpha$ to $\mathbf{p}_\alpha + \hbar\mathbf{k}$. This quantity serves in the following way to determine the kernel of the collision integral, which obviously is still of the form (3.3):

$$I^{ij}_{\alpha\beta}(\mathbf{v}_\alpha, \mathbf{v}_\beta) = \int \frac{d\mathbf{k}}{(2\pi)^3} k_i k_j \delta(\mathbf{k}\mathbf{v}_\alpha - \mathbf{k}\mathbf{v}_\beta) \lim_{\hbar\to 0} w_{\alpha\beta}(\mathbf{p}_\alpha, \mathbf{p}_\alpha + \hbar\mathbf{k})\hbar/2. \tag{4.2}$$

Equation (4.2) can easily be obtained by going to the limit $\hbar = 0$ in the quantum collision integral.

For the calculation of the scattering probability we must determine the fields. Using the gauge in which the scalar potential is zero, we can write the following equations for the Fourier components of the vector potential of the field produced by the uniform motion of a charge e_β with the velocity $\mathbf{v}_\beta$ in a medium with the complex permittivity tensor $\epsilon_{ij}(\omega, \mathbf{k})$:

$$a_{ij}(\mathbf{k}\mathbf{v}_\beta, \mathbf{k}) A_j \equiv \{(\mathbf{k}\mathbf{v}_\beta)^2 c^{-2}\varepsilon_{ij}(\mathbf{k}\mathbf{v}_\beta, \mathbf{k}) - k^2\delta_{ij} + k_i k_j\} A_j = -4\pi c^{-1} e_\beta v^i_\beta, \tag{4.3}$$

from which we have

$$A_i = -(4\pi/c)\, e_\beta a^{-1}_{ij}(\mathbf{k}\mathbf{v}_\beta, k)\, v^j_\beta. \tag{4.4}$$

According to Møller's paper,[10] with our gauge

$$\lim_{\hbar=0} w_{\alpha\beta}(\mathbf{p}_\alpha, \mathbf{p}_\alpha + \hbar\mathbf{k})\hbar/2 = \pi |e_\alpha c^{-1}\mathbf{v}_\alpha \mathbf{A}|^2. \tag{4.5}$$

Therefore for the collision integral we get

$$I^{ij}_{\alpha\beta}(\mathbf{v}_\alpha, \mathbf{v}_\beta) = \frac{(4\pi e_\alpha e_\beta)^2}{c^4} \int \frac{d\mathbf{k}}{(2\pi)^3} \pi k_i k_j \delta(\mathbf{k}\mathbf{v}_\alpha - \mathbf{k}\mathbf{v}_\beta) \times |v^i_\alpha a^{-1}_{ij}(\mathbf{k}\mathbf{v}_\alpha, \mathbf{k})\, v^j_\beta|^2. \tag{4.6}$$

In the special case of an isotropic distribution the complex permittivity tensor has the form

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \varepsilon^{tr}(\omega, k)(\delta_{ij} - k^{-2}k_i k_j) + k^{-2}k_i k_j \varepsilon^l(\omega, k). \tag{4.7}$$

Here we have, according to Eq. (4.1)*

$$\varepsilon^{tr}(\omega, k) = 1 + \sum_\alpha \frac{2\pi e_\alpha^2}{\omega k^2} \int \frac{d\mathbf{p}_\alpha}{\omega - \mathbf{k}\mathbf{v}_\alpha} [\mathbf{k}[\mathbf{v}_\alpha \mathbf{k}]] \frac{\partial f_\alpha}{\partial \mathbf{p}_\alpha} \frac{N_\alpha}{V},$$

$$\varepsilon^l(\omega, k) = 1 + \sum_\alpha \frac{4\pi e_\alpha^2}{k^2} \int \frac{d\mathbf{p}_\alpha}{\omega - \mathbf{k}\mathbf{v}_\alpha} \left(\mathbf{k}\frac{\partial f_\alpha}{\partial \mathbf{p}_\alpha}\right) \frac{N_\alpha}{V}.$$

Then the formula (4.6) can be simplified and takes the following form:

$$I^{ij}_{\alpha\beta}(\mathbf{v}_\alpha \mathbf{v}_\beta) = (4\pi e_\alpha e_\beta)^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\pi k_i k_j}{k^4} \delta(\mathbf{k}\mathbf{v}_\alpha - \mathbf{k}\mathbf{v}_\beta) \times \left| \frac{1}{\varepsilon^l(\mathbf{k}\mathbf{v}_\alpha, \mathbf{k})} + \frac{k^2 \mathbf{v}_\alpha \mathbf{v}_\beta - (\mathbf{k}\mathbf{v}_\alpha)^2}{(\mathbf{k}\mathbf{v}_\alpha)^2 \varepsilon^{tr}(\mathbf{k}\mathbf{v}_\alpha, \mathbf{k}) - k^2 c^2} \right|^2. \tag{4.8}$$

In the limit $\epsilon^l = \epsilon^{tr} = 1$ Eq. (4.8) goes over into Eq. (22) of the paper of Klimontovich,[11] and therefore it corresponds to the relativistic collision integral of Belyaev and Budker.[12]

The integral in Eq. (4.8) must be cut off at large k for the same reasons as in the nonrelativistic treatment. At small k, corresponding to large impact parameters, the integral converges. We shall show in particular that in the case of relativistic temperatures the longitudinal and transverse permittivities lead to cutting off at distances of the order of the Debye radius. For this purpose we note that in the region of small values of $\mathbf{k}\mathbf{v}_\alpha$ we can use for the permittivities the approximate formulas[9,13]

$$\varepsilon^l \sim (kr_D)^{-2}, \qquad \varepsilon^{tr} \sim iv_T/(\mathbf{k}\mathbf{v}_\alpha)\, kr_D^2,$$

where v_T is the thermal velocity. The formula for ϵ^l corresponds to the Debye screening, and the formula for ϵ^{tr} corresponds to the region of the anomalous skin effect, for which $\epsilon^{tr}(\omega, k) \sim i/\omega k$.

It is clear that under conditions in which v_T is close to the speed of light, these approximate expressions for the permittivities lead to a cutting off at impact parameters of the order of the Debye radius. If, on the other hand, $v_T \ll c$, then unlike the longitudinal permittivity, which leads to a cutting off of the logarithmic divergence at the Debye radius, the transverse permittivity cuts off the divergence at parameters $\sim (c/v_T)\, r_D$. Under these conditions, however, the contribution of the transverse interaction to the collision integral is only a small correction. Therefore for the Boltzmann distribution there is no large error in cutting off both the transverse and the longitudinal interactions at the Debye radius.

We note that the kernel (4.8) can be used not only in the case of an isotropic distribution, but also in the case of a small departure from isotropy. For this purpose it is assumed to be possible to linearize the collision integral. In the case of a decidedly anisotropic distribution, such as occurs, for example, in the collision of beams of charged particles that are neutral taken on the whole, it is necessary to use the collision integral with the kernel (4.6).

Note added in proof (May 12, 1961). It was stated above that in Eq. (4.6), as indeed always when a collision integral of the Landau type[8] is being used, it is necessary to cut off the integration for large k. This shortcoming is absent for the ordinary Boltzmann collision integral, which holds also in our case and is written in the form (3.1) with the energy of the particle replaced by its relativistic value. One then has for the transition probability for distributions independent of the spin the following expression

*$[\mathbf{k}[\mathbf{v}_\alpha \mathbf{k}]] = \mathbf{k} \times [\mathbf{v}_\alpha \times \mathbf{k}]$.

$$W(\mathbf{p}_\alpha, \mathbf{p}_\beta; \mathbf{p}'_\alpha, \mathbf{p}'_\beta) = \frac{2\pi}{\hbar} \frac{(4\pi e_\alpha e_\beta)^2}{4E_\alpha(\mathbf{p}_\alpha) E_\alpha(\mathbf{p}'_\alpha) E_\beta(\mathbf{p}_\beta) E_\beta(\mathbf{p}'_\beta)}$$
$$\times \left\{c^2 [p'^r_\alpha p^i_\alpha + p'^i_\alpha p^r_\alpha] - \frac{1}{2}\delta_{ir}([E_\alpha(p'_\alpha) - E_\alpha(\mathbf{p}_\alpha)]^2 - c^2 [\mathbf{p}'_\alpha - \mathbf{p}_\alpha]^2)\right\}$$
$$\times \left\{c^2 [p'^l_\beta p^j_\beta + p'^j_\beta p^l_\beta] - \frac{1}{2}\delta_{lj}([E_\beta(\mathbf{p}'_\beta) - E_\beta(\mathbf{p}_\beta)]^2 - c^2 [\mathbf{p}'_\beta - \mathbf{p}_\beta]^2)\right\}$$
$$\times a^{-1}_{ij}\left(\frac{E_\beta(\mathbf{p}_\beta) - E_\beta(\mathbf{p}'_\beta)}{\hbar}, \frac{\mathbf{p}_\beta - \mathbf{p}'_\beta}{\hbar}\right)$$
$$\times a^{-1}_{rl}\left(\frac{E_\alpha(\mathbf{p}_\alpha) - E_\alpha(\mathbf{p}'_\alpha)}{\hbar}, \frac{\mathbf{p}_\alpha - \mathbf{p}'_\alpha}{\hbar}\right). \quad (4.9)$$

In these formulas one must use the quantum expression for the permittivity tensor.

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Translated by W. H. Furry
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